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GROUPS OF INFINITE RANK WITH A NORMALIZER CONDITION ON SUBGROUPS

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Abstract. Groups of infinite rank in which every subgroup is either normal or self-normalizing are characterized in terms of their subgroups of infinite rank.

1. Introduction

A group G is said to have *finite (Prüfer) rank* r if every finitely generated subgroup of G can be generated by at most r elements, and r is the least positive integer with such property. The investigation of the influence on a (generalized) soluble group of the behavior of its subgroups of infinite rank has been developed in a series of recent papers (see for instance [5],[4],[7],[6],[9],[11]). The aim of this paper is to provide some new contributions to this topic, considering groups G in which every subgroup of infinite rank is either normal or coincides with its normalizer in G. Groups satisfying such property will be called $\mathcal{E}_\infty$-groups, in analogy with the symbol $\mathcal{E}$ used in literature to denote the class of groups in which every non-normal subgroup is self-normalizing. The class of $\mathcal{E}$-groups was studied in [13], while groups in which the condition of being either normal or self-normalizing is imposed only to certain systems of subgroups were considered in [3].

We will work within the universe of strongly locally graded groups, a class of generalized soluble groups that can be defined as follows. Recall that a group G is *locally graded* if every finitely generated non-trivial subgroup of G contains a proper subgroup of finite index. Let $\mathfrak{D}$ be the class of all periodic locally graded groups, and let $\bar{\mathfrak{D}}$ be the closure of $\mathfrak{D}$ by the operators $\acute{\mathbf{P}}$, $\grave{\mathbf{P}}$, $\mathbf{R}$, $\mathbf{L}$ (we shall use the first chapter of the monograph [18] as a general reference for definitions and properties of closure operations

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on group classes). It is easy to prove that any $\bar{\mathfrak{D}}$-group is locally graded, and that the class $\bar{\mathfrak{D}}$ is closed with respect to forming subgroups. Moreover, N. S. Černikov proved that every $\bar{\mathfrak{D}}$ - group of finite rank contains a locally soluble subgroup of finite index. Obviously, all residually finite groups belong to $\bar{\mathfrak{D}}$, and hence the consideration of any free non-abelian group shows that the class $\bar{\mathfrak{D}}$ is not closed with respect to homomorphic images. For this reason, it is better in some cases to replace $\bar{\mathfrak{D}}$-groups by *strongly locally graded groups*, i.e. groups in which every section belongs to $\bar{\mathfrak{D}}$. The class of strongly locally graded groups has been introduced in [7].

2. $\mathcal{E}_\infty$-groups

A relevant result by A. I. Mal'cev [15] states that any locally nilpotent group of infinite rank must contain an abelian subgroup of infinite rank, and a corresponding result for locally finite groups was later proved by V. P. Šunkov [20]. Mal'cev's theorem was improved by R. Baer and H. Heineken [1], to the case of radical groups of infinite rank. As in many problems concerning groups of infinite rank, also in our case, the existence of abelian subgroups of infinite rank plays a crucial role.

Lemma 2.1. *Let G be an $\mathcal{E}_\infty$-group. If G contains an abelian subgroup of infinite rank, then G is an $\mathcal{E}$-group.*

Proof. Let A be an abelian subgroup of infinite rank of G. Let H be any subgroup of G of finite rank such that $H < N_G(H)$, and take an element $x \in N_G(H) \setminus H$. Then A contains a direct product $A_1 \times A_2$, such that the subgroups A_1 and A_2 have both infinite rank and $(A_1 \times A_2) \cap H\langle x\rangle = \{1\}$. Clearly the subgroups A_1 and A_2 are normal in G. Moreover, $HA_1 \cap H\langle x\rangle = H$, so that $x \in N_G(HA_1) \setminus HA_1$, and hence HA_1 is normal in G. Similary, the subgroup HA_2 is normal in G, so that $H = HA_1 \cap HA_2$ is normal in G, and G is an $\mathcal{E}$-group. □

Recall that a group G is said to be a *T-group* (or have the *T-property*) if normality in G is a transitive relation, i.e. if all subnormal subgroups of G are normal. The structure of soluble T-groups has been described by W. Gaschütz [12] in the finite case and by D. J. S. Robinson [17] for arbitrary groups. It turns out in particular that soluble groups with the property T are metabelian and hypercyclic, and that finitely generated soluble T-groups are either finite or abelian. Although the class of T-groups is not subgroup closed (because any simple group is obviously a T-group), it is known that subgroups of finite soluble T-groups have likewise the T-property. A group G is called a *$\bar{T}$-group* if all its subgroups are T-groups. It follows easily from the properties of T-groups, that any finite $\bar{T}$-group is soluble, while soluble non-periodic $\bar{T}$-groups are abelian. Obviously any $\mathcal{E}$-group has the property T, so that our investigation has connections with the theory of soluble T-groups.

Proposition 2.2. *Let G be a periodic locally graded $\mathcal{E}_\infty$-group of infinite rank. Then G is an $\mathcal{E}$-group.*

Proof. Assume that G contains a finitely generated subgroup H of infinite rank. If K is any normal subgroup of finite index of H, then K has infinite rank and hence the factor group H/K is an $\mathcal{E}$-group.

Therefore H/K is a finite $\bar{T}$-group, and hence it is metabelian. If R is the finite residual of H, then H/R is a finitely generated metabelian periodic group, and so it is finite. Thus R is a finitely generated subgroup of G which has no proper subgroups of finite index; it follows that $R = \{1\}$, so that H is finite. This contradiction shows that every finitely generated subgroup of G has finite rank and so it is finite by Černikov's theorem (see [2]). Therefore G is a locally finite group, so that it contains an abelian subgroup of infinite rank, and hence G is an $\mathcal{E}$-group by Lemma 2.1. □

Lemma 2.3. *Let G be a strongly locally graded $\mathcal{E}_\infty$-group of infinite rank. Then G contains a proper normal subgroup of infinite rank.*

Proof. Assume by contradiction that all proper normal subgroups of G have finite rank, so that G is not soluble, and hence it is not an $\mathcal{E}$-group (see [3], Theorem 2.4.). Moreover, it follows from Lemma 2.1 that G has no abelian subgroups of infinite rank, so that G is not locally nilpotent. Therefore G has a simple homomorphic image $\bar{G}$ of infinite rank (see [7], Lemma 2.4). Clearly every subgroup of infinite rank of $\bar{G}$ is self-normalizing in $\bar{G}$. Let $\bar{H}$ be any finitely generated subgroup of infinite rank of $\bar{G}$; since $\bar{G}$ is locally graded, $\bar{H}$ contains a proper normal subgroup $\bar{K}$ of finite index, so that $\bar{K}$ is properly contained in its normalizer, a contradiction. Therefore every finitely generated subgroup of $\bar{G}$ has finite rank, so that $\bar{G}$ is locally (soluble-by-finite) by Černikov's thoerem (see [2]). As $\bar{G}$ has infinite rank, it must contains a proper locally soluble subgroup $\bar{L}$ of infinite rank (see [8]). Since every subgroup of infinite rank of $\bar{G}$ is self-normalizing, $\bar{L}$ has no proper normal subgroups of infinite rank, and hence $\bar{L}$ has a simple homomorphic image of infinite rank (see [7], Lemma 2.4), a contradiction, since simple locally soluble groups have prime order. This contradiction completes the proof of the lemma. □

We are now ready to prove the main result of this section.

Theorem 2.4. *Let G be a strongly locally graded $\mathcal{E}_\infty$-group of infinite rank. Then G is an $\mathcal{E}$-group.*

Proof. It follows from Lemma 2.3 that G contains a proper normal subgroup N of infinite rank. The factor group G/N is an $\mathcal{E}$-group, so that $G'' \leq N$ (see [3], Theorem 2.4). If G'' has infinite rank then G'' is strongly locally graded $\mathcal{E}_\infty$-group of infinite rank and so G'' has a proper normal subgroup M of infinite rank. As above, $G^{(4)} \leq M$; if $G^{(4)}$ has infinite rank then $G/G^{(4)}$ has derived length at most 2, and hence $G'' = G^{(4)} \leq M$, a contradiction. Therefore in any case, $K = G^{(4)}$ has finite rank. So K is (locally soluble)-by-finite by Černikov's theorem (see [2]). If S is the locally soluble radical of K then K/S is finite, and therefore $G/C_G(K/S)$ is finite. Hence $C_G(K/S)$ has infinite rank and so $G/C_G(K/S)$ is metabelian; it follows that $K \leq C_G(K/S)$ and hence K is locally soluble (see [10] Lemma 2.7). There exists a positive integer n such that $K^{(n)}$ is a periodic hypercentral group with Černikov primary components (see [18], Lemma 10.39), so that the divisible radical R of $K^{(n)}$ is a divisible normal abelian subgroup of G and $K^{(n)}/R$ has finite primary components. In order to prove that K is soluble we may assume that $R = \{1\}$, so that each primary component of $K^{(n)}$ is finite.

Let P be any primary component of $K^{(n)}$. Since $G/C_G(P)$ is finite, $C_G(P)$ has infinite rank. Again $G/C_G(P)$ is metabelian and we have $P \leq K \leq G'' \leq C_G(P)$, so that P is abelian and, $K^{(n)}$ is abelian. Thus K is soluble and hence G is soluble. By Lemma 2.1 G is an $\mathcal{E}$-group. □

3. Groups with many metahamiltonian subgroups

In this section we will consider groups which are rich in $\mathcal{E}$-subgroups. In particular, in the main result of the section we will show that if G is a strongly locally graded group of infinite rank in which all proper subgroups of infinite rank are $\mathcal{E}$-groups, then G itself is forced to have the same property.

The proof is accomplished through some lemmas; the first of them shows that the class $\mathcal{E}$ is local, at least within the universe of locally graded groups.

Proposition 3.1. *Let G be a locally graded group such that every finitely generated subgroup is an $\mathcal{E}$-group. Then G is an $\mathcal{E}$-group.*

Proof. Assume first that G is not periodic, and let H be any finitely generated subgroup of G. If g is an element of G of infinite order, then the finitely generated subgroup $\langle H, g\rangle$ is a non-periodic $\mathcal{E}$-group, so that it is abelian. Therefore G is abelian. Assume now that G is a periodic group and let H be any finitely generated subgroup of G such that $H < N_G(H)$. Choose an element x in the set $N_G(H) \setminus H$, and let g be any element of G. The subgroup $K = \langle H, x, g\rangle$ is finitely generated and so it is an $\mathcal{E}$-group. Since $H < N_K(H)$, the subgroup H is normal in K. It follows that H is normal in G. Therefore G is an $\mathcal{E}$-group (see [3], Theorem 2.4). □

Lemma 3.2. *Let G be a locally graded group such that all its proper subgroups are $\mathcal{E}$-groups. Then G is soluble.*

Proof. We can assume that G is not an $\mathcal{E}$-group. It follows from Proposition 3.1 that G is finitely generated, so that there exists a proper normal subgroup N of G of finite index. The subgroup N is an $\mathcal{E}$-group and so it is soluble. On the other hand, all proper subgroups of G/N are finite $\mathcal{E}$-groups and hence they are supersoluble. Therefore G/N is soluble (see [19], 10.3.4), so that G is soluble. □

Lemma 3.3. *Let F be an infinite locally finite field. Then the simple groups $PSL(2, F)$ and $S_Z(F)$ contain proper subgroups of infinite rank which are not $\mathcal{E}$-groups.*

Proof. Let G be one of the groups $PSL(2, F)$ and $S_Z(F)$. In [16] it is proved that G contains a subgroup H such that H is not a Dedekind group and H/H' is not cyclic. It follows from Theorem 2.4 of [3] that H is not an $\mathcal{E}$-group. □

Lemma 3.4. *Let G be a locally soluble group of infinite rank whose proper subgroups of infinite rank are $\mathcal{E}$-groups. Then G is an $\mathcal{E}$-group.*

Proof. The group G is a soluble $\bar{T}$-group (see [4]). We can assume that G is not an abelian group, so that it is a periodic metabelian group.

Assume first that all proper subgroups of infinite rank of G are Dedekind groups, so that the commutator subgroup G' of G is finite (see [7], Proposition 3.1). Let H be any cyclic subgroup of G. Since the factor group G/G' has infinite rank, there exists a subgroup K of G such that $G' \leq K$ and the groups K and G/K have both infinite rank. Now HK is a proper subgroup of infinite rank of G and so it is a Dedekind group; therefore H is subnormal, and hence normal in G.

We can now assume that there exists a proper subgroup L of infinite rank of G which is not a Dedekind group. Since L is a locally finite $\mathcal{E}$-group, the factor group L/L' is finite (see [13], Theorem 3.4), so that the commutator subgroup L' has infinite rank, and hence G' has likewise infinite rank. Let H be any finitely generated subgroup of G. Since G' is abelian, it contains a subgroup A such that the groups A and G'/A have both infinite rank. The subgroup A is subnormal and hence normal in G; moreover, HA is a proper subgroup of infinite rank of G, and so it is an $\mathcal{E}$-group, so that H is likewise an $\mathcal{E}$-group. Therefore G is an $\mathcal{E}$-group (see [3], Theorem 2.4). □

Theorem 3.5. *Let G be a strongly locally graded group of infinite rank. If all proper subgroups of infinite rank of G are $\mathcal{E}$-groups, then G is an $\mathcal{E}$-group.*

Proof. Assume for a contradiction that G is not an $\mathcal{E}$-group, so that, by lemma 3.4, G is not locally soluble. If the commutator subgroup G' of G has finite rank, then all proper subgroups of G are $\mathcal{E}$-groups (see [7], Lemma 2.7), so that by Lemma 3.2 G is locally soluble. This contradiction shows that G' has infinite rank, and hence G/G' is finitely generated (see [7], Lemma 2.8). Since the commutator subgroup of any $\mathcal{E}$-group is locally finite (see [3], Theorem 2.4), then G' is locally finite (see [7], Theorem A). In particular the set T of all elements of finite order of G is a subgroup and the factor group G/T is a free abelian group of finite rank. Since G is not soluble, the subgroup T cannot be an $\mathcal{E}$-group; it follows that $G = T$ is a periodic group.

Assume now that every proper normal subgroup of G has finite rank, so that G has a simple section G/K of infinite rank (see [7], Lemma 2.4); since all proper subgroups of G are (locally soluble)-by-finite (see [2]), the factor group G/K must be isomorphic either to $PSL(2, F)$ or to $S_Z(F)$ for some infinite locally finite field F (see [14]), and this is impossible by Lemma 3.3. Therefore G is locally nilpotent. This contradiction shows that G contains a proper normal subgroup N of infinite rank; in particular, N is soluble so that G/N is not soluble. On the other hand, every proper subgroup of G/N is an $\mathcal{E}$-group, so that by Lemma 3.2, G/N is soluble. This last contradiction proves the theorem. □

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