www.theoryofgroups.ir

International Journal of Group Theory
ISSN (print): 2251-7650, ISSN (on-line): 2251-7669
Vol. 01 No. 3 (2012), pp. 33-37.

www.ui.ac.ir

ON VARIETAL CAPABILITY OF INFINITE DIRECT PRODUCTS OF GROUPS

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Communicated by Hamid Mousavi

Abstract. Recently, the authors gave some conditions under which a direct product of finitely many groups is $\mathcal{V}-$capable if and only if each of its factors is $\mathcal{V}-$capable for some varieties $\mathcal{V}$. In this paper, we extend this fact to any infinite direct product of groups. Moreover, we conclude some results for $\mathcal{V}-$capability of direct products of infinitely many groups in varieties of abelian, nilpotent and polynilpotent groups.

1. Introduction

R. Baer [1] initiated an investigation of the question "which conditions a group G must fulfill in order to be the group of inner automorphisms of a group E?", that is $G \cong E/Z(E)$. Following M. Hall and J. K. Senior [5], such a group G is called *capable*. Baer [1] determined all capable groups which are direct sums of cyclic groups. As P. Hall [4] mentioned, characterizations of capable groups are important in classifying groups of prime-power order.

F. R. Beyl, U. Felgner and P. Schmid [2] proved that every group G possesses a uniquely determined central subgroup $Z^*(G)$ which is minimal subject to being the image in G of the center of some central extension of G. This $Z^*(G)$ is characteristic in G and is the image of the center of every stem cover of G. Moreover, $Z^*(G)$ is the smallest central subgroup of G whose factor group is capable [2]. Hence G is capable if and only if $Z^*(G) = 1$. They showed that the class of all capable groups is closed under the direct products. Also, they presented a condition in which the capability of a direct product of finitely many of groups implies the capability of each of the factors. Moreover, they proved that if N

MSC(2010): Primary: 20E10; Secondary: 20K25, 20E34, 20D15, 20F18.

Keywords: Capable group, Direct product, Variety of groups, $\mathcal{V}-$capable group, direct limit.

Received: 19 Feburary 2012, Accepted: 14 May 2012.

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is a central subgroup of G, then $N \subseteq Z^*(G)$ if and only if the mapping $M(G) \to M(G/N)$ induced by the natural epimorphism, is monomorphism.

Then M. R. R. Moghadam and S. Kayvanfar [10] generalized the concept of capability to $\mathcal{V}-$capability for a group G. They introduced the subgroup $(V^*)^*(G)$ which is associated with the variety $\mathcal{V}$ defined by a set of laws V and a group G in order to establish a necessary and sufficient condition under which G can be $\mathcal{V}-$capable. They also showed that the class of all $\mathcal{V}-$capable groups is closed under the direct products. Moreover, they exhibited a close relationship between the groups $\mathcal{V}M(G)$ and $\mathcal{V}M(G/N)$, where N is a normal subgroup contained in the marginal subgroup of G with respect to the variety $\mathcal{V}$. Using this relationship, they gave a necessary and sufficient condition for a group G to be $\mathcal{V}-$capable.

The authors [7] presented some conditions in which the $\mathcal{V}-$capablity of a direct product of finitely many groups implies the $\mathcal{V}-$capablity of each of its factors. In this paper, we extend this fact to direct product of an infinite family of groups. Also, we deduce some new results about the $\mathcal{V}-$capability of direct product of infinitely many groups, where $\mathcal{V}$ is the variety of abelian, nilpotent, or polynilpotent groups.

2. Main Results

Suppose that $\mathcal{V}$ is a variety of groups defined by the set of laws V. A group G is said to be $\mathcal{V}-$capable if there exists a group E such that $G \cong E/V^*(E)$, where $V^*(E)$ is the marginal subgroup of E, which is defined as follows [6]:

$$\{g \in E \mid v(x_1, x_2, ..., x_n) = v(x_1, x_2, ..., gx_i, x_{i+1}, ..., x_n)$$

$$\forall x_1, x_2, ..., x_n \in E,\ \forall i \in \{1, 2, ..., n\}\}.$$

If $\psi : E \to G$ is a surjective homomorphism with $ker\psi \subseteq V^*(E)$, then the intersection of all subgroups of the form $\psi(V^*(E))$ is denoted by $(V^*)^*(G)$. It is obvious that $(V^*)^*(G)$ is a characteristic subgroup of G contained in $V^*(G)$. If $\mathcal{V}$ is the variety of abelian groups, then the subgroup $(V^*)^*(G)$ is the same as $Z^*(G)$ and in this case $\mathcal{V}-$capability is equal to capability [10]. In the following, there are some results which we need them in sequel.

Theorem 2.1. [10] *(i) A group G is $\mathcal{V}-$capable if and only if $(V^*)^*(G) = 1$.*
(ii) If $\{G_i\}_{i\in I}$ is a family of groups, then $(V^*)^*(\prod_{i\in I} G_i) \subseteq \prod_{i\in I}(V^*)^*(G_i)$.

As a consequence, if the G_i's are $\mathcal{V}-$capable groups, then $G = \prod_{i\in I} G_i$ is also $\mathcal{V}-$capable. In the above theorem, the equality does not hold in general (see Example 2.3 of [7]).

Theorem 2.2. [10] *Let $\mathcal{V}$ be a variety of groups with a set of laws V. Let G be a group and N be a normal subgroup with the property $N \subseteq V^*(G)$. Then $N \subseteq (V^*)^*(G)$ if and only if the homomorphism induced by the natural map $\mathcal{V}M(G) \to \mathcal{V}M(G/N)$ is a monomorphism.*

We recall that the Baer-invariant of a group G, with the free presentation F/R, with respect to the variety $\mathcal{V}$, denoted by $\mathcal{V}M(G)$, is

$$\mathcal{V}M(G)=\frac{R\cap V(F)}{[RV^*F]}\ ,$$

where $V(F)$ is the verbal subgroup of F with respect to $\mathcal{V}$ and

$$[RV^*F]=<v(f_1,\dots,f_{i-1},f_ir,f_{i+1},\dots,f_n)v(f_1,\dots,f_i,\dots,f_n)^{-1}|r\in R,$$

$$f_i\in F,v\in V,1\le i\le n,n\in\mathbf{N}>\ .$$

It is known that the Baer-invariant of a group G is always abelian and independent of the choice of the presentation of G. Also if $\mathcal{V}$ is the variety of abelian groups, then the Baer-invariant of G will be $R\cap F'/[R,F]\cong M(G)$, where $M(G)$ is the Schur multiplier of G (see [6]).

Theorem 2.3. [7] *Let $\mathcal{V}$ be a variety, A and B be two groups with $\mathcal{V}M(A\times B)\cong\mathcal{V}M(A)\times\mathcal{V}M(B)$, then $(V^*)^*(A\times B)=(V^*)^*(A)\times(V^*)^*(B)$. Consequently, $A\times B$ is $\mathcal{V}$-capable if and only if A and B are both $\mathcal{V}$-capable.*

Theorem 2.4. [8]*Let $\{G_i;\phi_i^j,I\}$ be a directed system of groups. Then, for a given variety $\mathcal{V}$, the Baer-invariant preserves direct limit, that is $\mathcal{V}M(\lim\limits_{\rightarrow}G_i)=\lim\limits_{\rightarrow}\mathcal{V}M(G_i)$.*

Lemma 2.5. *For any family of groups $\{G_i\}_{i\in I}$, consider the directed system $\{\mathcal{G}_{I_\lambda},\phi_\lambda^{\lambda'},\Lambda\}$ consisting of all finite direct products $\mathcal{G}_{I_\lambda}=\prod_{i\in I_\lambda}G_i$ (I_λ is a finite subset of I), with the natural embedding homomorphisms $\phi_\lambda^{\lambda'}:\mathcal{G}_{I_\lambda}\to\mathcal{G}_{I_{\lambda'}}$ ($I_\lambda\subseteq I_{\lambda'}$). Also, the index set Λ is ordered in a directed way so that for any $\lambda,\lambda'\in\Lambda$, $\lambda\le\lambda'$ if and only if $I_\lambda\subseteq I_{\lambda'}$. Then the direct product $\mathcal{G}_I=\prod_{i\in I}G_i$ is a direct limit of this directed system.*

Proof. Let $\mathcal{G}=\lim\limits_{\rightarrow}\mathcal{G}_{I_\lambda}$ be a direct limit of this directed system, with homomorphisms $\phi_\lambda:\mathcal{G}_{I_\lambda}\to\mathcal{G}$. Also, for any $\lambda\in\Lambda$, consider the embedding homomorphism $\tau_\lambda:\mathcal{G}_{I_\lambda}\to\mathcal{G}_I$. Clearly, for any $\lambda,\lambda'\in\Lambda$ with $\lambda\le\lambda'$, $\tau_{\lambda'}\tau_\lambda^{\lambda'}=\tau_\lambda$. Now, by universal property of $\mathcal{G}$, there exists a unique homomorphism $\phi:\mathcal{G}\to\mathcal{G}_I$ such that for any $\lambda\in\Lambda$, $\phi\phi_\lambda=\tau_\lambda$. To define the inverse homomorphism $\tau:\mathcal{G}_I\to\mathcal{G}$, recall that for any $x=\{x_i\}_{i\in I}\in\mathcal{G}_I$, there exists a finite subset I_λ of I that for any $i\in I\backslash I_\lambda$, x_i is trivial in G_i. Hence we can consider x as an element of $\mathcal{G}_{I_\lambda}$ and define $\tau(x)=\phi_\lambda(x)$. It is easy to see that for any $\lambda\in\Lambda$, $\tau\tau_\lambda=\phi_\lambda$. Finally, we see that for any $x\in\mathcal{G}_I$, $\phi\tau(x)=\phi\phi_\lambda(x)$, for some $\lambda\in\Lambda$; and so $\phi\tau(x)=\tau_\lambda(x)=x$. Conversely, the equation $\tau\phi=id_{\mathcal{G}}$ holds because of the universal property of the direct limit $\mathcal{G}$. □

By the above notations, we conclude that $\prod_{i\in I}G_i$, $\prod_{i\in I}V^{**}(G_i)$, and $\prod_{i\in I}G_i/V^{**}(G_i)$ are direct limits of directed systems $\{\prod_{i\in I_\lambda}G_i,\phi_\lambda^{\lambda'},\Lambda\}$, $\{\prod_{i\in I_\lambda}V^{**}(G_i),\bar{\phi}_\lambda^{\lambda'},\Lambda\}$, and $\{\prod_{i\in I_\lambda}G_i/V^{**}(G_i),\psi_\lambda^{\lambda'},\Lambda\}$ respectively, where $\bar{\phi}_\lambda^{\lambda'}$'s are restrictions of $\phi_\lambda^{\lambda'}$'s and $\psi_\lambda^{\lambda'}$'s are quotient homomorphisms induced by $\phi_\lambda^{\lambda'}$'s.

Now, suppose that $\{G_i\}_{i\in I}$ is a family of groups in which for any G_i and G_j $(i,j\in I)$, $\mathcal{V}M(G_i\times G_j)\cong \mathcal{V}M(G_i)\times\mathcal{V}M(G_j)$. By Theorem 2.3, $\prod_{i\in I_\lambda}(V^*)^*(G_i)\subseteq (V^*)^*(\prod_{i\in I_\lambda}G_i)$, for any finite subset I_λ of I. Thus, using Theorem 2.2, we have the following monomorphism

$$\mathcal{V}M(\prod_{i\in I_\lambda}G_i)\hookrightarrow \mathcal{V}M(\frac{\prod_{i\in I_\lambda}G_i}{\prod_{i\in I_\lambda}(V^*)^*(G_i)}).$$

By the fact that direct limit of a directed system preserves exactness of a sequence [8], we obtain the following monomorphism

$$\lim_{\rightarrow}\mathcal{V}M(\prod_{i\in I_\lambda}G_i)\hookrightarrow \lim_{\rightarrow}\mathcal{V}M(\frac{\prod_{i\in I_\lambda}G_i}{\prod_{i\in I_\lambda}(V^*)^*(G_i)}).$$

Using Theorem 2.4, we conclude the monomorphism

$$\mathcal{V}M(\lim_{\rightarrow}\prod_{i\in I_\lambda}G_i)\hookrightarrow \mathcal{V}M(\lim_{\rightarrow}\frac{\prod_{i\in I_\lambda}G_i}{\prod_{i\in I_\lambda}(V^*)^*(G_i)}),$$

and so we have the monomorphism

$$\mathcal{V}M(\prod_{i\in I}G_i)\hookrightarrow \mathcal{V}M(\frac{\prod_{i\in I}G_i}{\prod_{i\in I}(V^*)^*(G_i)}).$$

Finally, by Theorem 2.2, we conclude that

$$\prod_{i\in I}(V^*)^*(G_i)\subseteq (V^*)^*(\prod_{i\in I}G_i).$$

Using these notes, we deduce the following theorem.

Theorem 2.6. *Let $\mathcal{V}$ be a variety, $\{G_i\}_{i\in I}$ be a family of groups such that for any $i,j\in I$, $\mathcal{V}M(G_i\times G_j)\cong \mathcal{V}M(G_i)\times\mathcal{V}M(G_j)$. Then $(V^*)^*(\prod_{i\in I}G_i)=\prod_{i\in I}(V^*)^*(G_i)$. Consequently, $\prod_{i\in I}G_i$ is $\mathcal{V}$-capable if and only if each G_i is $\mathcal{V}$-capable.*

Remark 2.7. *(i) In the above theorem, the sufficient condition*

$$\mathcal{V}M(A\times B)\cong \mathcal{V}M(A)\times\mathcal{V}M(B)$$

is not necessary (see Example 2.3(iii) of [7]). Also, this condition is essential and can not be omitted (see Example 2.3(i),(ii) of [7]).
(ii) It is known that for varieties of abelian and nilpotent groups, and for any groups A and B, $\mathcal{V}M(A\times B)\cong\mathcal{V}M(A)\times\mathcal{V}M(B)\times T$, where T is an abelian group whose elements are tensor products of the elements of A^{ab} and B^{ab} [3], [9]. Hence in these known varieties, the isomorphism $\mathcal{V}M(A\times B)\cong\mathcal{V}M(A)\times\mathcal{V}M(B)$ holds, where both A^{ab} and B^{ab} have finite exponent with $(exp(A^{ab}),exp(B^{ab}))=1$.

In the following, using the main theorem and the above remark, we deduce some corollaries which are generalizations of some results of [7] (Remark 2.4(ii), Corollary 2.5 and Example 2.2).

Corollary 2.8. *Let $\{G_i\}_{i\in I}$ be a family of groups whose abelianizations have mutually coprime exponents. Then $\prod_{i\in I}G_i$ is capable ($\mathcal{N}_c$-capable) if and only if each G_i is capable ($\mathcal{N}_c$-capable).*

Corollary 2.9. *Suppose that $\{G_i\}_{i\in I}$ is a family of groups whose abelianizations have mutually coprime exponents. If $\prod_{i\in I} G_i$ is nilpotent of class at most c_1, then it is $\mathcal{N}_{c_1,\cdots,c_s}$-capable if and only if every G_i is $\mathcal{N}_{c_1,\cdots,c_s}$-capable.*

Corollary 2.10. *If $\{G_i\}_{i\in I}$ is a family of perfect groups, then $\prod_{i\in I} G_i$ is $\mathcal{V}$-capable if and only if each G_i is $\mathcal{V}$-capable, where $\mathcal{V}$ may be each of these three varieties:*

(1) *variety of abelian groups,*
(2) *variety of nilpotent groups,*
(3) *variety of polynilpotent groups.*

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