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## ON $p$ -SOLUBLE GROUPS WITH A GENERALIZED $p$ -CENTRAL OR POWERFUL SYLOW $p$ -SUBGROUP

E. I. KHUKHRO

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ABSTRACT. Let  $G$  be a finite  $p$ -soluble group, and  $P$  a Sylow  $p$ -subgroup of  $G$ . It is proved that if all elements of  $P$  of order  $p$  (or of order  $\leq 4$  for  $p = 2$ ) are contained in the  $k$ -th term of the upper central series of  $P$ , then the  $p$ -length of  $G$  is at most  $2m + 1$ , where  $m$  is the greatest integer such that  $p^m - p^{m-1} \leq k$ , and the exponent of the image of  $P$  in  $G/O_{p',p}(G)$  is at most  $p^m$ . It is also proved that if  $P$  is a powerful  $p$ -group, then the  $p$ -length of  $G$  is equal to 1.

### 1. Introduction

A finite  $p$ -group  $P$  is called  $p$ -central if all its elements of order  $p$  are contained in the centre:  $\Omega_1(P) \leq Z(P)$ . Sometimes this definition is modified in the case of  $p = 2$  to require that all elements of order  $\leq 4$  belong to  $Z(P)$ . Such  $p$ -groups are in many respects dual to powerful  $p$ -groups (and the above-mentioned modification for  $p = 2$  reflects the definition of powerful 2-groups). Although  $p$ -central  $p$ -groups received less attention in the literature than the very important case of powerful  $p$ -groups, there are several papers devoted to  $p$ -central  $p$ -groups and properties of their embeddings in finite groups; the reader can find relevant references in [4].

González-Sánchez and Weigel [4] initiated the study of more general classes: a finite  $p$ -group  $P$  is called  $p^i$ -central of height  $k$  if all its elements of order dividing  $p^i$  are contained in the  $k$ -th term of the upper central series:  $\Omega_i(P) \leq \zeta_k(P)$ . In particular, they proved [4, Theorem E] that if, for an odd prime  $p$ , a Sylow  $p$ -subgroup of a finite  $p$ -soluble group  $G$  is  $p$ -central of height  $p - 2$ , then  $G$  has  $p$ -length 1.

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In this note we generalize this result to arbitrary height (including the case  $p = 2$  with the above-mentioned proviso). Namely, we obtain a bound for the  $p$ -length of a  $p$ -soluble group  $G$  whose Sylow  $p$ -subgroup is  $p$ -central of height  $k$  (Theorem 3.1). This result is derived from a bound for the exponent of a Sylow  $p$ -subgroup of  $G/O_{p',p}(G)$  (Theorem 3.2), which is proved on the basis of Hall–Higman theorems.

We also prove the result “dual” to [4, Theorem E], that if a finite  $p$ -soluble group  $G$  has a powerful Sylow  $p$ -subgroup, then the  $p$ -length of  $G$  is equal to 1 (Theorem 4.1).

## 2. Preliminaries

We shall need the following well-known property of coprime action by automorphisms. Recall that for a finite  $p$ -group  $P$  by definition  $\Omega_i(P) = \langle g \in P \mid g^{p^i} = 1 \rangle$ .

**Lemma 2.1** ([6, Kap. IV, Satz 5.12]). *Suppose that a finite  $p'$ -group  $A$  acts by automorphisms on a finite  $p$ -group  $P$ . If  $A$  acts trivially on  $\Omega_1(P)$  for  $p \neq 2$ , or on  $\Omega_2(P)$  for  $p = 2$ , then  $A$  acts trivially on  $P$ .*

Some other well-known properties of coprime actions of groups of automorphisms will be used without special references.

Recall that if a finite group  $G$  acts by automorphisms on an elementary abelian  $p$ -group  $V$ , then  $V$  can be regarded as a vector space over the field of  $p$  elements  $\mathbb{F}_p$  and the action of  $G$  by conjugation on  $V$  can be regarded as action by linear transformations of this vector space. The linear transformation of  $V$  induced by an element  $g \in G$  is denoted by  $T(g)$ . We use the right operator notation for this action: for  $v \in V$  and  $g \in G$  the image of  $v$  under  $T(g)$  is denoted by  $vT(g)$ . For example, if  $V$  is a normal elementary abelian section of  $G$ , then  $G$  acts on  $V$  by conjugation and  $vT(g)$  is equal to the image of the group element  $\hat{v}^g$ , where  $\hat{v}$  is an inverse image of  $v$  in  $G$ . Note that  $v(T(g) - \mathbf{1}_V)$ , where  $\mathbf{1}_V$  is the identity transformation of  $V$ , is equal to the image of the group commutator  $[\hat{v}, g]$ , which is also equal to  $[v, g]$  in the natural semidirect product  $V \rtimes G$ .

We also recall Theorem B from the celebrated Hall–Higman paper [5].

**Theorem 2.2** ([5, Theorem B]). *Let  $H$  be a  $p$ -soluble linear group over a field of characteristic  $p$ , with no normal  $p$ -subgroup greater than 1. If  $h$  is an element of order  $p^m$  in  $H$ , then the minimal equation of  $h$  is  $(x - 1)^r = 0$ , where  $r = p^m$ , unless there is an integer  $m_0$ , not greater than  $m$ , such that  $p^{m_0} - 1$  is a power of a prime  $q$  for which a Sylow  $q$ -subgroup of  $H$  is non-abelian, in which case, if  $m_0$  is the least such integer,  $p^m - p^{m-m_0} \leq r \leq p^m$ .*

We shall only need the fact that we always have  $p^m - p^{m-1} \leq r \leq p^m$ .

When an element  $g \in GL(V)$  of order  $p^m$  acts as a linear transformation on a vector space  $V$  over a field of characteristic  $p$ , its minimal polynomial always has the form  $(x - 1)^r = 0$ , because  $x^{p^m} - 1 = (x - 1)^{p^m}$  in characteristic  $p$ . It follows that  $V$  has a basis in which the matrix of  $g$  has Jordan normal form, since the only eigenvalue is 1. The maximum size of Jordan blocks is  $p^m \times p^m$ .

It is well known that the natural semidirect product  $V\langle g \rangle$  of groups  $V$  and  $\langle g \rangle$  contains an element of order  $p^{m+1}$  if and only if there is at least one Jordan block of size  $p^m \times p^m$ .

### 3. Generalized $p$ -central Sylow $p$ -subgroup

Recall that  $O_{p'}(G)$  is the maximal normal  $p'$ -subgroup of a finite group  $G$ ; then  $O_{p',p}(G)$  is the full inverse image of the maximal normal  $p$ -subgroup of  $G/O_{p'}(G)$ , and so on, defining by induction the terms of the *upper  $p$ -series*  $O_{p',p,p',p,\dots}(G)$ . A finite group  $G$  is  *$p$ -soluble* if  $G = O_{p',p,p',p,\dots,p,p'}(G)$  and the minimum number of symbols  $p$  in this equation is called the  *$p$ -length* of  $G$ .

**Theorem 3.1.** *Let  $P$  be a Sylow  $p$ -subgroup of a finite  $p$ -soluble group  $G$ . Suppose that  $\Omega_1(P) \leq \zeta_k(P)$  for  $p \neq 2$ , or  $\Omega_2(P) \leq \zeta_k(P)$  for  $p = 2$ . Then the  $p$ -length of  $G$  is at most  $2m + 1$ , where  $m$  is the maximum integer such that  $p^m - p^{m-1} \leq k$ .*

In particular, as a rough estimate,  $m < 1 + \log_p k$ , so that the  $p$ -length is at most  $3 + 2 \log_p k$ .

Theorem 3.1 will follow from a bound for the exponent of a Sylow  $p$ -subgroup of  $G/O_{p',p}(G)$ .

**Theorem 3.2.** *Let  $P$  be a Sylow  $p$ -subgroup of a finite  $p$ -soluble group  $G$ . Suppose that  $\Omega_1(P) \leq \zeta_k(P)$  for  $p \neq 2$ , or  $\Omega_2(P) \leq \zeta_k(P)$  for  $p = 2$ . Then the exponent of a Sylow  $p$ -subgroup of  $G/O_{p',p}(G)$  is at most  $p^m$ , where  $m$  is the maximum integer such that  $p^m - p^{m-1} \leq k$ .*

*Proof.* We can obviously assume that  $O_{p'}(G) = 1$ .

Let  $Q$  be a Hall  $p'$ -subgroup of  $O_{p,p'}(G)$ , so that  $O_{p,p'}(G) = O_p(G)Q$ . By the generalized Frattini argument,

$$G = O_{p,p'}(G)N_G(Q) = O_p(G)N_G(Q),$$

so we need to obtain a bound for the exponent of the image of a Sylow  $p$ -subgroup of  $N_G(Q)$  in  $\bar{G} = G/O_p(G)$ . We use bars to denote images of elements or subsets in  $\bar{G}$ .

Let  $g$  be an element of a Sylow  $p$ -subgroup of  $N_G(Q)$ , so that  $\bar{g}$  is its image in  $\bar{G} = G/O_p(G)$ . Let  $|\bar{g}| = p^n$ . We must show that  $p^n - p^{n-1} \leq k$ . We can of course assume that  $n \geq 1$ .

Since  $O_p(\bar{G}) = 1$  and  $\bar{Q} = O_{p'}(\bar{G})$ , we have  $C_{\bar{G}}(\bar{Q}) \leq \bar{Q}$  (see, for example, [2, Theorem 6.3.2]). Hence the  $p$ -element  $\bar{g}$  acts faithfully on  $\bar{Q}$ ; in other words,  $[\bar{Q}, \bar{g}^{p^{n-1}}] \neq 1$ . Clearly,  $\bar{g}$  also acts on  $Q$  itself, and  $[Q, \bar{g}^{p^{n-1}}] = [\bar{Q}, \bar{g}^{p^{n-1}}] \neq 1$ .

Let  $\Omega$  denote  $\Omega_1(O_p(G))$  if  $p \neq 2$ , and  $\Omega_2(O_p(G))$  if  $p = 2$ .

Consider a series of normal subgroups of  $G$

$$(3.1) \quad 1 = U_0 < U_1 < \dots < U_n = \Omega$$

in which each factor  $U_{i+1}/U_i$  is an elementary abelian  $p$ -group contained in the centre of  $O_p(G)/U_i$ . Then the action of the semidirect product  $Q\langle \bar{g} \rangle$  on each factor  $U_{i+1}/U_i$  is well defined.

Since  $O_{p'}(G) = 1$ , the  $p'$ -subgroup  $[Q, \bar{g}^{p^{n-1}}] \neq 1$  acts faithfully on  $O_p(G)$ . By Lemma 2.1, moreover,  $[Q, \bar{g}^{p^{n-1}}]$  acts faithfully on  $\Omega$ . Since the action is coprime, we obtain that  $[Q, \bar{g}^{p^{n-1}}]$  acts nontrivially on at least one of the factors  $V$  of the series (3.1). Let  $H$  denote the image of  $Q\langle \bar{g} \rangle$  in the group of

linear transformations of the vector space  $V$  over  $\mathbb{F}_p$ , which consists of elements  $T(u)$  for  $u \in Q\langle\bar{g}\rangle$  in accordance with our notation.

Since the subgroup  $[Q, \bar{g}^{p^{n-1}}]$  acts non-trivially on  $V$ , we must have  $O_p(H) = 1$ . Indeed, otherwise  $T(\bar{g})^{p^{n-1}}$  would be in  $O_p(H)$  and then the image of  $[Q, \bar{g}^{p^{n-1}}]$  would be in  $O_q(H) \cap O_p(H) = 1$  and therefore trivial, contrary to the assumption. For the same reasons,  $T(\bar{g})$  has the same order  $p^n$ .

By the Hall–Higman Theorem 2.2 the minimal polynomial of  $T(\bar{g})$  is  $(x-1)^r = 0$ , where  $p^n - p^{n-1} \leq r \leq p^n$ . Therefore there is  $v \in V$  such that

$$(3.2) \quad v(T(\bar{g}) - \mathbf{1}_V)^{p^n - p^{n-1} - 1} \neq 0.$$

Since the image of an element  $u \in V$  under the linear transformation  $T(\bar{g}) - \mathbf{1}_V$  is equal to the group commutator  $[u, \bar{g}]$ , it follows from (3.2) that

$$\underbrace{[\dots[[v, \bar{g}], \bar{g}], \dots, \bar{g}]}_{p^n - p^{n-1} - 1} \neq 1.$$

But by the hypothesis of the theorem we have  $\Omega \leq \Omega_1(P) \leq \zeta_k(P)$  for  $p \neq 2$  (or  $\Omega \leq \Omega_2(P) \leq \zeta_k(P)$  for  $p = 2$ ). Therefore we must also have

$$\underbrace{[\dots[[v, \bar{g}], \bar{g}], \dots, \bar{g}]}_k = 1.$$

It follows that  $p^n - p^{n-1} - 1 < k$ , as required.  $\square$

*Proof of Theorem 3.2.* Once we know a bound for the exponent  $p^e$  of a Sylow  $p$ -subgroup of  $G/O_{p',p}(G)$ , we obtain a bound for the  $p$ -length  $l$  of  $G/O_{p',p}(G)$ . Indeed, for  $p \neq 2$  we have  $e \geq [(l+1)/2]$  by the Hall–Higman theorem [5, Theorem A], and for  $p = 2$  we have  $e \geq l$  by Bryukhanova’s theorem [1] (which is the best-possible improvement of the earlier estimate  $2e - 2 \geq l$  by Gross [3]). Since  $l+1$  is exactly the  $p$ -length of  $G$ , the result follows from Theorem 3.2.  $\square$

**Remark 3.3.** The Hall–Higman Theorem A gives a better bound  $e \geq l$  if  $p$  is not a Fermat prime. As noticed in the Hall–Higman paper [5], it follows from the proof that in the Hall–Higman Theorem 2.2 we have  $r = p^m$  if  $p$  is odd and not a Fermat prime. Thus, the estimates can be further improved in these cases.

**Remark 3.4.** Theorems 3.1 and 3.2 lend further support to the viewpoint that the “correct” definition of 2-central 2-groups (also those of height  $k$ ) must involve  $\Omega_2$  rather than  $\Omega_1$ . May be, this definition can also be used to extend to  $p = 2$  some other results involving  $p$ -central  $p$ -groups of height  $k$ , which do not hold for  $p = 2$  without this amendment.

#### 4. Powerful Sylow $p$ -subgroup

Recall that a finite  $p$ -group  $P$  is *powerful* if  $P^p \geq [P, P]$  for  $p \neq 2$ , or  $P^4 \geq [P, P]$  for  $p = 2$ . Properties of powerful  $p$ -groups that we need here are well known since the original paper by Lubotzky and Mann [7]. In particular, if  $P$  is a powerful  $p$ -group, then the subgroups  $P^{p^i} = \langle g^{p^i} \mid g \in P \rangle$  form a central series of  $P$ , and  $P^{p^i} = \{g^{p^i} \mid g \in P\}$  for all  $i$ .

**Theorem 4.1.** *If a finite  $p$ -soluble group  $G$  has a powerful Sylow  $p$ -subgroup, then the  $p$ -length of  $G$  is equal to 1.*

*Proof.* We argue by contradiction. Let  $G$  be a finite  $p$ -soluble group of minimal order with a powerful Sylow  $p$ -subgroup such that the  $p$ -length of  $G$  is greater than 1. By minimality we must have  $O_{p'}(G) = 1$ . Since homomorphic images of powerful  $p$ -groups are powerful, it follows by minimality that  $V := O_p(G)$  is an elementary abelian  $p$ -group. Then  $G/V$  acts faithfully on  $V$ , which we can also regard as an  $\mathbb{F}_p(G/V)$ -module.

Let  $Q$  be a Hall  $p'$ -subgroup of  $O_{p,p'}(G)$ . Then  $Q$  acts faithfully on  $V/C_V(Q)$ , since the action is coprime. Clearly,  $C_V(Q) = Z(O_{p,p'}(G))$ , and therefore  $C_V(Q)$  is normal in  $G$ . By minimality we must have  $C_V(Q) = 1$ .

By the generalized Frattini argument,  $VN_G(Q) = G$ . Let  $S$  be a Sylow  $p$ -subgroup of  $N_G(Q)$ . Then  $P := VS$  is a Sylow  $p$ -subgroup of  $G$ . Note that  $V \cap S = 1$ , since  $C_V(Q) = 1$ .

Choose an element  $g \in P$  of maximal possible order  $p^n$ , so that  $p^n$  is the exponent of  $P$ . From this moment on we consider separately the cases  $p \neq 2$  and  $p = 2$ .

**Case  $p \neq 2$ .** Then  $n \geq 2$ . Indeed, a powerful  $p$ -group of exponent  $p$  is abelian, and if we had  $n = 1$ , then  $P$  would be abelian and the  $p$ -length of  $G$  would be equal to 1, contrary to our assumption.

Hence the element  $h = g^{p^{n-2}}$  is well defined. By the properties of powerful  $p$ -groups,  $P^{p^{n-1}} \leq Z(P)$  and  $P^{p^{n-2}} \leq \zeta_2(P)$ . Therefore,  $1 \neq h^p \in Z(P) \leq V$  and  $h \in \zeta_2(P)$ . Since  $V$  is elementary abelian, we also have  $h \notin V$ .

Since  $P = VS$ , we can represent  $h$  as  $h = vs$  for  $v \in V$  and  $s \in S$ . Then  $|s| = p$ , because  $s^p \in V \cap S = 1$ . At the same time,  $|vs| = |h| = p^2$ . Hence the Jordan normal form of the linear transformation  $T(s)$  of  $V$  induced by the action of  $s$  by conjugation must have a block of size  $p \times p$ . Therefore there is a vector  $x \in V$  such that

$$x(T(s) - \mathbf{1}_V)^{p-1} \neq 0.$$

In terms of group commutators, this means that

$$\underbrace{[\dots[[x, s], s], \dots, s]}_{p-1} \neq 1.$$

But the action of  $s$  on  $V$  coincides with the action of  $h = vs$ . Therefore,

$$\underbrace{[\dots[[x, h], h], \dots, h]}_{p-1} \neq 1.$$

This contradicts the inclusion  $h \in \zeta_2(P)$ , since  $p \geq 3$ .

**Case  $p = 2$ .** Then  $n \geq 3$ . Indeed, a powerful 2-group of exponent 4 is abelian, and if we had  $n \leq 2$ , then  $P$  would be abelian and the  $p$ -length of  $G$  would be equal to 1, contrary to our assumption.

Hence the element  $h = g^{2^{n-3}}$  is well defined. By the properties of powerful 2-groups,  $P^{2^{n-1}} \leq Z(P)$ ,  $P^{2^{n-2}} \leq \zeta_2(P)$ , and  $P^{2^{n-3}} \leq \zeta_3(P)$ . Therefore,  $1 \neq h^4 \in Z(P) \leq V$  and  $h \in \zeta_3(P)$ . Since  $V$  is elementary abelian, we also have  $h^2 \notin V$ .

We again represent  $h$  as  $h = vs$  for  $v \in V$  and  $s \in S$ . Then  $|s| = 4$ , since  $s^4 \in V \cap S = 1$ . At the same time,  $|vs| = |h| = 8$ . Hence the Jordan normal form of the linear transformation  $T(s)$  of  $V$  induced by the action of  $s$  by conjugation must have a block of size  $4 \times 4$ . Therefore there is a vector  $x \in V$  such that

$$x(T(s) - \mathbf{1}_V)^3 \neq 0.$$

In terms of group commutators, this means that

$$[[[x, s], s], s] \neq 1.$$

Since the action of  $s$  on  $V$  coincides with the action of  $h = vs$ , we also have

$$[[[x, h], h], h] \neq 1.$$

This contradicts the inclusion  $h \in \zeta_3(P)$ . □

**Remark 4.2.** It is not immediately clear how to generalize the definition of powerful  $p$ -groups “dually” to the definition of  $p$ -central  $p$ -groups of height  $k$ . Probably, such a definition would also allow to prove a bound for the  $p$ -length of  $p$ -soluble group  $G$  with a Sylow  $p$ -subgroup satisfying this definition. A rough bound for the  $p$ -length would follow by Hall–Higman theorems if such generalized “ $k$ -powerful”  $p$ -groups had the following property: if the exponent is  $p^n$ , then the nilpotency class is bounded by a function of  $n$  and  $k$  that is subexponential (even linear) in  $n$ . This would of course generalize the property of powerful  $p$ -groups, where the nilpotency class is at most  $n$ . Indeed, let  $p^m$  be the exponent of the image of a Sylow  $p$ -subgroup  $P$  of  $G$  in  $G/O_{p',p}(G)$ . Let  $V$  be the Frattini quotient of  $O_{p',p}(G)/O_{p'}(G)$  regarded as an  $\mathbb{F}_p(G/O_{p',p}(G))$ -module. As we saw in the proof of Theorem 3.2, then by Hall–Higman theorems there are elements  $v \in V$  and  $g \in P$  such that

$$[\dots \underbrace{[v, g], g, \dots, g}_{p^m - p^{m-1} - 1}] \neq 1.$$

On the other hand, we would have

$$[\dots \underbrace{[v, g], g, \dots, g}_{f(k,m)}] = 1$$

with the hypothetical function  $f(k, n)$  bounding the nilpotency class. Hence,

$$p^m - p^{m-1} - 1 \leq f(k, m).$$

Provided the function  $f(k, m)$  is subexponential in  $m$  (and it is most likely and natural to have this function being linear in  $m$ ), an estimate for  $m$  would follow, which would in turn give an estimate for the  $p$ -length.

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**E. I. Khukhro**

Sobolev Institute of Mathematics, 630090, Novosibirsk, Russia

Email: khukhro@yahoo.co.uk