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ON A GROUP OF THE FORM $3^7{:}Sp(6,2)$

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Abstract. The purpose of this paper is the determination of the inertia factors, the computations of the Fischer matrices and the ordinary character table of the split extension $\overline{G} = 3^7{:}Sp(6,2)$ by means of Clifford-Fischer Theory. We firstly determine the conjugacy classes of $\overline{G}$ using the coset analysis method. The determination of the inertia factor groups of this extension involved looking at some maximal subgroups of the maximal subgroups of $Sp(6,2)$. The Fischer matrices of $\overline{G}$ are all listed in this paper and their sizes range between 2 and 10. The character table of $\overline{G}$, which is a 118×118 $\mathbb{C}$-valued matrix, is available in the PhD thesis of the first author, which could be accessed online.

1. Introduction

Let $G = Sp(6,2)$ be the symplectic group of order 1451520. By the electronic Atlas [16], the group G has a 7−dimensional (absolutely) irreducible module over $GF(3) = \mathbb{F}_3 = \{0,1,\xi\}$, where ξ is a primitive element of the field $\mathbb{F}_3$. Consequently a split extension of the form $3^7{:}Sp(6,2)$ does exist. Using the two 7×7 matrices over $\mathbb{F}_3$ that generate $Sp(6,2)$, supplied by the electronic Atlas, we were able to construct G and then $\overline{G}$ inside GAP [10]. In fact we constructed the group $\overline{G}$ in GAP, in terms of 8×8 matrices over $\mathbb{F}_3$. The following two elements $\overline{g}_1$ and $\overline{g}_2$ are 8−dimensional matrices over $\mathbb{F}_3$ that generate $\overline{G}$.

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$$\overline{g}_1 = \begin{pmatrix} 1 & 1 & \xi & 1 & 1 & \xi & \xi & 0 \\ 0 & \xi & 0 & 1 & 1 & 1 & \xi & 0 \\ \xi & 1 & \xi & \xi & \xi & 1 & \xi & 0 \\ 1 & 1 & 0 & 0 & \xi & 0 & 1 & 0 \\ 1 & 0 & 0 & \xi & 1 & \xi & 0 & 0 \\ 0 & 0 & \xi & 0 & \xi & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & \xi & \xi & \xi & 0 \\ 0 & \xi & 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \overline{g}_2 = \begin{pmatrix} \xi & 1 & 1 & 1 & \xi & 1 & 0 & 0 \\ \xi & 0 & 1 & 1 & 0 & 0 & \xi & 0 \\ \xi & 0 & 0 & 1 & \xi & 0 & \xi & 0 \\ 0 & \xi & 0 & 0 & \xi & \xi & \xi & 0 \\ \xi & 0 & 0 & \xi & 1 & 0 & \xi & 0 \\ \xi & \xi & 0 & 1 & \xi & 1 & 0 & 0 \\ \xi & 1 & \xi & \xi & \xi & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & \xi & 1 \end{pmatrix},$$

with $o(\overline{g}_1) = 45$, $o(\overline{g}_2) = 45$ and $o(\overline{g}_1\overline{g}_2) = 6$.

Corollary 1.1. *$\overline{G} \leq SL(8,3)$ with index 302 954 487 010 019 919 360.*

Proof. This is readily verified since $\det(\overline{g}_1) = \det(\overline{g}_2) = 1$ and consequently $\det(\overline{g}) = 1$ for all $\overline{g} \in \overline{G}$. □

Using GAP, one can easily check all the normal subgroups of $\overline{G}$. In fact the only proper normal subgroup of $\overline{G}$ is a group of order 2187 and thus must be isomorphic to the elementary abelian group $N = 3^7$. The following elements $n_1, n_2, \ldots, n_7$ are 8×8 matrices over $\mathbb{F}_3$ that are generators of N.

$$n_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \xi & 1 & 1 & \xi & 0 & 1 \end{pmatrix}, \quad n_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & \xi & 1 & \xi & 1 & 0 & 1 & 1 \end{pmatrix}, \quad n_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & \xi & \xi & 0 & \xi & 1 & 0 & 1 \end{pmatrix},$$

$$n_4 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ \xi & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad n_5 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ \xi & 0 & \xi & 0 & 0 & 1 & \xi & 1 \end{pmatrix}, \quad n_6 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ \xi & 1 & 0 & \xi & 1 & 1 & 1 & 1 \end{pmatrix},$$

$$n_7 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ \xi & \xi & \xi & \xi & 1 & \xi & 0 & 1 \end{pmatrix}.$$

In terms of 8×8 matrices over $\mathbb{F}_3$, the group $Sp(6,2)$ is generated by the following two elements g_1 and g_2 :

$$g_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \xi & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \xi & \xi & \xi & 0 & 0 & 0 \\ 0 & 0 & \xi & \xi & 0 & \xi & 0 & 0 \\ 0 & 0 & \xi & \xi & 0 & 0 & \xi & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \xi \end{pmatrix}, \quad g_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & \xi & \xi & \xi & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & \xi & 1 & \xi & 1 & 0 & 1 \end{pmatrix},$$

with $o(g_1) = 2$, $o(g_2) = 7$ and $o(g_1g_2) = 9$. Note that the group $Sp(6,2) = \langle g_1, g_2 \rangle$ together with the mentioned generators of N, gives the split extension $\overline{G} = 3^7{:}Sp(6,2)$.

For the notation used in this paper and the description of Clifford-Fischer theory technique, we follow [1, 2].

2. **Conjugacy Classes of** $\overline{G} = 3^7{:}Sp(6,2)$

In this section we calculate the conjugacy classes of $\overline{G}$ using the *coset analysis* technique (see [1] or [12, 13] for more details) as we are interested to organize the classes of $\overline{G}$ corresponding to the classes of $Sp(6,2)$.

We have used GAP to build a small subroutine to find the values of k_i's, which we list in Table 1. We supplied the values of $\chi(Sp(6,2)|3^7)$ on each of the 30 conjugacy classes of $Sp(6,2)$. In fact the subroutine we have used to find the values of k_i's can be developed further to find the values of f_{ij}'s for each coset corresponding to $[g_i]_G$. A complete set of the f_{ij}'s and representatives for the conjugacy classes of $\overline{G}$ are given in Table 1. To each class of $\overline{G}$, we have attached some weight m_{ij}, which will be used later in computing the Fischer matrices of the extension. These weights are computed through the formula

$$m_{ij} = [N_{\overline{G}}(N\overline{g}_i) : C_{\overline{G}}(g_{ij})] = |N|\frac{|C_G(g_i)|}{|C_{\overline{G}}(g_{ij})|}. \tag{2.1}$$

Example 2.1. *Consider the identity coset* $N = 3^7$ *as this coset is so important. Recall that* N *is abelian and thus each orbit of the action of* N *on itself consists of singleton. Therefore* $k_1 = |N| = 2178$. *Since we can present* $\overline{G}$ *and* N *in GAP in terms of* 8×8 *matrices over* $\mathbb{F}_3$, *it is easy for* $\overline{G}$ *to act on* N. *In fact this action yielded six orbits of lengths 1, 56, 126, 576, 672 and 756 with representatives* $g_{11}, g_{12}, g_{13}, g_{14}, g_{15}$ *and* g_{16} *defined as follows:*

$$g_{11} = \begin{pmatrix} 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1 \end{pmatrix}, \quad g_{12} = \begin{pmatrix} 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1&0\\ 0&1&0&0&1&0&1&1 \end{pmatrix}, \quad g_{13} = \begin{pmatrix} 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1&0\\ 1&0&\xi&\xi&\xi&1&0&1 \end{pmatrix},$$

$$g_{14} = \begin{pmatrix} 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1&0\\ 1&0&\xi&1&0&0&1&1 \end{pmatrix}, \quad g_{15} = \begin{pmatrix} 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1&0\\ \xi&1&0&1&\xi&1&1&1 \end{pmatrix}, \quad g_{16} = \begin{pmatrix} 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1&0\\ 0&\xi&\xi&\xi&0&0&\xi&1 \end{pmatrix}.$$

Thus the identity coset affords six conjugacy classes in $\overline{G}$. *These classes have sizes equal to the orbits lengths of* $\overline{G}$ *on* N *with respective representatives* $g_{11}, g_{12}, g_{13}, g_{14}, g_{15}$ *and* g_{16}. *Also the values of the* f_{1j}*'s are same as lengths of the corresponding conjugacy classes for all* $1 \leq j \leq 6$. *Clearly* $g_{11} = 1_{\overline{G}}$ *and thus* $o(g_{11}) = 1$. *Since* $g_1 = 1_{Sp(6,2)}$ *and* $o(g_1) = 1$, *it follows by applications of Proposition 2.3.3 of* [1] *that* $o(g_{12}) = o(g_{13}) = o(g_{14}) = o(g_{15}) = o(g_{16}) = 3$. *The orders of the preceding elements can also be seen directly since* N *is an elementary abelian* $3-$*group. Similar arguments can be applied to all the other cosets* $Ng_i,\ 2 \leq i \leq 30$.

We list the conjugacy classes of $\overline{G}$ in Table 1.

Table 1: The conjugacy classes of $\overline{G} = 3^7{:}Sp(6,2)$

$[g_i]_G$	k_i	f_{ij}	m_{ij}	$[g_{ij}]_{\overline{G}}$	$o(g_{ij})$	$\lvert[g_{ij}]_{\overline{G}}\rvert$	$\lvert C_{\overline{G}}(g_{ij})\rvert$
		$f_{11} = 1$	$m_{11} = 1$	g_{11}	1	1	3174474240
		$f_{12} = 56$	$m_{12} = 56$	g_{12}	3	56	56687040
$g_1 = 1A$	$k_1 = 2187$	$f_{13} = 126$	$m_{13} = 126$	g_{13}	3	126	25194240
		$f_{14} = 576$	$m_{14} = 576$	g_{14}	3	576	5511240
		$f_{15} = 672$	$m_{15} = 672$	g_{15}	3	672	4723920
		$f_{16} = 756$	$m_{16} = 756$	g_{16}	3	756	4199040
$g_2 = 2A$	$k_2 = 3$	$f_{21} = 1$	$m_{21} = 729$	g_{21}	2	45927	69120
		$f_{22} = 2$	$m_{22} = 1458$	g_{22}	6	91854	34560
		$f_{31} = 1$	$m_{31} = 81$	g_{31}	2	25515	124416
$g_3 = 2B$	$k_3 = 27$	$f_{32} = 6$	$m_{32} = 486$	g_{32}	6	153090	20736
		$f_{33} = 8$	$m_{33} = 648$	g_{33}	6	204120	15552
		$f_{34} = 12$	$m_{34} = 972$	g_{34}	6	306180	10368
		$f_{41} = 1$	$m_{41} = 9$	g_{41}	2	8505	373248
		$f_{42} = 2$	$m_{42} = 18$	g_{42}	6	17010	186624
		$f_{43} = 8$	$m_{43} = 72$	g_{43}	6	68040	46656
		$f_{44} = 16$	$m_{44} = 144$	g_{44}	6	136080	23328
$g_4 = 2C$	$k_4 = 243$	$f_{45} = 16$	$m_{45} = 144$	g_{45}	6	136080	23328
		$f_{46} = 24$	$m_{46} = 216$	g_{46}	6	204120	15552
		$f_{47} = 32$	$m_{47} = 288$	g_{47}	6	272160	11664
		$f_{48} = 32$	$m_{48} = 288$	g_{48}	6	272160	11664
		$f_{49} = 48$	$m_{49} = 432$	g_{49}	6	408240	7776
		$f_{4,10} = 64$	$m_{4,10} = 576$	$g_{4,10}$	6	544320	5832
		$f_{51} = 1$	$m_{51} = 81$	g_{51}	2	306180	10368
$g_5 = 2D$	$k_5 = 27$	$f_{52} = 6$	$m_{52} = 486$	g_{52}	6	1837080	1728
		$f_{53} = 8$	$m_{53} = 648$	g_{53}	6	2449440	1296
		$f_{54} = 12$	$m_{54} = 972$	g_{54}	6	3674160	864
		$f_{61} = 1$	$m_{61} = 9$	g_{61}	3	6048	524880
		$f_{62} = 20$	$m_{62} = 180$	g_{62}	3	120960	26244
		$f_{63} = 30$	$m_{63} = 270$	g_{63}	3	181440	17496
$g_6 = 3A$	$k_6 = 243$	$f_{64} = 30$	$m_{64} = 270$	g_{64}	3	181440	17496
		$f_{65} = 12$	$m_{65} = 108$	g_{65}	9	72576	43740
		$f_{66} = 30$	$m_{66} = 270$	g_{66}	9	181440	17496
		$f_{67} = 120$	$m_{67} = 1080$	g_{67}	9	725760	4374
		$f_{71} = 1$	$m_{71} = 81$	g_{71}	3	181440	17496
$g_7 = 3B$	$k_7 = 27$	$f_{72} = 8$	$m_{72} = 648$	g_{72}	3	1451520	2187
		$f_{73} = 9$	$m_{73} = 729$	g_{73}	9	1632960	1944
		$f_{74} = 9$	$m_{74} = 729$	g_{74}	9	1632960	1944
		$f_{81} = 1$	$m_{81} = 81$	g_{81}	3	1088640	2916
		$f_{82} = 2$	$m_{82} = 162$	g_{82}	3	2177280	1458
		$f_{83} = 2$	$m_{83} = 162$	g_{83}	9	2177280	1458
$g_8 = 3C$	$k_8 = 27$	$f_{84} = 4$	$m_{84} = 324$	g_{84}	9	4354560	729
		$f_{85} = 6$	$m_{85} = 486$	g_{85}	9	6531840	486
		$f_{86} = 12$	$m_{86} = 972$	g_{86}	9	13063680	243
		$f_{91} = 1$	$m_{91} = 81$	g_{91}	4	306180	10368
$g_9 = 4A$	$k_9 = 27$	$f_{92} = 6$	$m_{92} = 486$	g_{92}	12	1837080	1728
		$f_{93} = 8$	$m_{93} = 648$	g_{93}	12	2449440	1296
		$f_{94} = 12$	$m_{94} = 972$	g_{94}	12	3674160	864
		$f_{10,1} = 1$	$m_{10,1} = 81$	$g_{10,1}$	4	612360	5184
$g_{10} = 4B$	$k_{10} = 27$	$f_{10,2} = 6$	$m_{10,2} = 486$	$g_{10,2}$	12	3674160	864

continued on next page

Table 1 (continued from previous page)

$[g_i]_G$	k_i	f_{ij}	m_{ij}	$[g_{ij}]_{\overline{G}}$	$o(g_{ij})$	$\|[g_{ij}]_{\overline{G}}\|$	$\|C_{\overline{G}}(g_{ij})\|$
		$f_{10,3}=8$	$m_{10,3}=648$	$g_{10,3}$	12	4898880	648
		$f_{10,4}=12$	$m_{10,4}=972$	$g_{10,4}$	12	734820	432
$g_{11}=4C$	$k_{11}=3$	$f_{11,1}=1$	$m_{11,1}=729$	$g_{11,1}$	4	5511240	576
		$f_{11,2}=2$	$m_{11,2}=1458$	$g_{11,2}$	12	11022480	288
$g_{12}=4D$	$k_{12}=3$	$f_{12,1}=1$	$m_{12,1}=729$	$g_{12,1}$	4	8266860	384
		$f_{12,2}=2$	$m_{12,2}=1458$	$g_{12,2}$	12	16533720	192
		$f_{13,1}=1$	$m_{13,1}=81$	$g_{13,1}$	4	3674160	864
		$f_{13,2}=2$	$m_{13,2}=162$	$g_{13,2}$	12	7348320	432
		$f_{13,3}=2$	$m_{13,3}=162$	$g_{13,3}$	12	7348320	432
$g_{13}=4E$	$k_{13}=27$	$f_{13,4}=2$	$m_{13,4}=162$	$g_{13,4}$	12	7348320	432
		$f_{13,5}=4$	$m_{13,5}=324$	$g_{13,5}$	12	14696640	216
		$f_{13,6}=4$	$m_{13,6}=324$	$g_{13,6}$	12	14696640	216
		$f_{13,7}=4$	$m_{13,7}=324$	$g_{13,7}$	12	14696640	216
		$f_{13,8}=8$	$m_{13,8}=648$	$g_{13,8}$	12	29393280	108
		$f_{14,1}=1$	$m_{14,1}=81$	$g_{14,1}$	5	3919104	810
		$f_{14,2}=3$	$m_{14,2}=243$	$g_{14,2}$	15	11757312	270
		$f_{14,3}=3$	$m_{14,3}=243$	$g_{14,3}$	15	11757312	270
$g_{14}=5A$	$k_{14}=27$	$f_{14,4}=2$	$m_{14,4}=162$	$g_{14,4}$	15	7838208	405
		$f_{14,5}=6$	$m_{14,5}=486$	$g_{14,5}$	15	23514624	135
		$f_{14,6}=6$	$m_{14,6}=486$	$g_{14,6}$	15	23514624	135
		$f_{14,7}=6$	$m_{14,7}=486$	$g_{14,7}$	15	23514624	135
$g_{15}=6A$	$k_{15}=3$	$f_{15,1}=1$	$m_{15,1}=729$	$g_{15,1}$	6	7348320	432
		$f_{15,2}=2$	$m_{15,2}=1458$	$g_{15,2}$	6	14696640	216
		$f_{16,1}=1$	$m_{16,1}=81$	$g_{16,1}$	6	816480	3888
		$f_{16,2}=4$	$m_{16,2}=324$	$g_{16,2}$	6	3265920	972
$g_{16}=6B$	$k_{16}=27$	$f_{16,3}=4$	$m_{16,3}=324$	$g_{16,3}$	6	3265920	972
		$f_{16,4}=6$	$m_{16,4}=486$	$g_{16,4}$	6	4898880	648
		$f_{16,5}=12$	$m_{16,5}=972$	$g_{16,5}$	6	9797760	324
		$f_{17,1}=1$	$m_{17,1}=729$	$g_{17,1}$	6	14696640	216
$g_{17}=6C$	$k_{17}=3$	$f_{17,2}=1$	$m_{17,2}=729$	$g_{17,2}$	18	14696640	216
		$f_{17,3}=1$	$m_{17,3}=729$	$g_{17,3}$	18	14696640	216
		$f_{18,1}=1$	$m_{18,1}=81$	$g_{18,1}$	6	2449440	1296
		$f_{18,2}=2$	$m_{18,2}=162$	$g_{18,2}$	6	4898880	648
		$f_{18,3}=2$	$m_{18,3}=162$	$g_{18,3}$	6	4898880	648
$g_{18}=6D$	$k_{18}=27$	$f_{18,4}=4$	$m_{18,4}=324$	$g_{18,4}$	6	9797760	324
		$f_{18,5}=2$	$m_{18,5}=162$	$g_{18,5}$	18	4898880	648
		$f_{18,6}=4$	$m_{18,6}=324$	$g_{18,6}$	18	9797760	324
		$f_{18,7}=4$	$m_{18,7}=324$	$g_{18,7}$	18	9797760	324
		$f_{18,8}=8$	$m_{18,8}=648$	$g_{18,8}$	18	19595520	162
$g_{19}=6E$	$k_{19}=3$	$f_{19,1}=1$	$m_{19,1}=729$	$g_{19,1}$	6	39191040	108
		$f_{19,2}=2$	$m_{19,2}=1458$	$g_{19,2}$	6	78382080	54
$g_{20}=6F$	$k_{20}=3$	$f_{20,1}=1$	$m_{20,1}=729$	$g_{20,1}$	6	39191040	108
		$f_{20,2}=2$	$m_{20,2}=1458$	$g_{20,2}$	18	78382080	54
$g_{21}=6G$	$k_{21}=3$	$f_{21,1}=1$	$m_{21,1}=729$	$g_{21,1}$	6	88179840	36
		$f_{21,2}=2$	$m_{21,2}=1458$	$g_{21,2}$	18	176359680	18
		$f_{22,1}=1$	$m_{22,1}=729$	$g_{22,1}$	7	151165440	21
$g_{22}=7A$	$k_{22}=3$	$f_{22,2}=1$	$m_{22,2}=729$	$g_{22,2}$	21	151165440	21
		$f_{22,3}=1$	$m_{22,3}=729$	$g_{22,3}$	21	151165440	21

continued on next page

Table 1 (continued from previous page)

$[g_i]_G$	k_i	f_{ij}	m_{ij}	$[g_{ij}]_{\overline{G}}$	$o(g_{ij})$	$\vert[g_{ij}]_{\overline{G}}\vert$	$\vert C_{\overline{G}}(g_{ij})\vert$
$g_{23} = 8A$	$k_{23} = 3$	$f_{23,1} = 1$	$m_{23,1} = 729$	$g_{23,1}$	8	66134880	48
		$f_{23,2} = 2$	$m_{23,2} = 1458$	$g_{23,2}$	24	132269760	24
		$f_{24,1} = 1$	$m_{24,1} = 729$	$g_{24,1}$	8	66134880	48
$g_{24} = 8B$	$k_{24} = 3$	$f_{24,2} = 1$	$m_{24,2} = 729$	$g_{24,2}$	24	66134880	48
		$f_{24,3} = 1$	$m_{24,3} = 729$	$g_{24,3}$	24	66134880	48
		$f_{25,1} = 1$	$m_{25,1} = 729$	$g_{25,1}$	9	117573120	27
$g_{25} = 9A$	$k_{25} = 3$	$f_{25,2} = 1$	$m_{25,2} = 729$	$g_{25,2}$	9	117573120	27
		$f_{25,3} = 1$	$m_{25,3} = 729$	$g_{25,3}$	9	117573120	27
		$f_{26,1} = 1$	$m_{26,1} = 729$	$g_{26,1}$	10	105815808	30
$g_{26} = 10A$	$k_{26} = 3$	$f_{26,2} = 1$	$m_{26,2} = 729$	$g_{26,2}$	30	105815808	30
		$f_{26,3} = 1$	$m_{26,3} = 729$	$g_{26,3}$	30	105815808	30
$g_{27} = 12A$	$k_{27} = 3$	$f_{27,1} = 1$	$m_{27,1} = 729$	$g_{27,1}$	12	44089920	72
		$f_{27,2} = 2$	$m_{27,2} = 1458$	$g_{27,2}$	36	88179840	36
$g_{28} = 12B$	$k_{28} = 3$	$f_{28,1} = 1$	$m_{28,1} = 729$	$g_{28,1}$	12	44089920	72
		$f_{28,2} = 2$	$m_{28,2} = 1458$	$g_{28,2}$	12	88179840	36
		$f_{29,1} = 1$	$m_{29,1} = 729$	$g_{29,1}$	12	88179840	36
$g_{29} = 12C$	$k_{29} = 3$	$f_{29,2} = 1$	$m_{29,2} = 729$	$g_{25,2}$	36	88179840	36
		$f_{29,3} = 1$	$m_{29,3} = 729$	$g_{29,3}$	36	88179840	36
		$f_{30,1} = 1$	$m_{30,1} = 729$	$g_{30,1}$	15	70543872	45
$g_{30} = 15A$	$k_{30} = 3$	$f_{30,2} = 1$	$m_{30,2} = 729$	$g_{30,2}$	45	70543872	45
		$f_{30,3} = 1$	$m_{30,3} = 729$	$g_{30,3}$	45	70543872	45

3. Inertia Factor Groups of $\overline{G} = 3^7{:}Sp(6,2)$

In this section, through some computations, we determine the inertia factor groups of $3^7{:}Sp(6,2)$. This determination is achieved by investigating the number of irreducible characters and fusions of conjugacy classes of some of the maximal subgroups of the maximal subgroups of $Sp(6,2)$ (sometimes we may go further and look at some of the maximal subgroups of the maximal subgroups of the maximal subgroups of $Sp(6,2)$).

We have seen in Section 2 that the action of $\overline{G} = 3^7{:}Sp(6,2)$ (or just $G = Sp(6,2)$) on 3^7 produces six orbits of lengths 1, 56, 126, 576, 672 and 756. By a theorem of Brauer (for example see Theorem 5.1.5 of [14]) the number of orbits of $\overline{G}$ (or just G) on $\mathrm{Irr}(3^7)$ will also be 6. Since $N = 3^7$ is a vector space, the action of G on $\mathrm{Irr}(3^7)$ can be viewed as the action of G on N^*, where N^* is the dual space of N. In fact we have found that the orbit lengths of G on $\mathrm{Irr}(3^7)$ are 1, 56, 126, 576, 672 and 756. Let $H_1, H_2, \ldots, H_6$ be the respective inertia factor groups of the representatives of characters from the previous orbits. We notice that these inertia factors have indices 1, 56, 126, 576, 672 and 756 respectively in $Sp(6,2)$. Clearly $H_1 = Sp(6,2)$. By looking at the $\mathbb{ATLAS}$ [5], the group $Sp(6,2)$ has 8 conjugacy classes of maximal subgroups. Let $M[1], M[2], \ldots, M[8]$ be representatives of these classes. That is $M[1] = U_4(2){:}2$, $M[2] = S_8$, $M[3] = 2^5{:}S_6$, $M[4] = U_3(3){:}2$, $M[5] = 2^6{:}L_3(2)$,

$M[6] = (2^{1+4} \times 2^2){:}(S_3 \times S_3)$, $M[7] = S_3 \times S_6$ and $M[8] = PSL(2,8){:}3$. In Table 2 we give some information about the maximal subgroups of $M[1], M[2], \ldots, M[8]$. Now by considering the indices of the maximal subgroups of $Sp(6,2)$, we get the following possibilities for $H_1, H_2, \ldots, H_6$:

- $H_1 = Sp(6,2)$,
- $H_2 \leq M[1] = U_4(2){:}2$ with index 2,
- $H_3 \leq M[3] = 2^5{:}S_6$ with index 2,
- $H_4 \leq M[2] = S_8$ with index 16,
- $H_5 \leq M[1] = U_4(2){:}2$ with index 24 or a subgroup of $M[7] = S_3 \times S_6$ with index 2,
- $H_6 \leq M[1]$ with index 27, or a subgroup of $M[2]$ of index 21, or a subgroup of $M[3]$ with index 12.

Recall from Table 1 that the total number of conjugacy classes of $\overline{G}$ is 118. We deduce that the total contribution of irreducible characters from the six inertia factor groups must also be 118. That is

$$|\mathrm{Irr}(H_1)| + |\mathrm{Irr}(H_2)| + |\mathrm{Irr}(H_3)| + |\mathrm{Irr}(H_4)| + |\mathrm{Irr}(H_5)| + |\mathrm{Irr}(H_6)| = 118. \tag{3.1}$$

3.1. **First, Second, Third and Fourth Inertia Factor Groups.** We recall that in Section 1, we represented the group $G = Sp(6,2)$ in terms of 8−dimensional matrices. For the sake of convenience in computations with GAP, we use 6−dimensional representations over $\mathbb{F}_2$ of $Sp(6,2)$.

We have used the following 6−dimensional matrices $\widetilde{\alpha}_1$ and $\widetilde{\alpha}_2$ over $\mathbb{F}_2$, that generate $Sp(6,2)$ (see [16]) to represent $Sp(6,2)$ in GAP and then locate the maximal subgroups and the other required subgroups.

$$\widetilde{\alpha}_1 = \begin{pmatrix} 0&1&0&0&0&0\\ 1&0&0&0&0&0\\ 1&1&1&0&0&0\\ 1&1&0&1&0&0\\ 0&0&0&0&1&0\\ 1&1&0&0&0&1 \end{pmatrix}, \quad \widetilde{\alpha}_2 = \begin{pmatrix} 0&0&1&0&0&0\\ 0&1&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1\\ 1&1&1&0&1&0 \end{pmatrix}.$$

We have mentioned that the first inertia factor group H_1 is $Sp(6,2)$, which has 30 irreducible characters. Since H_2 has an index 2 in $U_4(2){:}2$, it is readily verified that $H_2 = M[11] = U_4(2)$, which has 20 irreducible characters. As a 6−dimensional subgroup of $Sp(6,2)$ over $\mathbb{F}_2$, the group H_2 is generated by σ_1 and σ_2, where

$$\sigma_1 = \begin{pmatrix} 0&1&1&1&1&1\\ 0&0&0&1&0&0\\ 0&1&1&0&1&0\\ 1&1&0&1&0&1\\ 0&1&0&0&0&0\\ 0&0&0&1&1&1 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0&1&1&1&1&1\\ 0&1&0&0&0&0\\ 0&0&0&0&1&1\\ 0&1&1&0&0&1\\ 1&0&0&0&0&0\\ 1&1&0&0&1&0 \end{pmatrix}.$$

The character table of $H_2 = U_4(2)$ is available in the $\mathbb{ATLAS}$ and also appears as Table 11.1 of [1].

The third inertia factor group H_3 has an index 2 in $2^5{:}S_6$. In Table 2 we list some information on the maximal subgroups of the maximal subgroups of $Sp(6,2)$.

TABLE 2. Some information on the maximal subgroups of the maximal subgroups of $Sp(6,2)$

Maximal Subgroups of $Sp(6,2)$	$M[ij]$	$\|M[ij]\|$	$[M[i]:M[ij]]$	$\|\mathrm{Irr}(M[ij])\|$
$M[1]=U_4(2){:}2$	$M[11]=U_4(2)$	25920	2	20
	$M[12]=(2^4{:}A_5){:}2$	1920	27	18
	$M[13]=2\times S_6$	1440	36	22
	$M[14]=((((3\times(3^2{:}2)){:}2){:}3){:}2){:}2$	1296	40	22
	$M[15]=((3^{1+2}{:}Q_8){:}3){:}2$	1296	40	18
	$M[16]=(((((2\times D_8){:}2){:}3){:}3){:}2){:}2$	1152	45	25
$M[2]=S_8$	$M[21]=A_8$	20160	2	14
	$M[22]=S_7$	5040	8	15
	$M[23]=2\times S_6$	1440	28	22
	$M[24]=(S_4\times S_4){:}2$	1152	35	20
	$M[25]=S_5\times S_3$	720	56	21
	$M[26]=((((2\times D_8){:}2){:}3){:}2){:}2$	384	105	20
	$M[27]=PSL(3,2){:}2$	336	120	9
$M[3]=2^5{:}S_6$	$M[31]=2^5{:}A_6$	11520	2	23
	$M[32]=(2^5{:}A_5){:}2$	3840	6	23
	$M[33]=2\times((2^4{:}A_5){:}2)$	3840	6	36
	$M[34]=2\times((S_4\times S_4){:}2)$	2304	10	40
	$M[35]=(2^2\times((((2\times D_8){:}2){:}3){:}2)){:}2$	1536	15	53
	$M[36]=((((2\times(2^4{:}2)){:}2){:}3){:}2){:}2$	1536	15	40
	$M[37]=2\times S_6$	1440	16	22
	$M[38]=2\times S_6$	1440	16	22
$M[4]=U_3(3){:}2$	$M[41]=PSU(3,3)$	6048	2	14
	$M[42]=(3^{1+2}{:}8){:}2$	432	28	14
	$M[43]=PSL(3,2){:}2$	336	36	9
	$M[44]=((4^2{:}3){:}2){:}2$	192	63	14
	$M[45]=(SL(2,3){:}4){:}2$	192	63	17
$M[5]=2^6{:}L_3(2)$	$M[51]=(((2^2\times((2\times D_8){:}2)){:}3){:}2){:}2$	1536	7	40
	$M[52]=((((2\times(2^4{:}2)){:}2){:}3){:}2){:}2$	1536	7	40
	$M[53]=2^3{:}PSL(3,2)$	1344	8	11
	$M[34]=2^3{\cdot}PSL(3,2)$	1344	8	11
	$M[55]=(2^6{:}7){:}3$	1344	8	16
$M[6]=(2^{1+4}\times 2^2){:}(S_3\times S_3)$	$M[61]=((2^2\times(((2\times D_8){:}2){:}3)){:}3){:}2$	2304	2	36
	$M[62]=((2^2\times(((2\times D_8){:}2){:}3)){:}3){:}2$	2304	2	33
	$M[63]=((2^2\times(((2\times D_8){:}2){:}3)){:}3){:}2$	2304	2	39
	$M[64]=(2^2\times((((2\times D_8){:}2){:}3){:}2)){:}2$	1536	3	40
	$M[65]=(((2^2\times((2\times D_8){:}2)){:}3){:}2){:}2$	1536	3	53
	$M[66]=(((((2\times D_8){:}2){:}3){:}3){:}2){:}2$	1152	4	25
	$M[67]=2\times S_4\times S_3$	288	16	30
$M[7]=S_3\times S_6$	$M[71]=A_6\times S_3$	2160	2	21
	$M[72]=(3\times A_6){:}2$	2160	2	18
	$M[73]=3\times S_6$	2160	2	33
	$M[74]=2\times S_6$	1440	3	22
	$M[75]=S_3\times S_5$	720	6	21
	$M[76]=S_5\times S_3$	720	6	21
	$M[77]=((S_3\times S_3){:}2)\times S_3$	432	10	27
	$M[78]=2\times S_4\times S_3$	288	15	30
	$M[79]=2\times S_3\times S_4$	288	15	30
$M[8]=PSL(2,8){:}3$	$M[81]=PSL(2,8)$	504	3	9
	$M[82]=(2^3{:}7){:}3$	168	9	8
	$M[83]=(9{:}3){:}2$	54	28	10
	$M[84]=(7{:}3){:}2$	42	36	7

By checking the indices of the maximal subgroups of $M[3]=2^5{:}S_6$ (supplied in Table 2) we can see that H_3 is in the conjugacy class of maximal subgroups of $M[3]$ containing $M[31]=2^5{:}A_6$. Thus we can take $H_3=M[31]$. Note that $M[31]=2^5{:}A_6=\langle \xi_1,\xi_2\rangle$, where

$$\xi_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \xi_2 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}.$$

The character table of $H_3 = 2^5{:}A_6$ appears as Table 11.2 of [1] or can easily be obtained using GAP.

Next we determine the fourth inertia factor group H_4, which sits inside $M[2] = S_8$. Note that $[S_8 : H_4] = 16$ and thus (using Table 2) H_4 is either a subgroup of $M[21] = A_8$ with index 8, or it is a subgroup of $M[22] = S_7$ with index 2. In Table 3, we give some information about the maximal subgroups of $M[21]$ and $M[22]$.

Table 3. Some information on the maximal subgroups of $M[21]$ and $M[22]$

Maximal Subgroups of $M[12]$ & $M[22]$	$M[ijk]$	$\lvert M[ijk]\rvert$	$[M[ij] : M[ijk]]$	$\lvert \mathrm{Irr}(M[ijk])\rvert$
$M[21] = A_8$	$M[211] = A_7$	2520	8	9
	$M[212] = 2^3{:}PSL(3,2)$	1344	15	11
	$M[213] = 2^3{:}PSL(3,2)$	1344	15	11
	$M[214] = S_6$	720	28	11
	$M[215] = ((A_4 \times A_4){:}2){:}2$	576	35	16
	$M[216] = GL(2,4){:}2$	360	56	12
$M[22] = S_7$	$M[221] = A_7$	2520	2	9
	$M[222] = S_6$	720	7	11
	$M[223] = 2 \times S_5$	240	21	14
	$M[224] = S_4 \times S_3$	144	35	15
	$M[225] = (7{:}3){:}2$	42	120	7

From Table 3, we can see that a subgroup of $M[21] = A_8$ of index 8 must be isomorphic to A_7, while a subgroup of $M[22] = S_7$ of index 2 must also be isomorphic to A_7. Note that $M[211] = A_7 = \langle \beta_1, \beta_2 \rangle$ and $M[221] = A_7 = \langle \gamma_1, \gamma_2 \rangle$, where

$$\beta_1 = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}, \quad \beta_2 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \end{pmatrix},$$

$$\gamma_1 = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \end{pmatrix}.$$

Thus for the construction of the character table of $\overline{G}$, it will not make difference to which A_7 we choose. Hence we may take H_4 to be $M[211] \cong A_7$ and note that $|\mathrm{Irr}(H_4)| = |\mathrm{Irr}(A_7)| = 9$. The character table of A_7 appears as Table 11.3 of [1].

3.2. **Fifth and Sixth Inertia Factor Groups.** From the last subsection we have seen that $|\mathrm{Irr}(H_1)| = 30$, $|\mathrm{Irr}(H_2)| = 20$, $|\mathrm{Irr}(H_3)| = 23$ and $|\mathrm{Irr}(H_4)| = 9$. Substituting these into Equation (3.1), we get that $|\mathrm{Irr}(H_5)| + |\mathrm{Irr}(H_6)| = 36$.

Recall that H_5 is either an index 24 subgroup of $M[1] = U_4(2){:}2$ or it is an index 2 subgroup of $M[7] = S_3 \times S_6$. If $H_5 \leq U_4(2){:}2$ with index 24, then the only possibility (see Table 2) is that

$H_5 \leq U_4(2)$ with $[U_4(2) : H_5] = 12$. However by looking at the $\mathbb{ATLAS}$ we can see that the group $U_4(2)$ does not contain a subgroup of index 12. This leaves us with the other possibility, that is $H_5 \leq S_3 \times S_6$ and $[S_3 \times S_6 : H_5] = 2$. From Table 2, we can see that there are three classes of non-conjugate maximal subgroups of $S_3 \times S_6$, such that a subgroup of each class has an index 2. Therefore H_5 is either $M[71] = A_6 \times S_3$, $M[72] = (3 \times A_6){:}2$ or $M[73] = 3 \times S_6$. Hence by the last column of Table 2, it follows that $|\mathrm{Irr}(H_5)| \in \{21, 18, 33\}$. We take this point into our considerations and we look at the group H_6.

The index of the sixth inertia factor group H_6 in $Sp(6, 2)$ is 756. This forces H_6 to be either a subgroup of $U_4(2){:}2$ with index 27, or a subgroup of S_8 of index 21, or a subgroup of $2^5{:}S_6$ with index 12. However the second possibility ($H_6 \leq S_8$) is not feasible since S_8 does not contain a subgroup of index that is a divisor of 27 (see Table 2). If H_6 is a subgroup of $U_4(2){:}2$ of index 27, then it must be in the conjugacy class of maximal subgroups of $U_4(2){:}2$ containing $M[12] = (2^4{:}A_5){:}2$. The group $M[12] = (2^4{:}A_5){:}2$ is generated by π_1 and π_2, where

$$\pi_1 = \begin{pmatrix} 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{pmatrix}, \quad \pi_2 = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

and $|\mathrm{Irr}(H_6)| = 18$. On the other hand if $H_6 \leq 2^5{:}S_6$ such that $[2^5{:}S_6 : H_6] = 12$, then three possibilities arise (see Table 2):

- $H_6 \leq M[31] = 2^5{:}A_6$ with index 6,
- $H_6 \leq M[32] = (2^5{:}A_5){:}2$ with index 2 or
- $H_6 \leq M[33] = 2 \times ((2^4{:}A_5){:}2)$ with index 2.

In Table 4, we provide some information on the maximal subgroups of $M[31]$, $M[32]$ and $M[33]$.

Table 4. Some information on the maximal subgroups of $M[31], M[32]$ and $M[33]$

Maximal Subgroups of $M[31]$, $M[32]$ & $M[33]$	$M[ijk]$	$\lvert M[ijk]\rvert$	$[M[ij] : M[ijk]]$	$\lvert\mathrm{Irr}(M[ijk])\rvert$
$M[31] = 2^5{:}A_6$	$M[311] = 2^5{:}A_5$	1920	6	16
	$M[312] = 2 \times (2^4{:}A_5)$	1920	6	24
	$M[313] = 2 \times ((A_4 \times A_4){:}4)$	1152	10	26
	$M[314] = (2 \times ((((2 \times D_8){:}2){:}3){:}2)){:}2$	768	15	31
	$M[315] = (((2 \times (2^4{:}2)){:}2){:}3){:}2$	768	15	23
	$M[316] = 2 \times A_6$	720	16	14
	$M[317] = 2 \times A_6$	720	16	14
$M[32] = (2^5{:}A_5){:}2$	$M[321] = 2^5{:}A_5$	1920	2	16
	$M[322] = (((2 \times (2^4{:}2)){:}2){:}3){:}2$	768	5	23
	$M[323] = 2 \times ((2^4{:}5){:}4)$	640	6	22
	$M[324] = 2 \times (((2^4{:}3){:}2){:}2)$	384	10	28
	$M[325] = 2 \times S_5$	240	16	14
	$M[326] = 2 \times S_5$	240	16	14
$M[33] = 2 \times ((2^4{:}A_5){:}2)$	$M[331] = (2^4{:}A_5){:}2$	1920	2	18
	$M[332] = 2 \times (2^4{:}A_5)$	1920	2	24
	$M[333] = (2^4{:}A_5){:}2$	1920	2	18
	$M[334] = 2 \times (((((2 \times D_8){:}2){:}3){:}2){:}2)$	768	5	40
	$M[335] = 2 \times ((2^4{:}5){:}4)$	640	6	22
	$M[336] = 2 \times S_4 \times D_8$	384	10	50
	$M[337] = 2 \times S_5$	240	16	14

From Table 4, we can see that there are 6 possibilities for H_6. This together with the option $H_6 = M[12] = (2^4{:}A_5){:}2$ gives that $H_6 \in \{M[12], M[311], M[312], M[321], M[331], M[332], M[333]\}$ and we notice that $|\mathrm{Irr}(H_6)| \in \{18, 16, 24, 16, 18, 24, 18\}$ respectively. We recall that $|\mathrm{Irr}(H_5)| \in \{21, 18, 33\}$. Therefore possible pairs representing (H_5, H_6) are

$$\begin{aligned}(H_5, H_6) \in \; & \{(M[71], M[12]), (M[71], M[311]), (M[71], M[312]), (M[71], M[321]),\\ & (M[71], M[331]), (M[71], M[332]), (M[71], M[333]), (M[72], M[12]),\\ & (M[72], M[311]), (M[72], M[312]), (M[72], (M[321]), (M[72], M[331]),\\ & (M[72], M[332]), (M[72], M[333]), (M[73], M[12]), (M[73], (M[311]),\\ & (M[73], M[312]), (M[73], M[321]), (M[71], M[331]), (M[71], M[332]),\\ & (M[73], M[333])\}\end{aligned}$$

and it follows respectively that

$$\begin{aligned}(|\mathrm{Irr}(H_5)|, |\mathrm{Irr}(H_6)|) \in \; & \{(21, 18), (21, 16), (21, 24), (21, 16), (21, 18), (21, 24), (21, 18),\\ & (18, 18), (18, 16), (18, 24), (18, 16), (18, 18), (18, 24), (18, 18),\\ & (33, 18), (33, 16), (33, 24), (33, 16), (33, 18), (33, 24), (33, 18)\}.\end{aligned}$$

Since $|\mathrm{Irr}(H_5)| + |\mathrm{Irr}(H_6)| = 36$, we deduce that

$$(|\mathrm{Irr}(H_5)|, |\mathrm{Irr}(H_6)|) \in \{(18, 18), (18, 18), (18, 18)\},$$

that is

$$(H_5, H_6) \in \{(M[72], M[12]), (M[72], M[331]), (M[72], M[333])\}.$$

Hence $H_5 = M[72] = (3 \times A_6){:}2$ (in all the cases) and $H_6 = M[12] = (2^4{:}A_5){:}2$, $M[331] = (2^4{:}A_5){:}2$ or $M[333] = (2^4{:}A_5){:}2$. Note that the group $M[72] = (3 \times A_6){:}2$ is generated by μ_1 and μ_2, where

$$\mu_1 = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 1\\ 0 & 1 & 1 & 0 & 0 & 1\\ 0 & 1 & 0 & 1 & 0 & 1\\ 1 & 1 & 0 & 0 & 1 & 1\\ 1 & 1 & 1 & 0 & 1 & 1\\ 0 & 1 & 0 & 1 & 1 & 1 \end{pmatrix}, \quad \mu_2 = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 1\\ 1 & 1 & 1 & 0 & 1 & 1\\ 1 & 1 & 0 & 1 & 1 & 1\\ 1 & 0 & 0 & 0 & 1 & 1\\ 1 & 0 & 1 & 0 & 0 & 1\\ 1 & 1 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

The full character table of H_5 appears as Table 11.4 of [1] or can easily be obtained using GAP. Next we determine the group H_6.

The groups $M[12]$, $M[331]$ and $M[333]$ are generated as follows: $M[12] = \langle\theta_1, \theta_2\rangle$, $M[331] = \langle\epsilon_1, \epsilon_2\rangle$ and $M[333] = \langle\delta_1, \delta_2\rangle$, where

$$\theta_1 = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0\\ 0 & 1 & 0 & 1 & 1 & 0\\ 1 & 0 & 0 & 0 & 1 & 1\\ 0 & 0 & 0 & 1 & 0 & 0\\ 1 & 0 & 1 & 1 & 1 & 0\\ 1 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}, \quad \theta_2 = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 1\\ 0 & 1 & 0 & 0 & 1 & 1\\ 1 & 0 & 0 & 1 & 1 & 1\\ 0 & 0 & 0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix},$$

$$\epsilon_1 = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}, \quad \epsilon_2 = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \end{pmatrix},$$

$$\delta_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \quad \delta_2 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 \end{pmatrix}.$$

By looking at the fusion of the conjugacy classes of $M[12] = \langle \theta_1, \theta_2 \rangle$, we see that the unique conjugacy class of involutions with size 20, fuses into the class $g_2 = 2A$ of $Sp(6,2)$. From Table 2 we see that $c(g_2) = 2$ and hence from the properties of the Fischer matrices (Proposition 3.6 of [2]), we have $\sum_{k=1}^{t} c(g_{ik}) = c(g_i)$ and for $i = 2$ we get $\sum_{k=1}^{6} c(g_{2k}) = c(g_2) = 2$. From Table 6.11 of [1], we can see that the classes $2A$ of $H_1 = Sp(6,2)$ and $2a$ of H_3 are both fusing into the class $2A$ of $Sp(6,2)$. Therefore if $M[12]$ is the sixth inertia factor group, then we get a contradiction (the Fischer matrix $\mathcal{F}_2$ corresponds to $g_2 = 2A$ will be of size 3×2 contradicting Proposition 3.6(i) of [2]). By similar arguments we can show that the group $M[333]$ can not be H_6. Hence we deduce that $M[331] = (2^4{:}A_5){:}2$, with the generators ϵ_1 and ϵ_2, is the sixth inertia factor group H_6. The fusion of classes of $H_6 = (2^4{:}A_5){:}2$ into classes of $Sp(6,2)$ can be viewed in Table 6.11 of [1]. The full character table of H_6 appears as Table 11.5 of [1]. This completes our determination of the inertia factor groups of $\overline{G} = 3^7{:}Sp(6,2)$.

Note 3.1. *If θ_i is an orbit representative of the action of G on $\mathrm{Irr}(N)$, then H_i can be obtained by GAP as G_{θ_i}, the set stabilizer of θ_i in G.*

3.3. **Fusions of the Inertia Factor Groups into $Sp(6,2)$.** In this section we determine the fusions of classes of the inertia factor groups H_2, H_3, H_4, H_5 and H_6 into classes of $Sp(6,2)$. We have used the permutation characters of $Sp(6,2)$ on the inertia factor groups and the centralizer sizes to determine these fusions. We have found the following proposition is very helpful in calculating the permutation characters $\chi(Sp(6,2)|H_i),\ 2 \leq i \leq 6$.

Proposition 3.2. *Let $K_1 \leq K_2 \leq K_3$ and let ψ be a class function on K_1. Then $(\psi\uparrow_{K_1}^{K_2})\uparrow_{K_2}^{K_3} = \psi\uparrow_{K_1}^{K_3}$. More generally if $K_1 \leq K_2 \leq \cdots \leq K_n$ is a nested sequence of subgroups of K_n and ψ is a class function on K_1, then $(\psi\uparrow_{K_1}^{K_2})\uparrow_{K_2}^{K_3} \cdots \uparrow_{K_{n-1}}^{K_n} = \psi\uparrow_{K_1}^{K_n}$.*

Proof. See Proposition 3.5.6 of [1]. □

The decompositions of the permutation characters $\chi(Sp(6,2)|M[i]),\ 1 \leq i \leq 8$ are all given in the 𝔸𝕋𝕃𝔸𝕊. Also it is not difficult to calculate $\chi(M[i]|M[ij]),\ 1 \leq i \leq 8,\ j$ is the number of conjugacy classes of maximal subgroups of $M[i]$. Where it is needed, it is also not difficult to calculate the permutation character $\chi(M[ij]|M[ijk]),\ 1 \leq i \leq 8,\ j$ is the number of conjugacy classes of maximal subgroups of $M[i]$, k is the number of conjugacy classes of maximal subgroups of $M[ij]$. Thus with the applications of Proposition 3.2, the values of $\chi(Sp(6,2)|H_i),\ 2 \leq i \leq 6$ are easy to calculate. In fact we have listed these values at the bottom of Tables 6.6, 6.7, 6.8, 6.9 and 6.10 of [1] respectively. Now with the aid of the permutation characters, centralizer sizes and matrix conjugation in $Sp(6,2)$, we are able to determine all the fusions of classes of the inertia factor groups into classes of $Sp(6,2)$. We list these fusions in Table 5.

TABLE 5. The fusions of conjugacy classes of the inertia factor groups into classes of $Sp(6,2)$

Inertia Factor Groups H_2, H_3, H_4, H_5 & H_6	Class of H_i $\hookrightarrow$	Class of $Sp(6,2)$	Class of H_i $\hookrightarrow$	Class of $Sp(6,2)$
$H_2 = M[11] = U_4(2)$	$1a = g_{121}$	$1A$	$6a = g_{16,21}$	$6B$
	$2a = g_{421}$	$2C$	$6b = g_{16,22}$	$6B$
	$2b = g_{321}$	$2B$	$6c = g_{17,21}$	$6C$
	$3a = g_{621}$	$3A$	$6d = g_{17,22}$	$6C$
	$3b = g_{721}$	$3B$	$6e = g_{20,21}$	$6F$
	$3c = g_{722}$	$3B$	$6f = g_{18,21}$	$6D$
	$3d = g_{821}$	$3C$	$9a = g_{25,21}$	$9A$
	$4a = g_{13,21}$	$4E$	$9b = g_{25,22}$	$9A$
	$4b = g_{921}$	$4A$	$12a = g_{29,21}$	$12C$
	$5a = g_{14,21}$	$5A$	$12b = g_{29,22}$	$12C$
$H_3 = M[31] = 2^5{:}A_6$	$1a = g_{131}$	$1A$	$4e = g_{10,31}$	$4B$
	$2a = g_{231}$	$2A$	$5a = g_{14,31}$	$5A$
	$2b = g_{331}$	$2B$	$5b = g_{14,32}$	$5A$
	$2c = g_{431}$	$2C$	$6a = g_{16,31}$	$6B$
	$2d = g_{432}$	$2C$	$6b = g_{18,31}$	$6D$
	$2e = g_{531}$	$2D$	$6c = g_{15,31}$	$6A$
	$3a = g_{631}$	$3A$	$6d = g_{19,31}$	$6E$
	$3b = g_{831}$	$3C$	$8a = g_{24,31}$	$8B$
	$4a = g_{12,31}$	$4D$	$8b = g_{24,32}$	$8B$
	$4b = g_{931}$	$4A$	$10a = g_{26,31}$	$10A$
	$4c = g_{13,31}$	$4E$	$10b = g_{26,32}$	$10A$
	$4d = g_{13,32}$	$4E$		
$H_4 = M[211] = A_7$	$1a = g_{141}$	$1A$	$5a = g_{14,41}$	$5A$
	$2a = g_{441}$	$2C$	$6a = g_{18,41}$	$6D$
	$3a = g_{841}$	$3C$	$7a = g_{22,41}$	$7A$
	$3b = g_{641}$	$3A$	$7b = g_{22,42}$	$7A$
	$4a = g_{13,41}$	$4E$		
$H_5 = M[72] = (3 \times A_6){:}2$	$1a = g_{151}$	$1A$	$4a = g_{10,51}$	$4B$
	$2a = g_{451}$	$2C$	$4b = g_{13,51}$	$4E$
	$2b = g_{452}$	$2C$	$5a = g_{14,51}$	$5A$
	$2c = g_{551}$	$2D$	$6a = g_{18,51}$	$6D$
	$3a = g_{651}$	$3A$	$6b = g_{18,52}$	$6D$
	$3b = g_{652}$	$3A$	$6c = g_{21,51}$	$6G$
	$3c = g_{851}$	$3C$	$12a = g_{27,51}$	$12A$
	$3d = g_{751}$	$3B$	$15a = g_{30,51}$	$15A$
	$3e = g_{852}$	$3C$	$15b = g_{30,52}$	$15A$
$H_6 = M[331] = (2^4{:}A_5){:}2$	$1a = g_{161}$	$1A$	$4c = g_{10,61}$	$4B$
	$2a = g_{461}$	$2C$	$4d = g_{11,61}$	$4C$
	$2b = g_{361}$	$2B$	$4e = g_{13,62}$	$4E$
	$2c = g_{462}$	$2C$	$5a = g_{14,61}$	$5A$
	$2d = g_{561}$	$2D$	$6a = g_{16,61}$	$6B$
	$2e = g_{463}$	$2C$	$6b = g_{18,61}$	$6D$
	$3a = g_{661}$	$3A$	$6c = g_{18,62}$	$6D$
	$4a = g_{961}$	$4A$	$8a = g_{23,61}$	$8A$
	$4b = g_{13,61}$	$4E$	$12a = g_{28,61}$	$12B$

4. Character Tables of the Inertia Factor Groups

We recall that knowledge of the appropriate character tables of inertia factor groups is pivotal in calculating the full character table of any group extension. Since in our extension $\overline{G} = 3^7{:}Sp(6,2)$, the normal subgroup 3^7 is abelian and the extension splits, it follows by Mackey's Theorem (see Theorem 5.1.4 of Basheer [1] for example), that every character θ_k of 3^7 is extendible to a character of its inertia group $\overline{H}_k$. Thus all the factor sets $\overline{\alpha}_k$ are trivial and all the character tables of the inertia factor groups that we will use to construct the character table of $\overline{G}$, are the ordinary ones.

The character table of $H_1 = Sp(6,2)$ is available in the $\mathbb{ATLAS}$. We have used GAP to construct the character tables of $H_2 = U_4(2)$, $H_3 = 2^5{:}A_6$, $H_4 = A_7$, $H_5 = (3 \times A_6){:}2$ and $H_6 = (2^4{:}A_5){:}2$ since we know from Section 3 that $H_2 = \langle \sigma_1, \sigma_2 \rangle$, $H_3 = \langle \xi_1, \xi_2 \rangle$, $H_4 = \langle \beta_1, \beta_2 \rangle$, $H_5 = \langle \mu_1, \mu_2 \rangle$ and $H_6 = \langle \epsilon_1, \epsilon_2 \rangle$. Also the character tables of H_2, H_3, H_4, H_5 and H_6 are given as Tables 11.1, 11.2, 11.3, 11.4 and 11.5 of [1] respectively.

5. Fischer Matrices of $\overline{G} = 3^7{:}Sp(6,2)$

The theory of Clifford-Fischer matrices, which is based on Clifford Theory (see [6]), was developed by B. Fischer [7, 8, 9]. For the general definition of Fischer matrices we refer to [1, 2]. We recall that we label the top and bottom of the columns of the Fischer matrix $\mathcal{F}_i$, corresponding to g_i, by the sizes of the centralizers of g_{ij}, $1 \leq j \leq c(g_i)$ in $\overline{G}$ and m_{ij} respectively. In Table 1 we supplied $|C_{\overline{G}}(g_{ij})|$ and m_{ij}, $1 \leq i \leq 30$, $1 \leq j \leq c(g_i)$. Also having obtained the fusions of the inertia factor groups H_2, H_3, H_4, H_5 and H_6 into $Sp(6,2)$, we are able to label the rows of the Fischer matrices as described in [1]. We have used the properties of Fischer matrices, given in Proposition 3.6 of [2] to calculate some of the entries of the Fischer matrices and also to build an algebraic system of equations. For example since the extension is split, then every coset $N\overline{g}_i$ (or just Ng_i) is a split coset (see [15]) and it results that $a_{i1}^{(k,m)} = \frac{|C_{Sp(6,2)}(g_i)|}{|C_{H_k}(g_{ikm})|}$, for all $i \in \{1, 2, \ldots, 30\}$. With the help of the symbolic mathematical package Maxima [11], we were able to solve these systems of equations and hence we have computed all the Fischer matrices of $\overline{G}$, which we list below.

$\mathcal{F}_1$

g_1		g_{11}	g_{12}	g_{13}	g_{14}	g_{15}	g_{16}
$o(g_{1j})$		1	3	3	3	3	3
$\lvert C_{\overline{G}}(g_{1j})\rvert$		3174474240	56687040	25194240	5511240	4723920	4199040
(k,m)	$\lvert C_{H_k}(g_{1km})\rvert$						
$(1,1)$	1451520	1	1	1	1	1	1
$(2,1)$	25920	56	-25	20	-7	2	2
$(3,2)$	11520	126	45	27	0	-9	0
$(4,1)$	2520	576	-72	0	27	-18	0
$(5,1)$	2160	672	24	-48	-21	-3	24
$(6,1)$	1920	756	27	0	0	27	-27
m_{1j}		1	56	126	576	672	756

$\mathcal{F}_2$

g_2		g_{21}	g_{22}
$o(g_{2j})$		2	6
$\lvert C_{\overline{G}}(g_{2j})\rvert$		69120	34560
(k,m)	$\lvert C_{H_k}(g_{2km})\rvert$		
$(1,1)$	23040	1	1
$(3,1)$	11520	2	-1
m_{2j}		729	1458

$\mathcal{F}_3$

g_3		g_{31}	g_{32}	g_{33}	g_{34}
$o(g_{3j})$		2	6	6	6
$\lvert C_{\overline{G}}(g_{2j})\rvert$		124416	20736	15552	10368
(k,m)	$\lvert C_{H_k}(g_{2km})\rvert$				
$(1,1)$	4608	1	1	1	1
$(2,1)$	576	8	-4	-1	2
$(3,1)$	768	6	3	-3	0
$(6,1)$	384	12	0	3	-3
m_{2j}		81	486	648	972

$\mathcal{F}_4$

g_4		$g_{4,1}$	$g_{4,2}$	$g_{4,3}$	$g_{4,4}$	$g_{4,5}$	$g_{4,6}$	$g_{4,7}$	$g_{4,8}$	$g_{4,9}$	$g_{4,10}$
$o(g_{4j})$		2	6	6	6	6	6	6	6	6	6
$\lvert C_{\overline{G}}(g_{4j})\rvert$		373248	186624	46656	23328	23328	15552	11664	11664	7776	5832
(k,m)	$\lvert C_{H_k}(g_{4km})\rvert$										
$(1,1)$	1536	1	1	1	1	1	1	1	1	1	1
$(2,1)$	96	16	-8	10	-8	-5	4	4	-2	-2	1
$(3,1)$	768	2	-1	2	2	-1	2	-1	2	-1	-1
$(3,2)$	64	24	24	6	6	6	-3	6	-3	-3	-3
$(4,1)$	24	64	-32	-8	-8	4	-8	4	10	4	-5
$(5,1)$	48	32	32	-4	-4	-4	-4	-4	5	-4	5
$(5,2)$	48	32	-16	-16	2	8	8	-1	-4	-4	2
$(6,1)$	192	8	8	5	-4	5	2	-4	-1	2	-1
$(6,2)$	32	48	-24	12	-15	12	3	-15	-6	3	-6
$(6,3)$	96	16	16	-17	19	-17	-5	19	7	-5	7
m_{4j}		9	18	72	144	144	216	288	288	432	576

$\mathcal{F}_5$

g_5		g_{51}	g_{52}	g_{53}	g_{54}
$o(g_{5j})$		2	6	6	6
$\lvert C_{\overline{G}}(g_{5j})\rvert$		10368	1728	1296	864
(k,m)	$\lvert C_{H_k}(g_{5km})\rvert$				
$(1,1)$	384	1	1	1	1
$(3,1)$	64	6	3	-3	0
$(5,1)$	48	8	-4	-1	2
$(6,1)$	32	12	0	3	-3
m_{5j}		81	486	648	972

$\mathcal{F}_6$

g_6		g_{61}	g_{62}	g_{63}	g_{64}	g_{65}	g_{66}	g_{67}
$o(g_{6j})$		3	3	3	3	9	9	9
$\lvert C_{\overline{G}}(g_{6j})\rvert$		524880	26244	17496	17496	43740	17496	4374
(k,m)	$\lvert C_{H_k}(g_{6km})\rvert$							
$(1,1)$	2160	1	1	1	1	1	1	1
$(2,1)$	108	20	-7	2	2	-10	8	-1
$(3,1)$	72	30	3	-6	3	15	6	-3
$(4,1)$	36	60	6	-12	6	-15	-6	3
$(5,1)$	1080	2	2	2	2	-1	-1	-1
$(5,2)$	54	40	-14	4	4	10	-8	1
$(6,1)$	24	90	9	9	-18	0	0	0
m_{6j}		9	180	270	270	108	270	1080

$\mathcal{F}_7$

g_7		g_{71}	g_{72}	g_{73}	g_{74}
$o(g_{7j})$		3	3	9	9
$\lvert C_{\overline{G}}(g_{7j})\rvert$		17496	2187	1944	1944
(k,m)	$\lvert C_{H_k}(g_{7km})\rvert$				
$(1,1)$	648	1	1	1	1
$(2,1)$	648	1	1	$-\frac{1}{2}+\frac{\sqrt{3}}{2}$	$-\frac{1}{2}-\frac{\sqrt{3}}{2}$
$(2,2)$	648	1	1	$-\frac{1}{2}-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}+\frac{\sqrt{3}}{2}$
$(5,1)$	27	24	-3	0	0
m_{7j}		81	648	729	729

$\mathcal{F}_8$

g_8		g_{81}	g_{82}	g_{83}	g_{84}	g_{85}	g_{86}
$o(g_{8j})$		3	3	9	9	9	9
$\lvert C_{\overline{G}}(g_{8j})\rvert$		2916	1458	1458	729	486	243
(k,m)	$\lvert C_{H_k}(g_{8km})\rvert$						
$(1,1)$	108	1	1	1	1	1	1
$(2,1)$	54	2	2	-1	-1	2	-1
$(3,1)$	18	6	-3	6	-3	0	0
$(4,1)$	9	12	-6	-6	3	0	0
$(5,1)$	27	4	4	-2	-2	-2	1
$(5,2)$	54	2	2	2	2	-1	-1
m_{8j}		81	162	162	324	486	972

$\mathcal{F}_9$

g_9		g_{91}	g_{92}	g_{93}	g_{94}
$o(g_{9j})$		4	12	12	12
$\lvert C_{\overline{G}}(g_{9j})\rvert$		10368	1728	1296	864
(k,m)	$\lvert C_{H_k}(g_{9km})\rvert$				
$(1,1)$	384	1	1	1	1
$(2,1)$	48	8	-4	-1	2
$(3,1)$	64	6	3	-3	0
$(6,1)$	32	12	0	3	-3
m_{9j}		81	486	648	972

$\mathcal{F}_{10}$

g_{10}		$g_{10,1}$	$g_{10,2}$	$g_{10,3}$	$g_{10,4}$
$o(g_{10j})$		4	12	12	12
$\lvert C_{\overline{G}}(g_{10j})\rvert$		5184	864	648	432
(k,m)	$\lvert C_{H_k}(g_{10km})\rvert$				
$(1,1)$	192	1	1	1	1
$(3,1)$	16	12	0	3	-3
$(5,1)$	24	8	-4	-1	2
$(6,1)$	32	6	3	-3	0
m_{10j}		81	486	648	972

$\mathcal{F}_{11}$

g_{11}		$g_{11,1}$	$g_{11,2}$
$o(g_{11j})$		4	12
$\lvert C_{\overline{G}}(g_{11,j})\rvert$		576	288
(k,m)	$\lvert C_{H_k}(g_{11km})\rvert$		
$(1,1)$	192	1	1
$(6,1)$	96	2	-1
m_{11j}		729	1458

$\mathcal{F}_{12}$

g_{12}		$g_{12,1}$	$g_{12,2}$
$o(g_{12j})$		4	12
$\lvert C_{\overline{G}}(g_{12,j})\rvert$		384	192
(k,m)	$\lvert C_{H_k}(g_{12km})\rvert$		
$(1,1)$	128	1	1
$(3,1)$	64	2	-1
m_{12j}		729	1458

$\mathcal{F}_{13}$

g_{13}		$g_{13,1}$	$g_{13,2}$	$g_{13,3}$	$g_{13,4}$	$g_{13,5}$	$g_{13,6}$	$g_{13,7}$	$g_{13,8}$
$o(g_{13j})$		4	12	12	12	12	12	12	12
$\lvert C_{\overline{G}}(g_{13j})\rvert$		864	432	432	432	216	216	216	108
(k,m)	$\lvert C_{H_k}(g_{13km})\rvert$								
$(1,1)$	32	1	1	1	1	1	1	1	1
$(2,1)$	8	4	4	-2	-2	-2	1	-2	1
$(3,1)$	16	2	2	-1	2	-1	-1	2	-1
$(3,2)$	16	2	-1	2	2	-1	2	-1	-1
$(4,1)$	4	8	-4	-4	-4	2	2	2	-1
$(5,1)$	8	4	-2	4	-2	-2	-2	1	1
$(6,1)$	16	2	2	2	-1	2	-1	-1	-1
$(6,2)$	8	4	-2	-2	4	1	-2	-2	1
m_{13j}		81	162	162	162	324	324	324	648

$\mathcal{F}_{14}$

g_{14}		$g_{14,1}$	$g_{14,2}$	$g_{14,3}$	$g_{14,4}$	$g_{14,5}$	$g_{14,6}$	$g_{14,7}$
$o(g_{14j})$		5	15	15	15	15	15	15
$\lvert C_{\overline{G}}(g_{14j})\rvert$		810	270	270	405	135	135	135
(k,m)	$\lvert C_{H_k}(g_{14km})\rvert$							
$(1,1)$	30	1	1	1	1	1	1	1
$(2,1)$	5	6	0	0	-3	0	3	-3
$(3,1)$	10	3	$-\frac{3}{2}-\frac{3\sqrt{3}}{2}$	$-\frac{3}{2}+\frac{3\sqrt{3}}{2}$	3	0	0	0
$(3,2)$	10	3	$-\frac{3}{2}+\frac{3\sqrt{3}}{2}$	$-\frac{3}{2}-\frac{3\sqrt{3}}{2}$	3	0	0	0
$(4,1)$	5	6	0	0	-3	3	-3	0
$(5,1)$	15	2	2	2	2	-1	-1	-1
$(6,1)$	5	6	0	0	-3	-3	0	3
m_{14j}		81	243	243	162	486	486	486

$\mathcal{F}_{15}$

g_{15}		$g_{15,1}$	$g_{15,2}$
$o(g_{15j})$		6	6
$\|C_{\overline{G}}(g_{15,j})\|$		432	216
(k,m)	$\|C_{H_k}(g_{15km})\|$		
$(1,1)$	144	1	1
$(3,1)$	72	2	-1
m_{15j}		729	1458

$\mathcal{F}_{16}$

g_{16}		$g_{16,1}$	$g_{16,2}$	$g_{16,3}$	$g_{16,4}$	$g_{16,5}$
$o(g_{16j})$		6	6	6	6	6
$\|C_{\overline{G}}(g_{16j})\|$		3888	972	972	648	324
(k,m)	$\|C_{H_k}(g_{16km})\|$					
$(1,1)$	144	1	1	1	1	1
$(2,1)$	36	4	$-\frac{1}{2}-\frac{3\sqrt{3}}{2}$	$-\frac{1}{2}+\frac{3\sqrt{3}}{2}$	-2	1
$(2,2)$	36	4	$-\frac{1}{2}+\frac{3\sqrt{3}}{2}$	$-\frac{1}{2}-\frac{3\sqrt{3}}{2}$	-2	1
$(3,1)$	24	6	-3	-3	3	0
$(6,1)$	12	12	3	3	0	-3
m_{16j}		81	324	324	486	972

$\mathcal{F}_{17}$

g_{17}		$g_{17,1}$	$g_{17,2}$	$g_{17,3}$
$o(g_{17j})$		6	18	18
$\|C_{\overline{G}}(g_{17j})\|$		216	216	216
(k,m)	$\|C_{H_k}(g_{17km})\|$			
$(1,1)$	72	1	1	1
$(2,1)$	72	1	$-\frac{1}{2}+\frac{\sqrt{3}}{2}$	$-\frac{1}{2}-\frac{\sqrt{3}}{2}$
$(2,2)$	72	1	$-\frac{1}{2}-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}+\frac{\sqrt{3}}{2}$
m_{17j}		729	729	729

$\mathcal{F}_{18}$

g_{18}		$g_{18,1}$	$g_{18,2}$	$g_{18,3}$	$g_{18,4}$	$g_{18,5}$	$g_{18,6}$	$g_{18,7}$	$g_{18,8}$
$o(g_{18j})$		6	6	6	6	18	18	18	18
$\|C_{\overline{G}}(g_{18j})\|$		1296	648	648	324	648	324	324	162
(k,m)	$\|C_{H_k}(g_{18km})\|$								
$(1,1)$	48	1	1	1	1	1	1	1	1
$(2,1)$	12	4	-2	-2	1	4	-2	-2	1
$(3,1)$	24	2	2	-1	-1	2	-1	2	-1
$(4,1)$	12	4	4	-2	-2	-2	1	-2	1
$(5,1)$	24	2	2	2	2	-1	-1	-1	-1
$(5,2)$	6	8	-4	-4	2	-4	2	2	-1
$(6,1)$	24	2	-1	2	-1	2	2	-1	-1
$(6,2)$	12	4	-2	4	-2	-2	-2	1	1
m_{18j}		81	162	162	324	162	324	324	648

$\mathcal{F}_{19}$

g_{19}		$g_{19,1}$	$g_{19,2}$
$o(g_{19j})$		6	6
$\|C_{\overline{G}}(g_{19,j})\|$		108	54
(k,m)	$\|C_{H_k}(g_{19km})\|$		
$(1,1)$	36	1	1
$(3,1)$	18	2	-1
m_{19j}		729	1458

$\mathcal{F}_{20}$

g_{20}		$g_{20,1}$	$g_{20,2}$
$o(g_{20j})$		6	18
$\|C_{\overline{G}}(g_{20,j})\|$		108	54
(k,m)	$\|C_{H_k}(g_{20km})\|$		
$(1,1)$	36	1	1
$(2,1)$	18	2	-1
m_{20j}		729	1458

$\mathcal{F}_{21}$

g_{21}		$g_{21,1}$	$g_{21,2}$
$o(g_{21j})$		6	18
$\|C_{\overline{G}}(g_{21,j})\|$		36	18
(k,m)	$\|C_{H_k}(g_{21km})\|$		
$(1,1)$	12	1	1
$(5,1)$	6	2	-1
m_{21j}		729	1458

$\mathcal{F}_{22}$

g_{22}		$g_{22,1}$	$g_{22,2}$	$g_{22,3}$
$o(g_{22j})$		7	21	21
$\|C_{\overline{G}}(g_{22j})\|$		21	21	21
(k,m)	$\|C_{H_k}(g_{22km})\|$			
$(1,1)$	7	1	1	1
$(4,1)$	7	1	$-\frac{1}{2}+\frac{\sqrt{3}}{2}$	$-\frac{1}{2}-\frac{\sqrt{3}}{2}$
$(4,2)$	7	1	$-\frac{1}{2}-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}+\frac{\sqrt{3}}{2}$
m_{22j}		729	729	729

$\mathcal{F}_{23}$

g_{23}		$g_{23,1}$	$g_{23,2}$
$o(g_{23j})$		8	24
$\|C_{\overline{G}}(g_{23,j})\|$		48	24
(k,m)	$\|C_{H_k}(g_{231km})\|$		
$(1,1)$	16	1	1
$(6,1)$	8	2	-1
m_{23j}		729	1458

$\mathcal{F}_{24}$

g_{24}		$g_{24,1}$	$g_{24,2}$	$g_{24,3}$
$o(g_{24j})$		8	24	24
$\|C_{\overline{G}}(g_{24j})\|$		48	48	48
(k,m)	$\|C_{H_k}(g_{24km})\|$			
$(1,1)$	16	1	1	1
$(3,1)$	16	1	$-\frac{1}{2}+\frac{\sqrt{3}}{2}$	$-\frac{1}{2}-\frac{\sqrt{3}}{2}$
$(3,2)$	16	1	$-\frac{1}{2}-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}+\frac{\sqrt{3}}{2}$
m_{24j}		729	729	729

$\mathcal{F}_{25}$

g_{25}		$g_{25,1}$	$g_{25,2}$	$g_{25,3}$
$o(g_{25j})$		9	9	9
$\|C_{\overline{G}}(g_{25j})\|$		27	27	27
(k,m)	$\|C_{H_k}(g_{25km})\|$			
$(1,1)$	9	1	1	1
$(2,1)$	9	1	$-\frac{1}{2}+\frac{\sqrt{3}}{2}$	$-\frac{1}{2}-\frac{\sqrt{3}}{2}$
$(2,2)$	9	1	$-\frac{1}{2}-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}+\frac{\sqrt{3}}{2}$
m_{25j}		729	729	729

$\mathcal{F}_{26}$

g_{26}		$g_{26,1}$	$g_{26,2}$	$g_{26,3}$
$o(g_{26j})$		10	30	30
$\|C_{\overline{G}}(g_{26j})\|$		30	30	30
(k,m)	$\|C_{H_k}(g_{26km})\|$			
$(1,1)$	10	1	1	1
$(3,1)$	10	1	$-\frac{1}{2}+\frac{\sqrt{3}}{2}$	$-\frac{1}{2}-\frac{\sqrt{3}}{2}$
$(3,2)$	10	1	$-\frac{1}{2}-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}+\frac{\sqrt{3}}{2}$
m_{26j}		729	729	729

$\mathcal{F}_{27}$

g_{27}		$g_{27,1}$	$g_{27,2}$
$o(g_{27j})$		12	36
$\|C_{\overline{G}}(g_{27,j})\|$		72	36
(k,m)	$\|C_{H_k}(g_{27km})\|$		
$(1,1)$	24	1	1
$(5,1)$	12	2	-1
m_{27j}		729	1458

$\mathcal{F}_{28}$

g_{28}		$g_{28,1}$	$g_{28,2}$
$o(g_{28j})$		12	12
$\|C_{\overline{G}}(g_{28,j})\|$		72	36
(k,m)	$\|C_{H_k}(g_{28km})\|$		
$(1,1)$	24	1	1
$(6,1)$	12	2	-1
m_{28j}		729	1458

$\mathcal{F}_{29}$

g_{29}		$g_{29,1}$	$g_{29,2}$	$g_{29,3}$
$o(g_{29j})$		12	36	36
$\|C_{\overline{G}}(g_{29j})\|$		36	36	36
(k,m)	$\|C_{H_k}(g_{29km})\|$			
$(1,1)$	12	1	1	1
$(2,1)$	12	1	$-\frac{1}{2}+\frac{\sqrt{3}}{2}$	$-\frac{1}{2}-\frac{\sqrt{3}}{2}$
$(2,2)$	12	1	$-\frac{1}{2}-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}+\frac{\sqrt{3}}{2}$
m_{29j}		729	729	729

$\mathcal{F}_{30}$

g_{30}		$g_{30,1}$	$g_{30,2}$	$g_{30,3}$
$o(g_{30j})$		15	45	45
$\|C_{\overline{G}}(g_{30j})\|$		45	45	45
(k,m)	$\|C_{H_k}(g_{30km})\|$			
$(1,1)$	15	1	1	1
$(5,1)$	15	1	$-\frac{1}{2}+\frac{\sqrt{3}}{2}$	$-\frac{1}{2}-\frac{\sqrt{3}}{2}$
$(5,2)$	15	1	$-\frac{1}{2}-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}+\frac{\sqrt{3}}{2}$
m_{30j}		729	729	729

6. Character Table of $\overline{G}$

Now we have

- the conjugacy classes of $\overline{G} = 3^7{:}Sp(6,2)$ (Table 1),
- the character tables of all the inertia factors (Tables 11.1, 11.2, 11.3, 11.4 and 11.5 of [1]),
- the fusions of classes of the inertia factors into classes of $Sp(6,2)$ (Table 5),
- the Fischer matrices of $\overline{G}$ (see Section 5).

By Section 3 of [2], it follows that the full character table of $\overline{G}$ can be constructed easily. The character table of $\overline{G}$ is a 118×118 $\mathbb{C}-$valued matrix. The full character table of $\overline{G}$ is available in the PhD thesis [1] of the first author, which could be accessed online. This character table is not yet incorporated into the GAP library but our aim is to do so.

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