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CHARACTERIZATION OF SOME SIMPLE K_4-GROUPS BY SOME IRREDUCIBLE COMPLEX CHARACTER DEGREES

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Abstract. In this paper, we examine that some finite simple K_4-groups can be determined uniquely by their orders and one or two irreducible complex character degrees.

1. Introduction

Throughout this paper, let G be a finite group and let all characters be complex characters. Also, let $l(G)$ be the largest irreducible character degree of G, $s(G)$ be the second largest irreducible character degree of G and $t(G)$ be the third largest irreducible character degree of G. The set of all irreducible characters of G is shown by $\mathrm{Irr}(G)$ and the set of all irreducible character degrees of G is shown by $\mathrm{cd}(G)$. In [4], B. Huppert conjectured that if G is a finite group and S is a finite non-abelian simple group such that $\mathrm{cd}(G) = \mathrm{cd}(S)$, then $G \cong S \times A$, where A is an abelian group. In [7], [11] and [12], it is shown that $L_2(p)$, simple K_3-groups and Mathieu simple groups are determined uniquely by their orders and one or two irreducible character degrees. In [6], it is proved that if $2^a + 1$ or $2^a - 1$ is prime, then $L_2(2^a)$ is determined uniquely by its order and the largest irreducible character degree. Also, in [3], the finite groups with the same order and the same largest and second largest irreducible character degrees as $PGL(2, p)$ have been determined. In this paper, we are going to prove that:

Main Theorem. Let G be a finite group.
(I) Let M be A_7, $L_2(25)$, $L_2(81)$ or $Sz(8)$. Then $G \cong M$ if and only if $|G| = |M|$ and $l(G) = l(M)$.
(II) Let M be $U_3(7)$, $Sz(32)$, $U_3(8)$ or $U_3(9)$. Then $G \cong M$ if and only if $|G| = |M|$ and $s(G) = s(M)$.

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(III) Let M be $S_6(2)$, $L_3(8)$, $L_3(5)$, $O_8^+(2)$, $L_3(7)$, $L_2(49)$, $G_2(3), U_3(5)$ or $S_4(7)$. Then $G \cong M$ if and only if $|G| = |M|$, $l(G) = l(M)$ and $t(G) = t(M)$.
(IV) Let M be $L_3(4)$, $U_4(3)$, ${}^2F_4(2)'$ or $U_5(2)$. Then $G \cong M$ if and only if $|G| = |M|$, $l(G) = l(M)$ and $s(G) = s(M)$.
(V) Let M be $U_3(4)$, ${}^3D_4(2)$, $S_4(4)$ or $S_4(5)$. Then $G \cong M$ if and only if $|G| = |M|$, $s(G) = s(M)$ and $t(G) = t(M)$.
(VI) Let $L_2(3^m)$ be a simple K_4-group as Lemma 1.6(ii)(4), which appears below. Then $G \cong L_2(3^m)$ if and only if $|G| = |L_2(3^m)|$, $l(G) = l(L_2(3^m))$ and $s(G) = s(L_2(3^m))$.

Note that by [7] and [6], the finite simple K_4-groups $L_2(p)$ and $L_2(2^a)$ are determined uniquely by their orders and one or two irreducible character degrees. Also, the statements in the main theorem do not hold for the remaining finite simple K_4-groups that are not covered in the main theorem. So they have been studied in the separate work.

Throughout this paper, we use the following notations: For two natural numbers b and n a natural number n, $\pi(n)$ is the set of all prime divisors of n. The set of all prime divisors of $\pi(|G|)$ is denoted by $\pi(G)$ and also, if $|\pi(G)| = n$, then we say that G is a K_n-group. For two natural numbers b and n and a prime a, we write $|b|_a = a^n$ when $a^n \parallel b$ i.e., $a^n \mid b$ while $a^{n+1} \nmid b$. For $p \in \pi(G)$, the set of all p-Sylow subgroups of G is denoted by $\mathrm{Syl}_p(G)$ and we write $n_p = |\mathrm{Syl}_p(G)|$. For the subgroup H of G, we set $H_G = \cap_{g \in G} H^g$. If H is a characteristic subgroup of G, then we write H ch G. If $\chi = \sum_{i=1}^{N} n_i \chi_i$, where for every $1 \leq i \leq N$, $\chi_i \in \mathrm{Irr}(G)$, then those χ_i with $n_i > 0$ are called irreducible constituents of χ.

In the following, we bring some lemmas, which are used in the proof of the main theorem. Also, we mention that throughout this paper, for the orders, character degrees, the orders of the outer automorphism groups of finite simple groups, we refer the reader to [1].

Lemma 1.1. (Ito's theorem) [5, Theorem 6.15] *Let $A \trianglelefteq G$ be abelian. Then $\chi(1) \mid [G : A]$, for all $\chi \in \mathrm{Irr}(G)$.*

Lemma 1.2. (Clifford's theorem)[5, Theorem 6.2 and Corollary 11.29] *Let $N \trianglelefteq G$ and $\chi \in \mathrm{Irr}(G)$. Let θ be an irreducible constituent of χ_N and suppose that $\theta_1 = \theta, \ldots, \theta_t$ are the distinct conjugates of θ in G. Then $\chi_N = e \sum_{i=1}^{t} \theta_i$, where $e = [\chi_N, \theta]$. Also, $\chi(1)/\theta(1) \mid [G : N]$.*

Lemma 1.3. [12] *Let G be a non-solvable group. Then G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K/H is a direct product of isomorphic non-abelian simple groups and $|G/K| \mid |\mathrm{Out}(K/H)|$.*

Lemma 1.4. [12] *Let G be a finite solvable group of order $\prod_{i=1}^{n} {p_i}^{a_i}$, where $p_1, p_2, \ldots, p_n$ are distinct primes. If $kp_n + 1 \nmid {p_i}^{a_i}$ for each $i \leq n-1$ and $k > 0$, then the p_n-Sylow subgroup of G is normal in G.*

Lemma 1.5. *Let G be a finite group and for $t \in \pi(G)$, let N be a t-elementary abelian subgroup of G. If $|C_G(N)|_t = |N|$, then $C_G(N) = N \times C$, where C is a $(\pi(C_G(N)) - \{t\})$-Hall subgroup of $C_G(N)$.*

Proof. Since $N \leq Z(C_G(N))$ and $N \in \mathrm{Syl}_t(C_G(N))$, lemma follows. □

Lemma 1.6. (i) [2] *If G is a simple K_3-group, then G is isomorphic to one of the following groups: A_5, A_6, $L_2(7)$, $L_2(8)$, $L_2(17)$, $L_3(3)$, $U_3(3)$ or $U_4(2)$.*

(ii) [8, 10] *Let G be a simple K_4-group. Then G is isomorphic to one of the following groups:*

(1) A_7, A_8, A_9, A_{10}, M_{11}, M_{12}, J_2, $L_2(16)$, $L_2(25)$, $L_2(49)$, $L_2(81)$, $L_3(4)$, $L_3(5)$, $L_3(7)$, $L_3(8)$, $L_3(17)$, $L_4(3)$, $S_4(4)$, $S_4(5)$, $S_4(7)$, $S_4(9)$, $S_6(2)$, $O_8^+(2)$, $G_2(3)$, $U_3(4)$, $U_3(5)$, $U_3(7)$, $U_3(8)$, $U_3(9)$, $U_4(3)$, $U_5(2)$, $Sz(8)$, $Sz(32)$, ${}^3D_4(2)$, ${}^2F_4(2)'$;

(2) $L_2(r)$, *where r is a prime,* $17 \neq r \geq 11$, $r^2-1 = 2^a3^bv^c$, $v > 3$ *is a prime,* $a, b \in \mathbb{N}$, *and c is either* 1 *or* 2 *for* $r \in \{97, 577\}$;

(3) $L_2(2^m)$, *where* $m \geq 5$, $u = 2^m - 1$, *and* $t = (2^m+1)/3$ *are primes;*

(4) $L_2(3^m)$, *where m and* $u = (3^m+1)/4$ *are odd primes and* $(3^m-1)/2$ *is either a prime or* 11^2 *(for* $m = 5$*).*

Lemma 1.7. *For $n \in \{3,4\}$, let G be a finite K_n-group. If there is not any finite simple group L in Lemma 1.6 such that $\pi(L) \subseteq \pi(G)$, then G is solvable.*

Proof. It follows immediately from Lemmas 1.3 and 1.6. □

Lemma 1.8. *If G is one of the groups in the main theorem, then G is non-solvable.*

Proof. On the contrary, let G be solvable. We are going to complete the proof in the following cases:
Case a. Let M be one of the groups mentioned in the main theorem (I) or (II).
Note that $l(A_7) = 5 \cdot 7$, $l(L_2(25)) = 2 \cdot 13$, $l(L_2(81)) = 2 \cdot 41$, $l(Sz(8)) = 7 \cdot 13$, $s(U_3(7)) = 2^3 \cdot 43$, $s(Sz(32)) = 5^2 \cdot 41$, $s(U_3(8)) = 3^3 \cdot 19$ and $s(U_3(9)) = 2 \cdot 5 \cdot 73$. Let $\chi \in \mathrm{Irr}(G)$ such that for $M = A_7$, $L_2(25)$, $L_2(81)$ and $Sz(8)$, $\chi(1) = l(G) = l(M)$ and for $M = U_3(7)$, $Sz(32)$, $U_3(8)$, $U_3(9)$, $\chi(1) = s(G) = s(M)$. For $M = A_7$, set $p = 5$, for $M = L_2(25)$ and $M = Sz(8)$, set $p = 13$, for $M = L_2(81)$, set $p = 41$, for $M = U_3(7)$, set $p = 43$, for $M = Sz(32)$, set $p = 41$, for $M = U_3(8)$, set $p = 19$ and for $M = U_3(9)$, set $p = 73$. Let $P \in \mathrm{Syl}_p(G)$. Then since $|M| = |G|$, P is a cyclic group of order p. Since G is solvable and, for every $t \in \pi(M)$ and every natural number k, $pk+1 \nmid |M|_t = |G|_t$, Lemma 1.4 shows that the p-Sylow subgroup P of G is normal in G. Thus Ito's theorem implies that $\chi(1) \mid [G : P]$, which is impossible. So, G is non-solvable.
Case b. Let M be one of the groups mentioned in the main theorem (III), (IV) or (V).
Note that $|S_6(2)| = 2^9 \cdot 3^4 \cdot 5 \cdot 7$, $|L_3(5)| = 2^5 \cdot 3 \cdot 5^3 \cdot 31$, $|O_8^+(2)| = 2^{12} \cdot 3^5 \cdot 5^2 \cdot 7$, $|L_2(49)| = 2^4 \cdot 3 \cdot 5^2 \cdot 7^2$, $|G_2(3)| = 2^6 \cdot 3^6 \cdot 7 \cdot 13$, $|U_3(5)| = 2^4 \cdot 3^2 \cdot 5^3 \cdot 7$, $|L_3(7)| = 2^5 \cdot 3^2 \cdot 7^3 \cdot 19$, $|S_4(7)| = 2^8 \cdot 3^2 \cdot 5^2 \cdot 7^4$, $|L_3(8)| = 2^9 \cdot 3^2 \cdot 7^2 \cdot 73$, $|L_3(4)| = 2^6 \cdot 3^2 \cdot 5 \cdot 7$, $|U_5(2)| = 2^{10} \cdot 3^5 \cdot 5 \cdot 11$, $|U_4(3)| = 2^7 \cdot 3^6 \cdot 5 \cdot 7$, $|{}^2F_4(2)'| = 2^{11} \cdot 3^3 \cdot 5^2 \cdot 13$, $|S_4(4)| = 2^8 \cdot 3^2 \cdot 5^2 \cdot 17$, $|U_3(4)| = 2^6 \cdot 3 \cdot 5^2 \cdot 13$, $|{}^3D_4(2)| = 2^{12} \cdot 3^4 \cdot 7^2 \cdot 13$, $|S_4(5)| = 2^6 \cdot 3^2 \cdot 5^4 \cdot 13$, $l(S_6(2)) = 2^9$, $t(S_6(2)) = 3^4 \cdot 5$, $l(L_3(5)) = 2 \cdot 3 \cdot 31$, $t(L_3(5)) = 5^3$, $l(O_8^+(2)) = 3^5 \cdot 5^2$, $t(O_8^+(2)) = 2^{12}$, $l(L_3(7)) = 2^3 \cdot 3 \cdot 19$, $t(L_3(7)) = 7^3$, $l(L_2(49)) = 2 \cdot 5^2$, $t(L_2(49)) = 2^4 \cdot 3$, $l(S_4(7)) = 2^7 \cdot 5^2$, $t(S_4(7)) = 7^4$ (see [9]), $l(G_2(3)) = 2^6 \cdot 13$, $t(G_2(3)) = 3^6$, $l(U_3(5)) = 2^4 \cdot 3^2$, $t(U_3(5)) = 5^3$, $l(L_3(8)) = 3^2 \cdot 73$, $t(L_3(8)) = 2^9$, $l({}^2F_4(2)') = 2^{11}$, $s({}^2F_4(2)') = 2^6 \cdot 3^3$, $l(L_3(4)) = 2^6$,

$s(L_3(4)) = 3^2 \cdot 7$, $l(U_4(3)) = 2^7 \cdot 7$, $s(U_4(3)) = 3^6$, $l(U_5(2)) = 3^5 \cdot 5$, $s(U_5(2)) = 2^{10}$, $s({}^3D_4(2)) = 2^{12}$, $t({}^3D_4(2)) = 3^4 \cdot 7^2$, $s(U_3(4)) = 13 \cdot 5$, $t(U_3(4)) = 2^6$, $s(S_4(4)) = 2^8$, $t(S_4(4)) = 3 \cdot 5 \cdot 17$, $s(S_4(5)) = 5^4$ and $t(S_4(5)) = 2^4 \cdot 3 \cdot 13$.

Let $\chi, \beta \in \mathrm{Irr}(G)$ such that for $M = S_6(2), L_3(8), L_3(5), O_8^+(2), L_3(7), L_2(49), G_2(3), U_3(5)$ and $S_4(7)$, $\chi(1) = l(G) = l(M)$ and $\beta(1) = t(G) = t(M)$, for $M = L_3(4), U_4(3), {}^2F_4(2)'$ and $U_5(2)$, $\chi(1) = l(G) = l(M)$ and $\beta(1) = s(G) = s(M)$ and for $M = U_3(4), {}^3D_4(2)$, $S_4(4)$ and $S_4(5)$, $\chi(1) = s(G) = s(M)$ and $\beta(1) = t(G) = t(M)$. First, let $M \notin \{L_3(8), S_4(4), U_3(4)\}$. Let N be a normal minimal subgroup of G. Then for some $t \in \pi(G)$, N is a t-elementary abelian subgroup of G. Thus applying Ito's theorem to N, χ and β shows that if $M = S_6(2)$, $O_8^+(2)$, $G_2(3)$ or $U_3(5)$, then $t = 7 = |N|$, if $M = L_3(5)$, then $t = 2$ and $|N| \leq 2^4$, if $M = L_2(49)$, then $t = 7$ and $|N| \mid 7^2$, if $M = L_3(4)$ or $U_4(3)$, then $|N| = t = 5$, if $M = U_5(2)$, then $|N| = t = 11$, if $M = {}^3D_4(2)$, then $t = 13 = |N|$, if $M = {}^2F_4(2)'$, then $t \in \{5, 13\}$ and if $M \in \{L_3(7), S_4(5), S_4(7)\}$, then $t \in \{2, 3\}$. Let $M = S_6(2)$, $O_8^+(2), G_2(3), U_3(5), L_3(4), U_4(3), U_5(2)$ or ${}^3D_4(2)$. Then since $G/C_G(N) \hookrightarrow \mathrm{Aut}(N) \cong Z_6, Z_4, Z_{10}$ or Z_{12}, we deduce that $N < C_G(N)$. But $t \| |G|$ and hence by Lemma 1.5, $C_G(N) = N \times C$, where C is a $(\pi(C_G(N)) - \{t\})$-Hall subgroup of $C_G(N)$, which is a normal subgroup of G. Therefore, there exists a normal minimal subgroup N_1 of G such that $N_1 \leq C$ and hence, $|N_1| \mid |G|/t = |M|/t$, which is a contradiction with the fact that every normal minimal subgroup of G is a t-group. Now, let $M = L_3(5)$, $t = 2$ and $1 \neq |N| \leq 2^4$. Then applying Lemma 1.4 to G/N shows that for $P \in \mathrm{Syl}_{31}(G)$, $PN/N \trianglelefteq G/N$ and hence, $PN \trianglelefteq G$. But $|PN| \mid 2^4 \cdot 31$ and hence, applying Lemma 1.4 to PN shows that P ch PN. Thus P is a normal minimal subgroup of G, which is a contradiction with the fact that every normal minimal subgroup of G is a 2-group. Let $M = L_3(7)$ and let L_1 be a maximal normal 2-subgroup of G and L_2 be a maximal normal 3-subgroup of G. Set $L = L_1 \times L_2$ and let $\theta \in \mathrm{Irr}(L_1)$ such that $[\chi_{L_1}, \theta] \neq 0$. Then by Lemma 1.2, $|\chi(1)|_2/[G : L_1]_2 \mid \theta(1)$. Also, $\theta^2(1) + 1 \leq |L_1|$. Thus we can check at once that either $|L_1| \leq 4$ or $|L_1| = 8$ and $\theta(1) = 2$, and hence, in the latter case $L_1 = D_8$ or Q_8. The same argument guarantees that $|L_2| \leq 3$. Thus $G/C_G(L) \hookrightarrow \mathrm{Aut}(L_1) \times \mathrm{Aut}(L_2)$, where $\mathrm{Aut}(L_1) \in \{D_8, S_4, GL_2(2), Z_2, \{1\}\}$ and $\mathrm{Aut}(L_2) \in \{Z_2, \{1\}\}$. This forces $7^3 \cdot 19 \mid |C_G(L)|$. If $|L_1| = 8$, then $|C_G(L)L_1/L|_2 \leq 2^2$ and hence, PL/L ch $C_G(L)L_1/L \trianglelefteq G/L$, where $P \in \mathrm{Syl}_7(C_G(L))$. Thus $P \times L \trianglelefteq G$ and hence, $P \trianglelefteq G$, which is a contradiction with the above statements. Now, let $|L_1| \leq 4$. Then L is abelian and hence, $L \leq C_G(L)$. Also, by the above statements, $|C_G(L)/L| \neq 1$. Now, let S/L be a normal minimal subgroup of G/L such that $S/L \leq C_G(L)/L$. Then for some $s \in \pi(C_G(L))$, S/L is s-elementary abelian and hence, there exists a s-subgroup S_1 of G such that $S = S_1 \times L \trianglelefteq G$. Thus $S_1 \trianglelefteq G$. But by assumptions on L, $s \notin \{2, 3\}$, which is a contradiction with the fact that every normal minimal subgroup of G is a 2-group or a 3-group. Let $M = L_2(49)$ and let L be a maximal normal 7-subgroup of G. Then $|L| \mid 7^2$. On the other hand, $G/C_G(L) \hookrightarrow \mathrm{Aut}(L)$ and hence, $|G/C_G(L)| \mid 1, 6, |GL_2(7)|$ or $6 \cdot 7$. Thus $L < C_G(L)$. Now, applying the argument given for $M = L_3(7)$ leads us to get a contradiction. Also, if $M \in \{S_4(7), {}^2F_4(2)', S_4(5)\}$, then repeating the argument given for $L_3(7)$ leads us to get a contradiction. Let $M = L_3(8)$. Suppose that H is a Hall-subgroup of G of order $2^9 \cdot 3^2 \cdot 73$. Then $G/H_G \leq S_{49}$ and hence, $73 \in \pi(H_G)$. So, H_G is a solvable group such that $|H_G| \mid 2^9 \cdot 3^2 \cdot 73$. Let N be a normal minimal subgroup of G such that $N \leq H_G$. Then for some $t \in \pi(H_G) \subseteq \{2, 3, 73\}$,

N is a t-elementary abelian subgroup of G. On the other hand, by Ito's theorem, $\chi(1) = 3^2 \cdot 73$ and $\beta(1) = 2^9$ divide $|G|/|N|$, which is impossible. Now, let $M = S_4(4)$ and let H be a $(|M|/9)$-Hall subgroup of G. Then $G/H_G \hookrightarrow S_9$ and hence, $5 \cdot 17 \mid |H_G|$. Let L be a normal minimal subgroup of G such that $L \leq H_G$. Then Ito's theorem shows that L is a 5-elementary abelian group. Note that L is a maximal normal 5-subgroup of G and $|L| = 5$. Also, $H_G/C_{H_G}(L) \hookrightarrow \mathrm{Aut}(L)$ and hence, we can see at once that $5 \cdot 17 \mid |C_{H_G}(L)|$. If E/L is a normal minimal subgroup of $C_{H_G}(L)/L$, then for some $e \in \pi(C_{H_G}(L))$, E/L is e-elementary abelian and hence, there exists an e-subgroup E_1 of G such that $E = E_1 \times L \trianglelefteq G$. Thus $E_1 \trianglelefteq G$. But by assumption on L, $e \neq 5$, which is a contradiction with the fact that every normal minimal subgroup of G which is a subgroup of H_G is a 5-group. Finally let $M = U_3(4)$. Then since for every natural number k, $13k + 1 \nmid 2^6, 3$ or 5^2, Lemma 1.4 implies that $P \trianglelefteq G$, where $P \in \mathrm{Syl}_{13}(G)$. On the other hand, P is abelian and hence, Ito's theorem shows that $\chi(1) = 13 \cdot 5 \mid [G : P]$, which is impossible.

Case c. Let $M = L_2(3^m)$ be a simple K_4-group as Lemma 1.6(ii)(4).

Let $\chi, \beta \in \mathrm{Irr}(G)$ such that $\chi(1) = l(G) = l(L_2(3^m)) = 3^m + 1$ and $\beta(1) = s(G) = s(L_2(3^m)) = 3^m$. Note that by Lemma 1.6(ii)(4), m and $u = (3^m + 1)/4$ are odd primes and $(3^m - 1)/2$ is either a prime or 11^2 (for $m = 5$). Thus $\pi((3^m - 1)/2) = \{t\}$. Also, $|G| = |L_2(3^m)| = (3^m \cdot (3^m - 1)(3^m + 1))/2$. Let $P \in \mathrm{Syl}_u(G)$. Then since for every natural number k, $ku + 1 \nmid |G|_2, |G|_3$ or $|G|_t$, we conclude by Lemma 1.4 that $P \trianglelefteq G$. Moreover, $|P| = u$. Thus P is abelian and hence, Ito's theorem shows that $\chi(1) = 3^m + 1 = 4u \mid [G : P]$, which is impossible.

These contradictions show that G is non-solvable. □

2. Proof of the Main Theorem

We only need to prove the sufficiency of the main theorem. Note that since $|G| = |M|$ and M is a K_4-group, G is a K_4-group.

Proof of (I). Let $\chi \in \mathrm{Irr}(G)$ such that $\chi(1) = l(G) = l(M)$. Then Lemma 1.8 shows that G is non-solvable. Therefore by Lemma 1.3, there exists a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K/H is a direct product of isomorphic non-abelian simple K_3 or K_4-groups and $|G/K| \mid |\mathrm{Out}(K/H)|$. Now, by comparing the order of G and the orders of the non-abelian simple groups mentioned in Lemma 1.6, we can see that K/H is a simple group, which is isomorphic to one of the following groups:

i. Let $M = A_7$. Then $|G| = |A_7| = 2^3 \cdot 3^2 \cdot 5 \cdot 7$, $\chi(1) = l(G) = l(A_7) = 5 \cdot 7$ and K/H is isomorphic to $A_5, A_6, L_2(7), L_2(8)$ or A_7. If $K/H \cong A_5$, then $|G/K||H| = 2 \cdot 3 \cdot 7$ and $|G/K| \mid |\mathrm{Out}(A_5)| = 2$. Thus $|H| = 2^t \cdot 3 \cdot 7$, where $0 \leq t \leq 1$. Let $P \in \mathrm{Syl}_7(H)$. Then we can see that P ch $H \trianglelefteq G$. Thus P is a normal abelian subgroup of G and hence, Ito's theorem shows that $\chi(1) = 5 \cdot 7 \mid [G : P]$, which is impossible. If $K/H \cong A_6$, then $|G/K||H| = 7$ and $|G/K| \mid |\mathrm{Out}(A_6)| = 4$. Thus $|H| = 7$ and hence, Ito's theorem implies that $\chi(1) = 5 \cdot 7 \mid [G : H]$, which is impossible. If $K/H \cong L_2(7)$ or $L_2(8)$, then since $|\mathrm{Out}(L_2(7))| = 2$ and $|\mathrm{Out}(L_2(8))| = 3$, Lemma 1.3 implies that $|H| = 3 \cdot 5$ or 5. Let $P \in \mathrm{Syl}_5(H)$. Then we can see that P ch $H \trianglelefteq G$. Thus P is a normal abelian subgroup of G and hence, Ito's theorem shows that $\chi(1) = 5 \cdot 7 \mid [G : P]$, which is impossible. If $K/H \cong A_7$, then $|G/K| = 1 = |H|$ and hence, $G = K \cong A_7$, as desired.

ii. Let $M = L_2(25)$. Then $|G| = |L_2(25)| = 2^3 \cdot 3 \cdot 5^2 \cdot 13$, $\chi(1) = l(G) = l(L_2(25)) = 2 \cdot 13$ and $K/H \cong A_5$ or $L_2(25)$. If $K/H \cong A_5$, then $|H||G/K| = 2 \cdot 5 \cdot 13$ and $|G/K| \mid |\mathrm{Out}(A_5)| = 2$. Thus $|H| = 2^t \cdot 5 \cdot 13$, where $0 \leq t \leq 1$ and hence, we can see at once that for $P \in \mathrm{Syl}_{13}(H)$, P ch $H \trianglelefteq G$. Moreover, P is abelian and hence, Ito's theorem shows that $\chi(1) = 2 \cdot 13 \mid [G : P]$, which is impossible. Thus $K/H \cong L_2(25)$ and hence, $G \cong L_2(25)$.
iii. Let $M = L_2(81)$. Then $|G| = |L_2(81)| = 2^4 \cdot 3^4 \cdot 5 \cdot 41$, $\chi(1) = l(G) = l(L_2(81)) = 2 \cdot 41$ and $K/H \cong A_5, A_6$ or $L_2(81)$. Suppose that $K/H \cong A_5$. Then $|H||G/K| = 2^2 \cdot 3^3 \cdot 41$ and $|G/K| \mid |\mathrm{Out}(A_5)| = 2$. Thus $|H| = 2^t \cdot 3^3 \cdot 41$, where $1 \leq t \leq 2$. Therefore we can see at once that a $P \in \mathrm{Syl}_{41}(H)$ is normal in H. Thus P ch $H \trianglelefteq G$. Moreover, P is abelian. So, Ito's theorem implies that $\chi(1) = 2 \cdot 41 \mid [G : P]$, which is impossible. Let $K/H \cong A_6$. Then $|H||G/K| = 2 \cdot 3^2 \cdot 41$ and $|G/K| \mid |\mathrm{Out}(A_6)| = 4$. Thus $|H| = 2 \cdot 3^2 \cdot 41$ or $3^2 \cdot 41$ and hence, a 41-Sylow subgroup P of H is normal in H. Therefore repeating the argument used in the case when $K/H \cong A_5$ leads us to get a contradiction. Thus $K/H \cong L_2(81)$ and hence, $G = K \cong L_2(81)$.
iv. Let $M = Sz(8)$. Then $|G| = |Sz(8)| = 2^6 \cdot 5 \cdot 7 \cdot 13$, $\chi(1) = l(G) = l(Sz(8)) = 7 \cdot 13$ and $K/H \cong Sz(8)$. It follows that $G = K \cong Sz(8)$, as desired.

Proof of (II). Let $\chi \in \mathrm{Irr}(G)$ such that $\chi(1) = s(G) = s(M)$. Then Lemma 1.8 shows that G is non-solvable and so, by Lemma 1.3, there exists a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K/H is a direct product of isomorphic non-abelian simple K_3 or K_4-groups and $|G/K| \mid |\mathrm{Out}(K/H)|$. Now by comparing the order of G and the orders of the non-abelian simple groups mentioned in Lemma 1.6, we can see that K/H is isomorphic to one of the following groups:
i. Let $M = U_3(7)$. Then $|G| = |U_3(7)| = 2^7 \cdot 3 \cdot 7^3 \cdot 43$, $\chi(1) = s(G) = s(U_3(7)) = 2^3 \cdot 43$ and $K/H \cong L_2(7)$ or $U_3(7)$. First, suppose that $K/H \cong L_2(7)$. Then $|G/K| \mid |\mathrm{Out}(L_2(7))| = 2$ and so, for some $3 \leq t \leq 4$, $|H| = 2^t \cdot 7^2 \cdot 43$. Hence, we can see at once that a 43-Sylow subgroup P of H is normal in H. Thus P ch $H \trianglelefteq G$ and since P is abelian, Ito's theorem implies that $\chi(1) = 2^3 \cdot 43 \mid [G : P]$, which is impossible. Therefore $K/H \cong U_3(7)$. Thus $H = \{1\} = G/K$ and hence, $G = K \cong U_3(7)$.
ii. Let $M = Sz(32)$. Then $|G| = |Sz(32)| = 2^{10} \cdot 5^2 \cdot 31 \cdot 41$ and $\chi(1) = s(G) = s(Sz(32)) = 5^2 \cdot 41$. Also, $Sz(32)$ is the only non-abelian simple group mentioned in Lemma 1.6 such that the set of prime divisors of its order is a subset of $\pi(G)$. It follows that $K/H \cong Sz(32)$ and hence, $G = K \cong Sz(32)$, as claimed.
iii. Let $M = U_3(8)$. Then $|G| = |U_3(8)| = 2^9 \cdot 3^4 \cdot 7 \cdot 19$, $\chi(1) = s(G) = s(U_3(8)) = 3^3 \cdot 19$ and $K/H \cong L_2(7), L_2(8), U_3(3)$ or $U_3(8)$. Let $K/H \cong L_2(7)$. Then $|G/K||H| = 2^6 \cdot 3^3 \cdot 19$ and $|G/K| \mid |\mathrm{Out}(L_2(7))| = 2$. Thus $|H| = 2^6 \cdot 3^3 \cdot 19$ or $2^5 \cdot 3^3 \cdot 19$. Hence, we can see at once by Lemma 1.7 that H is solvable and a $P \in \mathrm{Syl}_{19}(H)$ is normal in H. Therefore, P ch $H \trianglelefteq G$. Moreover, since $|P| = 19$, P is abelian. Now, Ito's theorem leads us to get a contradiction. Applying the same argument rules out the cases when $K/H \cong L_2(8)$ or $U_3(3)$. Thus $K/H \cong U_3(8)$, which implies that $G = K \cong U_3(8)$.
iv. Let $M = U_3(9)$. Then $|G| = |U_3(9)| = 2^5 \cdot 3^6 \cdot 5^2 \cdot 73$, $\chi(1) = s(G) = s(U_3(9)) = 2\dot{5} \cdot 73$ and $K/H \cong A_5, A_5 \times A_5, A_6$ or $U_3(9)$. If $K/H \cong A_5$, then since $|G| = |H||G/K||A_5|$ and $|G/K| \mid |\mathrm{Out}(A_5)| = 2$,

$|H| = 2^t \cdot 3^5 \cdot 5 \cdot 73$, where $2 \le t \le 3$. Let H be solvable and also, let P be a 73-Sylow subgroup of H. Then an easy calculation and Lemma 1.4 show P ch $H \trianglelefteq G$. Now, since P is cyclic, Ito's theorem leads us to get a contradiction. Therefore H is a non-solvable group of order $2^t \cdot 3^5 \cdot 5 \cdot 73$, where $2 \le t \le 3$. So, considering the order of H, and Lemmas 1.3 and 1.6 show that there exists a normal series as $1 \trianglelefteq H_1 \trianglelefteq K_1 \trianglelefteq H$ such that for $t = 3$, $K_1/H_1 \cong A_5$ or A_6 and for $t = 2$, $K_1/H_1 \cong A_5$. Thus $\pi(H_1) \subseteq \{2, 3, 73\}$ and hence, Lemma 1.7 guarantees that H_1 is solvable. Suppose that $P \in \mathrm{Syl}_{73}(H_1)$. Then by Lemma 1.4, P ch $H_1 \trianglelefteq H$. On the other hand, P is a 73-Sylow subgroup of H. Thus $P \trianglelefteq G$. Now, Ito's theorem leads us to get a contradiction. Also, applying the same argument rules out the cases when $K/H \cong A_5 \times A_5$ or A_6. Thus $K/H \cong U_3(9)$. It follows that $G \cong U_3(9)$, as claimed.

Proof of (III). Let $\chi, \beta \in \mathrm{Irr}(G)$ such that $\chi(1) = l(G) = l(M)$ and $\beta(1) = t(G) = t(M)$. Then Lemma 1.8 shows that G is non-solvable and hence, Lemma 1.3 implies that G has a normal series as Lemma 1.3 such that K/H is a direct product of isomorphic non-abelian simple K_3 or K_4-groups and $|G/K| \mid |\mathrm{Out}(K/H)|$. Now by comparing the order of G and the orders of the non-abelian simple groups mentioned in Lemma 1.6, we have the following cases:

i. Let $M = S_6(2)$. Then $|G| = |S_6(2)| = 2^9 \cdot 3^4 \cdot 5 \cdot 7$, $\chi(1) = l(G) = l(S_6(2)) = 2^9$, $\beta(1) = t(G) = t(S_6(2)) = 3^4 \cdot 5$. Let N be a normal minimal solvable subgroup of G. Then applying Ito's theorem to χ, β and N shows that $N = Z_7$ or $N = 1$. Since $G/C_G(N) \lesssim \mathrm{Aut}(Z_7) \cong Z_6$, $|C_G(N)/N| \neq 1$. Now, we claim that $C_G(N)/N$ is non-solvable. On the contrary, let $C_G(N)/N$ be solvable. Then we conclude that $C_G(N)$ is solvable and so, Lemma 1.5 shows that $C_G(N) = N \times C$, where C is a $(\pi(C_G(N)) - \{7\})$-Hall subgroup of $C_G(N)$. Thus C ch $C_G(N) \trianglelefteq G$. So, C is a normal solvable subgroup of G, which is a contradiction with the fact that every normal minimal solvable subgroup of G is a 7-group. This contradiction shows that $C_G(N)/N$ is non-solvable. Let L/N be a normal minimal subgroup of G/N such that $L/N \le C_G(N)/N$. Then the above statement shows that L/N is a direct product of t-copies of isomorphic non-abelian simple K_3 or K_4-groups. Thus considering Lemma 1.6 shows that

$$L/N \cong A_5, A_6, L_2(7), L_2(8), U_3(3), U_4(2), A_7, A_8, A_9, L_3(4) \text{ or } S_6(2). \tag{2.1}$$

Since L is a central extension of N by L/N, considering the schur multiplier of L/N shows that L is a central product of N and L/N. Also, $L \trianglelefteq G$ and hence, Lemma 1.2 applied to χ and β , respectively, allows us to assume that there exists $\alpha, \eta \in \mathrm{cd}(L)$ such that $|L|_2 \mid \alpha$ and $|L|_3 \cdot |L|_5 \mid \eta$. On the other hand, $\mathrm{cd}(L) = \mathrm{cd}(L/N)$, because L is a central product of N and L/N and N is abelian. Therefore, we can assume that $\alpha, \eta \in \mathrm{cd}(L/N)$ and hence, considering the character degrees of the groups mentioned in 2.1 forces

$$L/N \cong L_2(7), L_2(8), U_3(3), A_8, L_3(4) \text{ or } S_6(2). \tag{2.2}$$

Thus $7 \mid |L/N|$ and hence, $N = 1$. Now, let C be a normal minimal subgroup of G such that $C \le C_G(L)$. If $C \neq 1$, then since $7 \| |G|$ and $7 \mid |L|$, $7 \nmid |C|$. Also, repeating the argument given for L shows that C is one of the groups in (2.2) and hence, $7 \mid |C|$, which is a contradiction. Therefore,

$C_G(L) = 1$ and hence, $G \le \mathrm{Aut}(L)$. Thus considering the orders of the automorphism groups of the groups in 2.2 guarantees that $L \cong S_6(2)$ and so, $G = L \cong S_6(2)$, as desired.

ii. Let $M = L_3(8)$. Then $|G| = |L_3(8)| = 2^9 \cdot 3^2 \cdot 7^2 \cdot 73$, $\chi(1) = l(G) = l(L_3(8)) = 3^2 \cdot 73$, $\beta(1) = t(G) = t(L_3(8)) = 2^9$ and $K/H \cong L_2(7)$, $L_2(7) \times L_2(7)$, $L_2(8)$ or $L_3(8)$. If $K/H \cong L_2(7)$, then $|G/K| \mid |\mathrm{Out}(L_2(7))| = 2$ and hence, $|H| = 2^{6-t} \cdot 3 \cdot 7 \cdot 73$, where $0 \le t \le 1$. Let $P \in \mathrm{Syl}_{73}(H)$. Then an easy calculation shows that P ch $H \trianglelefteq G$. Now, since P is abelian, Ito's theorem forces $\chi(1) = 3^2 \cdot 73$ to divide $|G|/73 = |M|/73$, which is a contradiction. The same reasoning rules out the cases when $K/H \cong L_2(7) \times L_2(7)$ or $L_2(8)$. Therefore $K/H \cong L_3(8)$ and hence, $G = K \cong L_3(8)$.

iii. Let $M = L_3(5)$. Then $|G| = |L_3(5)| = 2^5 \cdot 3 \cdot 5^3 \cdot 31$, $\chi(1) = l(G) = l(L_3(5)) = 2 \cdot 3 \cdot 31$, $\beta(1) = t(G) = t(L_3(5)) = 5^3$ and $K/H \cong A_5, L_2(31)$ or $L_3(5)$. Let $K/H \cong A_5$. Then since $|\mathrm{Out}(A_5)| = 2$, $|H| = 2^t \cdot 5^2 \cdot 31$, where $2 \le t \le 3$ and hence, Lemma 1.7 shows that H is solvable. Let $P \in \mathrm{Syl}_{31}(H)$. Then since for every natural number k, $31k + 1 \nmid 5^2$ or 2^t for $t = 2$ or 3, Lemma 1.4 guarantees that P ch $H \trianglelefteq G$. But P is cyclic and hence, Ito's theorem shows that $2 \cdot 3 \cdot 31 = \chi(1) \mid [G : P]$, which is impossible. Also, if $K/H \cong L_2(31)$, then by Lemma 1.3, $|G/K| \mid |\mathrm{Out}(L_2(31))| = 2$ and so, $|H| = 5^2$. Now, applying Ito's theorem to H and β leads us to get a contradiction. Therefore $K/H \cong L_3(5)$ and hence, $G = K \cong L_3(5)$.

iv. Let $M = O_8^+(2)$. Then $|G| = |O_8^+(2)| = 2^{12} \cdot 3^5 \cdot 5^2 \cdot 7$, $\chi(1) = l(G) = l(O_8^+(2)) = 3^5 \cdot 5^2$ and $\beta(1) = t(G) = t(O_8^+(2)) = 2^{12}$. Let N be a normal minimal solvable subgroup of G. Then applying Ito's theorem to χ, β and N shows that $N = Z_7$ or $N = 1$. Let L/N be a normal minimal subgroup of G/N such that $L/N \le C_G(N)/N$. Then applying the same reasoning as that used for the non-solvability of $C_G(N)/N$ in (i) shows that $C_G(N)/N$ is non-solvable and so, L/N is a direct product of t-copies of isomorphic non-abelian simple K_3 or K_4-groups. Thus considering Lemma 1.6 shows that

$$L/N \cong A_5, A_5 \times A_5, A_6, A_6 \times A_6, L_2(7), L_2(8), U_3(3), \quad (2.3)$$
$$U_4(2), A_7, A_8, L_3(4), S_6(2), A_9, A_{10}, J_2 \text{ or } O_8^+(2).$$

Now, by the similar argument as that used in (i), we get $N = 1$ and

$$L/N \cong A_8, L_3(4), S_6(2), L_2(7), L_2(8), U_3(3) \text{ or } O_8^+(2). \quad (2.4)$$

and $C_G(L) = 1$. Thus $G \le \mathrm{Aut}(L)$. So, considering the order of the automorphism groups in 2.4 forces $L \cong O_8^+(2)$ and hence $G = L \cong O_8^+(2)$, as desired.

v. Let $M = L_3(7)$. Then $|G| = |L_3(7)| = 2^5 \cdot 3^2 \cdot 7^3 \cdot 19$, $\chi(1) = l(G) = l(L_3(7)) = 2^3 \cdot 3 \cdot 19$, $\beta(1) = t(G) = t(L_3(7)) = 7^3$ and $K/H \cong L_2(7)$, $L_2(8)$ or $L_3(7)$. If $K/H \cong L_2(7)$ or $L_2(8)$, then $|G/K| \mid |\mathrm{Out}(K/H)| = 2$ or 3. Thus $|H| \mid 2^2 \cdot 3 \cdot 7^2 \cdot 19$ or $|H| \mid 2^2 \cdot 7^2 \cdot 19$ and hence, Lemma 1.7 guarantees that H is solvable. Therefore, an easy calculation and Lemma 1.4 show that a 19-Sylow subgroup of H is normal in H and so, it is normal in G. Thus G has a normal abelian 19-Sylow subgroup and hence, Ito's theorem implies that $\chi(1) = 2^3 \cdot 3 \cdot 19 \mid [G : P]$, which is impossible. Therefore $K/H \cong L_3(7)$ and hence, $G = K \cong L_3(7)$, as desired.

vi. Let $M = L_2(49)$. Then $|G| = |L_2(49)| = 2^4 \cdot 3 \cdot 5^2 \cdot 7^2$, $\chi(1) = l(G) = l(L_2(49)) = 2 \cdot 5^2$, $\beta(1) = t(G) = t(L_2(49)) = 2^4 \cdot 3$ and $K/H \cong A_5, L_2(7)$ or $L_2(49)$. If $K/H \cong A_5$ or $L_2(7)$, then

$|G/K| \mid |\mathrm{Out}(K/H)| = 2$ and $|G/K||H| = 2^2 \cdot 5 \cdot 7^2$ or $2 \cdot 5^2 \cdot 7$. Therefore, $|H| \mid 2^2 \cdot 5^2 \cdot 7^2$ and hence, Lemma 1.7 guarantees that H is solvable. Let $P \in \mathrm{Syl}_5(H)$. Then since there is not any natural number k such that $5k+1 \mid 7^t$ or 2^t, where $1 \leq t \leq 2$, Lemma 1.4 shows that P ch $H \trianglelefteq G$. Thus P is an abelian normal subgroup of G and hence, Ito's theorem shows that $\chi(1) = 2 \cdot 5^2$ divides $[G : P]$, which is impossible. So, $K/H \cong L_2(49)$ and hence, $G = K \cong L_2(49)$, as desired.

vii. Let $M = G_2(3)$. Then $|G| = |G_2(3)| = 2^6 \cdot 3^6 \cdot 7 \cdot 13$, $\chi(1) = l(G) = l(G_2(3)) = 2^6 \cdot 13$ and $\beta(1) = t(G) = t(G_2(3)) = 3^6$. Let N be a normal minimal solvable subgroup of G. Then applying Ito's theorem to χ, β and N shows that $N = Z_7$ or $N = 1$. Let L/N be a normal minimal subgroup of G/N such that $L/N \leq C_G(N)/N$. Then the same reasoning as that used for the non-solvability of $C_G(N)/N$ in (i) shows that $C_G(N)/N$ is non-solvable and hence, L/N is a direct product of t-copies of isomorphic non-abelian simple K_3 or K_4-groups. Thus by considering Lemma 1.6, we can see that

$$L/N \cong L_2(7), L_2(8), L_3(3), U_3(3), L_2(27), L_2(13), \text{ or } G_2(3)$$

Now, repeating the argument given for (i) shows that $N = 1$,

$$L \cong L_2(7), L_2(8), U_3(3) \text{ or } G_2(3) \tag{2.5}$$

and $C_G(L) = 1$. Therefore, $G \leq \mathrm{Aut}(L)$. Thus considering the orders of the automorphism groups of the groups in (2.5) guarantees that $L \cong G_2(3)$ and hence, $L = G \cong G_2(3)$, as desired.

viii. Let $M = U_3(5)$. Then $|G| = |U_3(5)| = 2^4 \cdot 3^2 \cdot 5^3 \cdot 7$, $\chi(1) = l(G) = l(U_3(5)) = 2^4 \cdot 3^2$, $\beta(1) = t(G) = t(U_3(5)) = 5^3$ and $K/H \cong A_5$, A_6, A_7, $L_2(7)$, $L_2(8)$ or $U_3(5)$. Suppose that $K/H \cong A_5$. Then by Lemma 1.3, $|G/K| \mid |\mathrm{Out}(A_5)| = 2$ and hence, $|H| = 2^2 \cdot 3 \cdot 5^2 \cdot 7$ or $2 \cdot 3 \cdot 5^2 \cdot 7$. If H is solvable, then an easy calculation and Lemma 1.4 show that P ch $H \trianglelefteq G$, where $P \in \mathrm{Syl}_5(H)$. Thus by Ito's theorem, we get $\beta(1) = 5^3 \mid [G : P]$, which is impossible. Hence, H is a non-solvable group of order $2^2 \cdot 3 \cdot 5^2 \cdot 7$, by considering Lemma 1.7. Thus Lemma 1.3 shows that H has a normal series as $1 \trianglelefteq H_1 \trianglelefteq K_1 \trianglelefteq H$ such that $K_1/H_1 \cong A_5$. Now, since $|\mathrm{Out}(A_5)| = 2$, by Lemma 1.3, we get $|H_1| = 5 \cdot 7$. Let $Q \in \mathrm{Syl}_5(H_1)$. Then Q ch $H_1 \trianglelefteq H$. Let $\theta \in \mathrm{Irr}(H)$ such that $[\beta_H, \theta] \neq 0$. Then Lemma 1.2 forces $\theta(1) = 5^2$. Now, Ito's theorem implies that $\theta(1) = 5^2 \mid [H : Q]$, which is impossible. The same argument rules out the cases when $K/H \cong A_6$, A_7, $L_2(7)$ or $L_2(8)$. Thus $K/H \cong U_3(5)$, which implies that $G \cong U_3(5)$, as claimed.

ix. Let $M = S_4(7)$. Then $|G| = |S_4(7)| = 2^8 \cdot 3^2 \cdot 5^2 \cdot 7^4$, $\chi(1) = l(G) = l(S_4(7)) = 2^7 \cdot 5^2$, $\beta(1) = t(G) = t(S_4(7)) = 7^4$ and $K/H \cong A_5$, $A_5 \times A_5$, A_6, A_7, $L_2(49)$, $L_2(7)$, $L_2(7) \times L_2(7)$, $L_2(8)$, $L_3(4)$, $L_4(2)$ or $S_4(7)$. Suppose that $\eta, \theta \in \mathrm{Irr}(H)$ such that $[\chi_H, \eta] \neq 0$ and $[\beta_H, \theta] \neq 0$. Then by Lemma 1.2, $\chi(1)/\eta(1) \mid [G : H]$ and $\beta(1)/\theta(1) \mid [G : H]$. If $K/H \not\cong L_2(7)$ and $S_4(7)$, then since $|G/K| \mid |\mathrm{Out}(K/H)|$, we can check at once that $\eta^2(1) > |H|$, $\theta^2(1) > |H|$ or $\eta^2(1) + \theta^2(1) > |H|$, which is a contradiction. Now, let $K/H \cong L_2(7)$. Then either $|H| = 2^5 \cdot 3 \cdot 5^2 \cdot 7^3$ or $|H| = 2^4 \cdot 3 \cdot 5^2 \cdot 7^3$. If H is solvable, then let N be a normal minimal subgroup of G such that $N \leq H$. Thus Ito's theorem forces $|N| \mid 2$ or $|N| \mid 3$. Let L_1 be a maximal normal 2-subgroup of G and let L_2 be a maximal normal 3-subgroup of G such that $L_1, L_2 \leq H$. Then applying Lemma 1.2 to L_1 and η shows that $|L_1| \mid 2$. Also, $|L_2| \mid |H|_3 = 3$. Set $L = L_1 \times L_2$. Since $H/C_H(L) \hookrightarrow \mathrm{Aut}(L) = Z_2$, we deduce that

$|C_H(L)/L| \neq 1$. Now, the same reasoning as that used for the non-solvability of $L_3(7)$ leads us to get a contradiction. Thus H is non-solvable. Therefore by Lemma 1.3, there exists a normal series as $1 \trianglelefteq H_1 \trianglelefteq K_1 \trianglelefteq H$ such that K_1/H_1 is a direct product of isomorphic non-abelian simple K_3 or K_4-groups and $|H/K_1| \mid |\mathrm{Out}(K_1/H_1)|$. Now comparing the order of H and the orders of the non-abelian simple groups mentioned in Lemma 1.6 shows that $K_1/H_1 \cong L_2(7)$ or A_5. Therefore, $5 \cdot 7^2 \mid |H_1|$ and $|H_1| \mid 2^3 \cdot 5^2 \cdot 7^3$. Thus H_1 is solvable and Lemma 1.4 shows that $P \in \mathrm{Syl}_5(H_1)$ is normal in H_1 and hence, $P \trianglelefteq H$. So, Ito's theorem forces $\eta(1) \mid [H : P]$, which is impossible. Therefore $K/H \cong S_4(7)$ and hence, $G = K \cong S_4(7)$, as desired.

Proof of (IV) Let $\chi, \beta \in \mathrm{Irr}(G)$ such that $\chi(1) = l(G) = l(M)$ and $\beta(1) = s(G) = s(M)$. Then by Lemma 1.8, G is non-solvable and hence, Lemma 1.3 shows that there exists a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K/H is a direct product of isomorphic non-abelian simple K_3 or K_4-groups and $|G/K| \mid |\mathrm{Out}(K/H)|$. Now by comparing the order of G and the orders of the non-abelian simple groups mentioned in Lemma 1.6, we have the following cases:

i. Let $G = L_3(4)$. Then $|G| = |L_3(4)| = 2^6 \cdot 3^2 \cdot 5 \cdot 7$, $\chi(1) = l(G) = l(L_3(4)) = 2^6$, $\beta(1) = s(G) = s(L_3(4)) = 3^2 \cdot 7$ and $K/H \cong A_5$, A_6, $A_7, L_2(8)$, A_8, $L_2(7)$ or $L_3(4)$. Suppose that $\eta, \theta \in \mathrm{Irr}(H)$ such that $[\chi_H, \eta] \neq 0$ and $[\beta_H, \theta] \neq 0$. Then by Lemma 1.2, $\chi(1)/\eta(1) \mid [G : H]$ and $\beta(1)/\theta(1) \mid [G : H]$. Therefore, $|H|_2 \mid \eta(1)$ and $|H|_3 \cdot |H|_7 \mid \theta(1)$. Also, $|G/K| \mid |\mathrm{Out}(K/H)|$ and hence, if $K/H \cong A_5$, then $|H| = 2^t \cdot 3 \cdot 7$, where $3 \leq t \leq 4$. So, $\theta^2(1) = (3 \cdot 7)^2 > |H|$, which is a contradiction. If $K/H \cong A_6$, then $|G/K| \mid |\mathrm{Out}(A_6)| = 2^2$ and hence, $|H| = 2^t \cdot 7$, where $1 \leq t \leq 3$. So, either $t \leq 2$ and $\theta^2(1) = 7^2 > |H|$ or $\eta^2(1) = 2^6 > |H|$, which is a contradiction. If $K/H \cong A_7$, then $|H| = 2^2$ or 2^3 and $\eta(1) = |H|$, which is a contradiction. Let $K/H \cong L_2(8)$. Then $|H| = 2^3 \cdot 5$. Thus $\eta^2(1) = 2^6 > |H|$, which is a contradiction. If $K/H \cong A_8$, then $G \cong A_8$. Thus $3^2 \cdot 7 \in \mathrm{cd}(A_8)$, which is a contradiction. Finally, let $K/H \cong L_2(7)$. Then $|G/K| \mid |\mathrm{Out}(L_2(7))| = 2$ and hence, $|H| = 2^3 \cdot 3 \cdot 5$ or $2^2 \cdot 3 \cdot 5$. If H is solvable, then an easy calculation and Lemma 1.4 show that $P \in \mathrm{Syl}_5(G)$ is normal in G. Also, $H/C_H(P) \hookrightarrow \mathrm{Aut}(P) \cong Z_4$ and hence, $3, 5 \in \pi(C_H(P)) \subseteq \pi(H)$ and $C_H(P) = P \times C$, where $C \neq 1$ is a $(\pi(C_H(P)) - \{5\})$-Hall subgroup of $C_H(P)$. Thus C is a normal solvable subgroup of G such that $|C| \mid 2^3 \cdot 3$. Thus C contains a 2 or 3-elementary abelian normal subgroup of G which is impossible by considering Ito's theorem. Therefore H is a non-solvable group of order $2^3 \cdot 3 \cdot 5$ or $2^2 \cdot 3 \cdot 5$ and hence, Lemma 1.3 implies that H has a normal series as $1 \trianglelefteq H_1 \trianglelefteq K_1 \trianglelefteq H$ such that $K_1/H_1 \cong A_5$. Also, by Lemma 1.3, $|H/K_1| \mid |\mathrm{Out}(A_5)| = 2$ and hence, $|H_1| = 1$ or 2. If $|H_1| = 2$, then by Ito's theorem, we get $\eta(1) = |H|_2 = 2^3 \mid [H : H_1]$, which is impossible. Thus $|H_1| = 1$ and hence, $H = A_5 \cdot 2$ or A_5. Therefore, considering the fact that $G/C_G(H) \hookrightarrow \mathrm{Aut}(H)$ guarantees that $L_2(7) \leq C_G(H)$ and hence, $K = L_2(7) \times S_5$ or $K = L_2(7) \times A_5$. So, Lemma 1.2 shows that $\beta(1) = 3^2 \cdot 7 \in \mathrm{cd}(K) = \mathrm{cd}(L_2(7) \times \mathrm{cd}(A_5))$ or $\beta(1) = 3^2 \cdot 7 \in \mathrm{cd}(L_2(7) \times \mathrm{cd}(S_5))$, which is a contradiction. These contradictions show that $K/H \cong L_3(4)$ and hence, $G = K \cong L_3(4)$.

ii. Let $M = U_4(3)$. Then $|G| = |U_4(3)| = 2^7 \cdot 3^6 \cdot 5 \cdot 7$, $\chi(1) = l(G) = l(U_4(3)) = 2^7 \cdot 7$ and $\beta(1) = s(G) = s(U_4(3)) = 3^6$. Let N be a normal minimal solvable subgroup of G. Then applying Ito's theorem forces $N = Z_5$ or $N = 1$. Let L/N be a normal minimal subgroup of G/N such that $L/N \leq C_G(N)/N$. Then as mentioned in the previous cases and applying Lemma 1.2 to N, χ and

β show that L/N is a direct product of t-copies of isomorphic non-abelian simple K_3 or K_4-groups. Thus considering Lemma 1.6 shows that L/N is isomorphic to A_5, $L_2(7)$, A_6, $L_2(8)$, A_7, A_8, A_9, $L_3(4)$, $U_3(3)$, $U_4(2)$ or $U_4(3)$. Now repeating the argument given for (i) in the proof of (III) shows that $N = 1$,

$$L \cong A_5, A_6, U_4(2) \text{ or } U_4(3) \tag{2.6}$$

and $C_G(L) = 1$. Therefore, $G \leq \mathrm{Aut}(L)$. Thus considering the orders of the automorphism groups of the groups in (2.6) guarantees that $L \cong U_4(3)$ and hence, $L = G \cong U_4(3)$, as desired.

iii. Let $M = {}^2F_4(2)'$. Then $|G| = |{}^2F_4(2)'| = 2^{11} \cdot 3^3 \cdot 5^2 \cdot 13$, $\chi(1) = l(G) = l({}^2F_4(2)') = 2^{11}$, $\beta(1) = s(G) = s({}^2F_4(2)') = 2^6 \cdot 3^3$ and $K/H \cong A_5, A_5 \times A_5, A_6, L_3(3), L_2(25), U_3(4)$ or ${}^2F_4(2)'$. Suppose that $\theta, \eta \in \mathrm{Irr}(H)$ such that $[\chi_H, \theta] \neq 0$ and $[\beta_H, \eta] \neq 0$. Then Lemma 1.2 implies that $\theta(1) = |H|_2$ and $|\eta(1)|_3 = |H|_3$. Let $K/H \cong A_5$. Then since $|G/K| \mid |\mathrm{Out}(K/H)|$, we have, $|H| = 2^9 \cdot 3^2 \cdot 5 \cdot 13$ or $2^8 \cdot 3^2 \cdot 5 \cdot 13$. Let H be solvable. Then G has a normal minimal subgroup, namely N, such that $N \leq H$. Thus for some $t \in \pi(H)$, N is a t-elementary abelian group and so, applying Ito's theorem to N, χ and β shows that $t \in \{5, 13\}$. Now, by the similar argument as that used for the non-solvability of G, we get a contradiction. Hence, H is non-solvable. Thus Lemma 1.3 shows that H has a normal series as $1 \trianglelefteq H_1 \trianglelefteq K_1 \trianglelefteq H$ such that $K_1/H_1 \cong A_5$ or A_6. Now, since $|\mathrm{Out}(A_5)| = 2$ and $|\mathrm{Out}(A_6)| = 2^2$, $|H_1| = 2^s \cdot 3 \cdot 13$, where $5 \leq s \leq 7$ or $2^s \cdot 13$, where $3 \leq s \leq 6$. So, Lemma 1.7 shows that H_1 is solvable and hence, H_1 has a normal minimal subgroup namely L, such that $L \leq H_1$. Thus for some $t \in \pi(H_1)$, L is a t-elementary abelian group. Again, applying Ito's theorem to L, θ and η shows that $|L| = t = 13$ and by a similar argument as that used for the non-solvability of G , we get a contradiction. Also, the similar reasoning as the case when $K/H \cong A_5$, rules out $K/H \cong A_5 \times A_5$ or A_6. If $K/H \cong L_3(3)$, then for $t \in \{6, 7\}$, $|H| = 2^t \cdot 5^2$, if $K/H \cong L_2(25)$, then for $t \in \{6, 7, 8\}$, $|H| = 2^t \cdot 3^2$ and if $K/H \cong U_3(4)$, then for $t \in \{3, 4, 5\}$, $|H| = 2^t \cdot 3^2$. Thus by considering the order of H in the above cases, we can see that $\eta^2(1) > |H|$ or $\theta^2(1) > |H|$, which is a contradiction. Thus $K/H \cong {}^2F_4(2)'$, which implies that $G = K \cong {}^2F_4(2)'$, as claimed.

iv. Let $M = U_5(2)$. Then $|G| = |U_5(2)| = 2^{10} \cdot 3^5 \cdot 5 \cdot 11$, $\chi(1) = l(G) = l(U_5(2)) = 3^5 \cdot 5$, $\beta(1) = s(G) = s(U_5(2)) = 2^{10}$ and $K/H \cong A_5$, A_6, $U_4(2)$, M_{11}, $M_{12}, L_2(11)$ or $U_5(2)$. If $K/H \cong A_5$, then $|H| \mid 2^8 \cdot 3^4 \cdot 11$ and $2^7 \cdot 3^4 \cdot 11 \mid |H|$, if $K/H \cong A_6$, then $|H| \mid 2^7 \cdot 3^3 \cdot 11$ and $2^5 \cdot 3^3 \cdot 11 \mid |H|$, if $K/H \cong U_4(2)$, then $|H| \mid 2^4 \cdot 3 \cdot 11$ and $2^3 \cdot 3 \cdot 11 \mid |H|$, if $K/H \cong M_{11}$, then $|H| = 2^6 \cdot 3^3$, if $K/H \cong L_2(11)$, then $|H| \mid 2^8 \cdot 3^4$ and $2^7 \cdot 3^4 \mid |H|$ and if $K/H \cong M_{12}$, then $|H| \mid 2^4 \cdot 3^2$ and $2^3 \cdot 3^2 \mid |H|$. Hence, Lemma 1.7 shows that H is solvable. We note that in the above cases, the order of H has at least two prime divisors. It follows that H has a s-elementary abelian subgroup, namely N, which is normal in G. If $s \in \{2, 3, 5\}$, then applying Ito's theorem to N, β and χ leads us to get a contradiction. Thus $s = 11$. Since $|N| = |C_H(N)|_{11}$ and $H/C_H(N) \hookrightarrow \mathrm{Aut}(N) = Z_{10}$, the order of H in the above cases and Lemma 1.5 guarantee that $\pi(C_H(N)) - \{11\} \neq \emptyset$ and $C_H(N) = N \times C$, where C is a $(\pi(C_H(N)) - \{11\})$-Hall subgroup of $C_H(N)$. Thus we can see at once that $C \trianglelefteq G$ and hence, for some $u \in \pi(C_H(N)) - \{11\}$, C contains an u-elementary abelian normal subgroup of G, which is contradiction with the fact that the only normal minimal subgroups of G are 11-groups. Therefore $K/H \cong U_5(2)$, which implies that $G = K \cong U_5(2)$, as desired.

Proof of (V). Let $\chi, \beta \in \mathrm{Irr}(G)$ such that $\chi(1) = s(G) = s(M)$ and $\beta(1) = t(G) = t(M)$. Then Lemma 1.8 shows that G is non-solvable and hence, Lemma 1.3 implies that there exists a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K/H is a direct product of isomorphic non-abelian simple K_3 or K_4-groups and $|G/K| \mid |\mathrm{Out}(K/H)|$. Now, by comparing the order of G and the orders of the non-abelian simple groups mentioned in Lemma 1.6, we have the following cases:

i. Let $M = U_3(4)$. Then $|G| = |U_3(4)| = 2^6 \cdot 3 \cdot 5^2 \cdot 13$, $\chi(1) = s(G) = s(U_3(4)) = 13 \cdot 5$, $\beta(1) = t(G) = t(U_3(4)) = 2^6$ and $K/H \cong A_5, L_2(25)$ or $U_3(4)$. Suppose that $K/H \cong A_5$ or $L_2(25)$. Then since $|G/K| \mid |\mathrm{Out}(K/H)|$, $|H| = 2^t \cdot 5 \cdot 13$, where $3 \leq t \leq 4$ or $|H| = 2^t$, where $1 \leq t \leq 3$. Now, let $\theta,\ \eta \in \mathrm{Irr}(H)$ such that $[\chi_H, \eta] \neq 0$ and $[\beta_H, \theta] \neq 0$. Then Lemma 1.2 implies that $|H|_5 \cdot |H|_{13} \mid \eta(1)$ and $|H|_2 \mid \theta(1)$. Thus $\eta^2(1) > |H|$ or $\theta^2(1) > |H|$, a contradiction. Therefore $K/H \cong U_3(4)$. It follows that $G = K \cong U_3(4)$, as claimed.

ii. Let $M = {}^3D_4(2)$. Then $|G| = |{}^3D_4(2)| = 2^{12} \cdot 3^4 \cdot 7^2 \cdot 13$, $\chi(1) = s(G) = s({}^3D_4(2)) = 2^{12}$ and $\beta(1) = t(G) = t({}^3D_4(2)) = 3^4 \cdot 7^2$. Let N be a normal minimal solvable subgroup of G. Then applying Ito's theorem to χ, β and N shows that $N = Z_{13}$ or $N = 1$. Let L/N be a normal minimal subgroup of G/N such that $L/N \leq C_G(N)/N$. Then applying the same reasoning as that used for the non-solvability of G shows that L/N is a direct product of t-copies of isomorphic non-abelian simple K_3 or K_4-groups. Thus considering Lemma 1.6 shows that

(2.7) $L/N \cong L_2(7), L_2(8), L_3(3), U_3(3), L_2(13), L_2(27), L_2(7) \times L_2(7), L_2(8) \times L_2(8)$ or ${}^3D_4(2)$

Since L is a central extension of N by L/N, considering the schur multiplier of L/N shows that L is a central product of N and L/N. Suppose that $\eta, \theta \in \mathrm{Irr}(L) = \mathrm{Irr}(L/N) \times \mathrm{Irr}(N) = \mathrm{Irr}(L/N)$ such that $[\chi_L, \eta] \neq 0$ and $[\beta_L, \theta] \neq 0$. Then by Lemma 1.2, $\chi(1)/\eta(1) \mid [G : L]$ and $\beta(1)/\theta(1) \mid [G : L]$. Therefore, $|L|_2 \mid \eta(1)$ and $|L|_3 \cdot |L|_7 \mid \theta(1)$. Thus considering the character degrees of the groups mentioned in (2.7) shows that

(2.8) $$L/N \cong L_3(3) \text{ or } {}^3D_4(2)$$

This forces $N = 1$, because $13 \| |G|$. Now, let C be a normal minimal subgroup of G such that $C \leq C_G(L)$. Obviously, by repeating the argument given for L/N, $C = 1$ or C is isomorphic to one of the groups mentioned in (2.8). If $C \neq 1$, then $C \times L \trianglelefteq G$ and $|L|_3 \cdot |C|_3 > 3^4$, which is a contradiction. This forces $C = 1$ and hence, $C_G(L) = 1$. If $L \cong L_3(3)$, then $G \hookrightarrow \mathrm{Aut}(L_3(3))$. It follows that $|G| \mid |\mathrm{Aut}(L_3(3))| = 2^5 \cdot 3^3 \cdot 13$, which is a contradiction. Therefore $L/N \cong {}^3D_4(2)$ and hence, $G = L \cong {}^3D_4(2)$, as claimed.

iii. Let $M = S_4(4)$. Then $|G| = |S_4(4)| = 2^8 \cdot 3^2 \cdot 5^2 \cdot 17$, $\chi(1) = s(G) = s(S_4(4)) = 2^8$ and $\beta(1) = t(G) = t(S_4(4)) = 3 \cdot 5 \cdot 17$. Applying Ito's theorem to the normal minimal solvable subgroup of G, $s(G)$ and $t(G)$ shows that the normal minimal solvable subgroup of G is a 3-group or 5-group. Let N_1 be a maximal normal 3-subgroup of G, N_2 be a maximal normal 5-subgroup of G and $N = N_1 \times N_2$. Then $|N| \mid 3 \cdot 5$. Let L/N be a normal minimal subgroup of G/N such that $L \leq C_G(N)$. Then applying the same reasoning as that used in (III-i) shows that L/N is a direct product of t-copies of isomorphic non-abelian simple K_3 or K_4-groups. Thus considering Lemma 1.6 shows that

$$L/N \cong A_5, A_6, A_5 \times A_5, L_2(16), L_2(17) \text{ or } S_4(4). \tag{2.9}$$

Now, let C/N be a normal minimal subgroup of G/N such that $C \leq C_G(L)$. Obviously, $N \leq C$ and by our assumption on N and repeating the argument given for L/N, $C = N$ or C/N is isomorphic to one of the groups mentioned in (2.9). Let $C \neq N$. If $L/N \cong A_5 \times A_5$, then considering the order of G/N shows that $|C/N| \mid 2^4 \cdot 17$ and hence, C/N is solvable. Thus our assumption on N forces $C = N$, which ia a contradiction. Now, let $L/N \cong A_5$. Then repeating the above argument shows that $C/N \cong A_5$ or $L_2(16)$. Thus $L/N \times C/N \cong A_5 \times A_5 \trianglelefteq G/N$ or $L/N \times C/N \cong A_5 \times L_2(16) \trianglelefteq G/N$. If $L/N \times C/N \cong A_5 \times A_5$, then repeating the argument given for the case when $L/N \cong A_5 \times A_5$ leads us to get a contradiction. Now, let $L/N \times C/N \cong A_5 \times L_2(16)$. Then since $|A_5 \times L_2(16)|_5 |A_5 \times L_2(16)|_3 = |G|_5 |G|_3$, $N = 1$. Hence, $A_5 \times L_2(16) \trianglelefteq G$. Now, Lemma 1.2 forces $\beta(1) = 3 \cdot 5 \cdot 17 \in \mathrm{cd}(A_5 \times L_2(16))$, which is a contradiction. Also a similar argument shows that if $L/N \cong L_2(16)$, then $C/N \cong A_5$ and as above, we get a contradiction. If $L/N \cong A_6$ or $L/N \cong L_2(17)$, then $|C/N| \mid 2^5 \cdot 5 \cdot 17$ or $|C/N| \mid 2^4 \cdot 5^2$ and hence, considering the order of the finite simple K_3-groups shows that C/N is solvable. Therefore, our assumption on N forces $C = N$, which is a contradiction. Thus in the mentioned groups, $C/N = 1$, and so $G/N \hookrightarrow \mathrm{Aut}(L/N)$, which is a contradiction by considering the order of G and the order of $\mathrm{Aut}(L/N)$. Therefore, $L/N \cong S_4(4)$ and hence, $N = 1$ and $L = G \cong S_4(4)$.

iv. Let $M = S_4(5)$. Then $|G| = |S_4(5)| = 2^6 \cdot 3^2 \cdot 5^4 \cdot 13$, $\chi(1) = s(G) = s(S_4(5)) = 5^4$ and $\beta(1) = t(G) = t(S_4(5)) = 2^4 \cdot 3 \cdot 13$. Let N be a normal minimal solvable subgroup of G. Then applying Ito's theorem to χ, β and N shows that $|N| \mid 2^3 \cdot 3$. Let L/N be a normal minimal subgroup of G/N such that $L \leq C_G(N)$. Then applying the same reasoning as that used for the non-solvability of G shows L/N is a direct product of t-copies of isomorphic non-abelian simple K_3 or K_4-groups. Thus Lemma 1.6 shows that $L/N \cong A_5, A_5 \times A_5, A_6, L_2(25)$, $U_3(4)$ or $S_4(5)$. Then repeating the argument given for (iii) completes the proof.

Proof of (VI). Let $\chi, \beta \in \mathrm{Irr}(G)$ such that $\chi(1) = l(G) = l(L_2(3^m)) = 3^m + 1$ and $\beta(1) = s(G) = s(L_2(3^m)) = 3^m$. Then Lemma 1.8 shows that G is non-solvable. Now, since $L_2(3^m)$ is a simple K_4-group, $|\pi(G)| = 4$. Also, $4 \| |G|$ and $u, t \in \pi(G)$. Thus we can see by Lemma 1.3 that G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that $K/H \cong L_2(2^e)$, $L_2(r)$ or $L_2(3^e)$, where $L_2(2^e)$, $L_2(r)$ and $L_2(3^e)$ are simple K_4-groups. If $K/H \cong L_2(2^e)$, then since $4 \| |L_2(3^m)| = |G|$, we deduce that $2^e \leq 2^2$ and hence, $|K/H| = 2^2 \cdot 3 \cdot 5$, which is a contradiction. If $K/H \cong L_2(r)$, then $r = t$ or u. First let $m \neq 5$. Let $r = t = (3^m - 1)/2$. Then since $|L_2(r)| = r(r^2 - 1)/2 \mid |G|$, we deduce that $3(3^{m-1} - 1)/8 \mid 3^m$, which is impossible. Thus $r = u = (3^m + 1)/4$. Now, since $t = (3^m - 1)/2 \mid |K/H| = r(r^2 - 1)/2$, we deduce that $t \mid (u - 1)/2$ or $(u + 1)/2$, which is impossible. Also, if $m = 5$, then an easy calculation shows that $|L_2(r)| \nmid |G|$. Therefore $K/H \cong L_2(3^e)$, where e and $u' = (3^e + 1)/4$ are odd primes and $(3^e - 1)/2$ is either a prime or 11^2 (for $e = 5$). Hence, $u = u'$ or t'. But $(3^m + 1)/4 \nmid 3^e - 1$ and hence, $u = u'$. This forces $e = m$ and hence, $K/H \cong L_2(3^m)$. Thus $G = K \cong L_2(3^m)$, as desired. □

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