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ON GROUPS WITH SPECIFIED QUOTIENT POWER GRAPHS

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Abstract. In this paper we study some relations between the power and quotient power graph of a finite group. These interesting relations motivate us to find some graph theoretical properties of the quotient power graph and the proper quotient power graph of a finite group G. In addition, we classify those groups whose quotient (proper quotient) power graphs are isomorphic to trees or paths.

1. Introduction

There is a large literature which is devoted to studying the ways of associating a graph to a group for the purpose of investigating the algebraic structure using properties of the associated graph (see for example [2, 3, 11, 16, 15, 17]). The investigation of graphs related to groups as well as other algebraic structures is very important, because such graphs have valuable applications (see [18]) and are related to automata theory (see [12, 13]). Kelarev and Quinn [14] defined the *(directed) power graph* $\mathscr{G}(S)$ of a semigroup S as a (directed) graph in which the set of vertices is S and for $x, y \in S$, there is an arc from x to y if and only if $x \neq y$ and $y = x^m$, for some positive integer m. The *power graph* $\mathscr{G}(S)$ of a semigroup S was defined by Chakrabarty et al. [8] as a graph with vertex set S and two distinct vertices x and y joined if one is a power of the other. They proved that for a finite group G, the power graph $\mathscr{G}(G)$ is complete if and only if G is a cyclic group of order 1 or p^m, for some prime number p and some positive integer m. In [6, 7], Cameron and Ghosh obtained interesting results about power graphs of finite groups. In addition, Mirzargar et al. [19], considered some graph theoretical properties of the power graph $\mathscr{G}(G)$ that can be related to the group theoretical properties of G, such as clique number, independence number and chromatic number. A recent survey [1], has collected the main and

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beautiful results of the theory of power graphs, which seems to be a very interesting idea for future research.

If G is a finite group, then it can be easily shown that the power graph $\mathscr{G}(G)$ is connected of diameter 2, because the identity element 1_G is adjacent to every element of G. Hence in [4, 5], it is focused on the 2-connectivity of $\mathscr{G}(G)$. The 2-connectivity of $\mathscr{G}(G)$ can be studied by the *proper power graph* $\mathscr{G}^*(G)$, that is the 1-cut subgraph of $\mathscr{G}(G)$, which is obtained by omitting the vertex 1_G and all its incident edges. Actually, it is easy to check that $\mathscr{G}(G)$ is 2-connected if and only if $\mathscr{G}^*(G)$ is connected. In [4], we calculated the number of connected components of S_n and showed that $\mathscr{G}(S_n)$ is 2-connected if and only if $n = 2$ or both $n, n-1$ are not prime.

The complexity of a graph is reduced considerably when its quotient graph is considered instead. The possible equivalence relations imply very different levels of simplification and, as a consequence, have different impact on the properties of the graph. In [4], the *quotient power graph*, $\widetilde{\mathscr{G}}(G)$, and *proper quotient power graph*, $\widetilde{\mathscr{G}}^*(G)$, as quotient graphs of $\mathscr{G}(G)$ and $\mathscr{G}^*(G)$ respectively, are considered. Here, the elements of G which generate the same cyclic subgroup are identified in a unique vertex.

In this paper, we focus on the investigation of (proper) quotient power graphs of finite groups. We study some graph theoretical properties of the (proper) quotient power graphs of a finite group G. In addition, groups whose (proper) quotient power graphs are isomorphic to trees or paths are classified. The following three theorems are the main theorems of this paper.

Theorem A. *Let G be a finite group and $\mathcal{F}(G)$ be the Fitting subgroup of G. Then $\widetilde{\mathscr{G}}(G)$ is isomorphic to a tree if and only if G is one of the following groups:*
Case 1) G is a p-group of exponent p, p is a prime.
Case 2) G is a group of order p^mq as follows; where p and q are primes.

- *(i)* $|G| = p^mq$, *where* $3 \leq p < q, m \geq 3, |\mathcal{F}(G)| = p^{m-1}$ *and* $|G : G'| = p$.
- *(ii)* $|G| = p^mq$, *where* $3 \leq q < p, m \geq 1$ *and* $|\mathcal{F}(G)| = |G'| = p^m$.
- *(iii)* $|G| = 2^mp$, *where* $p \geq 3, m \geq 2$ *and* $|\mathcal{F}(G)| = |G'| = 2^m$.
- *(iv)* $|G| = 2p^m$, *where* $p \geq 3, m \geq 1, |\mathcal{F}(G)| = |G'| = p^m$ *and* $\mathcal{F}(G)$ *is elementary abelian.*

Case 3) $G \cong A_5$.

Theorem B. *Let G be a finite group. Then $\widetilde{\mathscr{G}}^*(G)$ is a path if and only if G is isomorphic to one of the groups $\mathbb{Z}_p, \mathbb{Z}_{p^2}$ and $\mathbb{Z}_{pq}$, where p, q are prime numbers.*

Theorem C. *Let G be a finite group. Then $\widetilde{\mathscr{G}}^*(G)$ is a bipartite graph if and only if $\widetilde{\mathscr{G}}^*(G)$ is connected and the order of each non-trivial element of G is a prime or a product of two primes (not necessary distinct).*

2. Definitions and preliminaries

All groups and graphs in this paper are assumed to be finite. Throughout the paper by a graph we mean a simple graphs which has no multiple edges or loops. We follow the terminology and notation of [21], for groups and [1], for (quotient) power graphs. All other notations for graphs are from [22].

Let $\Gamma = (V, E)$ be a graph and $a, b \in V$. The distance $d_\Gamma(a, b)$, between a and b in Γ is defined as the length of the shortest path connecting them. The *girth* of Γ, $\mathbf{g}(\Gamma)$, is the length of shortest cycle within the graph. Let $x \in V$ be a fixed vertex in the graph Γ. The *x-cut subgraph* of Γ is given by $\Gamma - x = (V \setminus \{x\}, E_x)$, where $E_x = E \setminus \{e \in E : e$ is incident to $x\}$. A graph is called 2-*connected* if, for each $x \in V$, the graph $\Gamma - x$ is connected. Suppose that $\overrightarrow{\Gamma}$ is a directed graph. The outdegree of a vertex a is the number of arcs which get out from a and is denoted by $d^+(a)$. Also, the indegree of a is the number of arcs entering to a denoted by $d^-(a)$. We use $a \to b$, to show an arc from a to b.

2.1. **The power graphs.** Let G be a finite group, with identity element 1_G and put $G_0 = G \setminus \{1_G\}$. The power graph $\mathscr{G}(G) = (V, E)$, of G is a graph with $V = G$ and for $x, y \in G$, with $x \neq y, \{x, y\} \in E$ if there exists $m \in \mathbb{N}$ such that $x = y^m$ or $y = x^m$. Since the cut graphs $\mathscr{G}(G) - x$, for $x \in G_0$, are trivially connected, then $\mathscr{G}(G)$ is 2-connected if and only if the proper power graph, defined as the cut graph $\mathscr{G}^*(G) = \mathscr{G}(G) - 1_G$ is connected.

2.2. **The quotient graphs.** Let $\Gamma = (V, E)$ be a graph. Assume that an equivalence relation, say $\sim$, is defined on the set V. Consider the quotient set $[V] = V/\sim$ and denote its elements with $[x]$, for $x \in V$. We say that there is an edge $\{[x], [y]\} \in [E]$ between $[x] \in [V]$ and $[y] \in [V]$ if $[x] \neq [y]$ and there exist $x', y' \in V$ such that $x' \sim x, y' \sim y$ and $\{x', y'\} \in E$. This defines the graph $\Gamma/\sim = ([V], [E])$, called the quotient graph of Γ with respect to $\sim$.

2.3. **The quotient power graphs.** To deal with the graphs $\mathscr{G}(G)$ and $\mathscr{G}^*(G)$ and simplify their complexity, we consider two quotient graphs in which the elements of G, generating the same cyclic subgroup, are identified in a unique vertex.

Define the relation $\sim$ on G as follows: if $x, y \in G$, then $x \sim y$ if and only if $\langle x \rangle = \langle y \rangle$. It is immediate to check that $\sim$ is an equivalence relation and that $[x] = \{x^m : 1 \leq m \leq o(x),\ (m, o(x)) = 1\}$ is of size $\phi(o(x))$, where ϕ denotes the Euler's totient function. We define the *order* of $[x] \in [G]$ as the order of x : this definition is well posed because if $\langle x \rangle = \langle y \rangle$, then $o(x) = o(y)$. The quotient graph $\mathscr{G}(G)/\sim$ will be denoted by $\widetilde{\mathscr{G}}(G)$ and is called the quotient power graph. Its vertex set is $[G] = G/\sim$ and its edge set $[E]$. By the definition of quotient graph, there is an edge between two distinct vertices $[x]$ and $[y]$ if and only if there exists $x' \in [x]$ and $y' \in [y]$ such that $\{x', y'\} \in E$, that is, x', y' are, in a suitable order, one the positive power of the other.

Remark 2.1. [4, Remark 1] *For each $x, y \in G$, such that $[x] \neq [y]$, $\{[x], [y]\} \in [E]$ if and only if $\{x, y\} \in E$.*

Since $[x] = [1_G]$ if and only if $x = 1_G$, G_0 is a union of equivalence classes. Thus it is possible to define the quotient graph $\mathscr{G}^*(G)/\sim$, called the proper quotient power graph and denoted by $\widetilde{\mathscr{G}}^*(G)$. Note that $\widetilde{\mathscr{G}}^*(G)$ may be viewed also as the cut graph $\widetilde{\mathscr{G}}(G) - [1_G]$. In particular its edge set is $[E]_0 = [E]_{[1_G]}$.

In this paper we set $\mathscr{G}(G) = (V, E), \widetilde{\mathscr{G}}(G) = ([V], [E]), \mathscr{G}^*(G) = (V_0, E_0)$ and $\widetilde{\mathscr{G}}^*(G) = ([V]_0, [E]_0)$.

We use the following lemma in the next sections.

Lemma 2.2. [4, Lemma 4.1] *Let* $c_0(G), \widetilde{c}_0(G)$, *be the number of connected components of* $\mathscr{G}^*(G), \widetilde{\mathscr{G}}^*(G)$ *respectively. Then* $c_0(G) = \widetilde{c}_0(G)$.

3. Relations between $\mathscr{G}(G)$ and $\widetilde{\mathscr{G}}(G)$

In this section we present some relations between $\mathscr{G}(G)$ and $\widetilde{\mathscr{G}}(G)$. These relations motivate us to investigate the quotient power graphs and the proper quotient power graphs of finite groups. Also we discuss the quotient power graphs of finite groups.

Definition 3.1. Let $\Gamma = (V, E)$ be a graph and $\sim$ be an equivalence relation on V. We say that $\sim$ is *strong-tame* if for each $x, y \in V$, with $x \neq y$, if $x \sim y$ then $\{x, y\} \in E$. Moreover we say that $[\Gamma] = \Gamma/\sim$ is a *strong-tame* quotient of Γ, with respect to $\sim$, if $\sim$ is strong-tame. *By Remark 2.1,* $\widetilde{\mathscr{G}}(G)$ *is strong-tame quotient of* $\mathscr{G}(G)$.

Theorem 3.2. *Let* $[\Gamma]$ *be a strong-tame quotient of* Γ.

(*i*) *Let* $x, y \in V$ *and* $[x] \neq [y]$. *Then* $d_{[\Gamma]}([x], [y]) = d_\Gamma(x, y)$.

(*ii*) *Let* d *be the length of a longest path in* Γ. *Then* $d = -1 + \sum_{[a]\in[P]} o([a])$, *where* $[P]$ *is a longest path in* $[\Gamma]$.

(*iii*) *Let* $[\Gamma]$ *be a Hamiltonian graph. Then* Γ *is also Hamiltonian.*

Proof. Suppose $d_{\mathscr{G}(G)}(x, y) = t$ and $P : x = v_0, v_1, \ldots, v_{t-1}, v_t = y$ is a shortest path between x and y in Γ. Consider the sequence $[P] : [x] = [v_0], [v_1], \ldots, [v_{t-1}], [v_t] = [y]$ in $[\Gamma]$. Suppose that $[v_j] = [v_k]$, where $0 \leq j < k \leq t$. Then $\{v_j, v_k\} \in E$. If $j \neq k-1$ then the path $P_0 : x = v_0, v_1, \ldots, v_j, v_k, \ldots, v_{t-1}, v_t = y$ is a path between x and y which is shorter than P, a contradiction. Now suppose that $j = k-1$. If $j = 0$ then there exists an edge between v_j and v_{k+1}. Thus the path $P_1 : x = v_j, v_{k+1}, \ldots, v_{t-1}, v_t = y$ is a shorter path between x and y, which is another contradiction. If $j > 0$ then it is easy to see that there exist an edges between v_{j-1} and v_k. So we may consider the path $P_2 : x = v_0, v_1, \ldots, v_{j-1}, v_k, \ldots, v_{t-1}, v_t = y$, of smaller length between x and y, a contradiction. Hence for every $0 \leq j \neq k \leq t$, $[v_j] \neq [v_k]$ and so the sequence $[P] : [x], [v_1], \ldots, [v_{t-1}], [y]$ is a shortest path between $[x]$ and $[y]$ in $[\Gamma]$. Therefore $d_{[\Gamma]}([x], [y]) = t$.

Conversely suppose $d_{[\Gamma]}([x], [y]) = t$ and $[Q] : [x], [v_1], \ldots, [v_{t-1}], [y]$ is a path of minimum length in $[\Gamma]$. By a similar argument as in the previous paragraph we may find the path $Q : x, v_1, \ldots, v_{t-1}, y$ between x and y in Γ of minimum length. Therefore $d_\Gamma(x, y) = t$. This proves (i).

For proving (ii), suppose that $[P] : [v_0], [v_1], \ldots, [v_{d'}], [v_{d'}]$. For every integer $0 \leq i \leq d'$, $[v_i]$ contain $o([v_i])$ vertices of Γ. Since $[\Gamma]$ is *strong-tame* then these vertices form a subgraph of Γ isomorphic to $K_{o([v_i])}$. For every integer $0 \leq i \leq d'$, we consider the path $v_{i_1} = v_i, v_{i_2}, \ldots, v_{i_{o([v_i])-1}}$ of length $o([v_i]) - 1$ which is a longest path in this subgraph. Now by adding these paths to the path $[P]$, we obtain a longest path in Γ. Hence $d = d' + \sum_{[a]\in[P]}(o([a]) - 1) = d' - (d'+1) + \sum_{[a]\in[P]} o([a]) = -1 + \sum_{[a]\in[P]} o([a])$.

Finally for item (iii), let $[\Gamma]$ be Hamiltonian. Since $[\Gamma]$ is Hamiltonian, we can find a Hamiltonian cycle, say $[C] : [v_0], [v_1], \ldots, [v_{d'}], [v_{d'}], [v_0]$. Let $[P] = [v_0], [v_1], \ldots, [v_{d'}], [v_{d'}]$. By a similar method used in (ii), we obtain the corresponding path to $[P]$ in Γ, say P. Suppose that the length of P is d.

By (ii), $d = -1 + \sum_{[a]\in[P]} o([a]) = |V| - 1$. This implies that P is a Hamiltonian path in Γ. Now since $\{[v_{d'}], [v_0]\} \in [E]$ and $[\Gamma]$ is *strong-tame*, then we have $\{v_{i_{o([v_{d'}])-1}}, v_0\} \in E$. By adding this edge to P, we obtain a Hamiltonian cycle in Γ. So Γ is a Hamiltonian graph. □

Corollary 3.3. (i) *Let* $x, y \in V$ *and* $[x] \neq [y]$. *Then*

$$d_{\widetilde{\mathscr{G}}(G)}([x],[y]) = d_{\mathscr{G}(G)}(x,y).$$

(ii) *Let d be the length of a longest path in* $\mathscr{G}(G)$. *Then*

$$d = -1 + \sum_{[a]\in[P]} \phi(o(a)),$$

where $[P]$ is a longest path in $\widetilde{\mathscr{G}}(G)$. (iii) *Let* $\widetilde{\mathscr{G}}(G)$ *be a Hamiltonian graph. Then* $\mathscr{G}(G)$ *is also Hamiltonian.*

Proof. These are immediate consequence of the fact that $\widetilde{\mathscr{G}}(G)$ is *strong-tame* quotient of $\mathscr{G}(G)$ and for every $[a] \in [G]$, $o([a]) = \phi(o(a))$. □

In [8], authors showed that the number of edges e of $\mathscr{G}(G)$ is given by $2e = \sum_{a\in G}\{2o(a) - \phi(o(a)) - 1\}$. Now we calculate, $[e]$, the number of edges in $\widetilde{\mathscr{G}}(G)$.

Theorem 3.4. *Suppose ϕ is the Euler's totient function. Then:*
(i) $|V| = \sum_{[a]\in[V]} \phi(o(a))$ *and* $|[V]| = \sum_{a\in V} \dfrac{1}{\phi(o(a))}$.
(ii) $[e] = \sum_{a\in G} \dfrac{1}{\phi(o(a))}(-1 + \sum_{i=1}^{o(a)} \dfrac{1}{\phi(o(a^i))})$, *where* $o(a^i) = \dfrac{o(a)}{(o(a), i)}$, *for every element* $a \in G$.

Proof. (i) Let $[a]$ be the class of a under relation " $\sim$ ". Then $o([a]) = \phi(o(a))$. Hence $|V| = \sum_{[a]\in[V]} \phi(o(a))$. Since $o([a]) = \phi(o(a))$, we have $\sum_{b\in[a]} \dfrac{1}{\phi(o(a))} = 1$. Hence $|[V]| = \sum_{a\in V} \dfrac{1}{\phi(o(a))}$.

(ii) In $\overrightarrow{\widetilde{\mathscr{G}}(G)}$, if there exist two arcs $[a] \to [b]$ and $[b] \to [a]$, then $a = b^n$ and $b = a^m$, for some positive integers n and m, respectively. This implies that $[a] = [b]$. Thus there are no bidirected arcs in $\overrightarrow{\widetilde{\mathscr{G}}(G)}$. So the number of edges in $\widetilde{\mathscr{G}}(G)$ is equal to the number of arcs in $\overrightarrow{\widetilde{\mathscr{G}}(G)}$. Hence instead of calculating the number of edges in $\widetilde{\mathscr{G}}(G)$, we calculate the number of arcs in $\overrightarrow{\widetilde{\mathscr{G}}(G)}$. Suppose $[\overrightarrow{e}]$ is the number of arcs in $\overrightarrow{\widetilde{\mathscr{G}}(G)}$. Since $[\overrightarrow{e}] = \sum_{[a]\in[G]} d^+([a])$, then for obtaining $[\overrightarrow{e}]$, it is enough to calculate the outdegree $d^+([a])$, for every $[a] \in [G]$. For this purpose, we construct the graph $\overrightarrow{\widetilde{\mathscr{G}}(G)}$ from the graph $\overrightarrow{\mathscr{G}(G)}$ by considering the equivalence relation " $\sim$ ". Let $t_a = d^+(a) - d^+([a])$. Since $d^+(a) = o(a) - 1$, for calculating $d^+([a])$, it is enough to obtain t_a. Suppose that $S = \{a, a^2, a^3, \ldots, a^{o(a)}\}$ and $[a^{i_1}] = [a], [a^{i_2}], \ldots, [a^{i_k}]$ be distinct classes of relation " $\sim$ " on the set S. For every integer $1 \leq j \leq k$, all vertices of the set $[a^{i_j}]$ are coincide with one vertex $[a^{i_j}]$ in $\overrightarrow{\widetilde{\mathscr{G}}(G)}$ and $\phi(o(a^{i_j})) - 1$ arcs are omitted from $\overrightarrow{\mathscr{G}(G)}$ to obtain $\overrightarrow{\widetilde{\mathscr{G}}(G)}$. (Note that for $[a^{i_1}] = [a]$, there exist $\phi(o(a)) - 1$ arcs from a to all vertices in $[a]$, which are omitted in $\overrightarrow{\widetilde{\mathscr{G}}(G)}$.) Thus $t_a = \sum_{j=1}^{k} \phi(o(a^{i_j})) - 1$. Since $o([a^{i_j}]) = \phi(o(a^{i_j}))$ and for every $a^t \in [a^{i_j}], \phi(o(a^t)) = \phi(o(a^{i_j}))$, then $t_a = \sum_{j=1}^{k} \sum_{a^t\in[a^{i_j}]} \dfrac{\phi(o(a^t)) - 1}{\phi(o(a^t))}$. Hence $t_a = \sum_{t=1}^{o(a)} \dfrac{\phi(o(a^t)) - 1}{\phi(o(a^t))} =$

$\sum_{t=1}^{o(a)}(1-\dfrac{1}{\phi(o(a^t))}) = o(a) - \sum_{t=1}^{o(a)}\dfrac{1}{\phi(o(a^t))}$. Since $d^+(a) = o(a)-1$, then $d^+([a]) = d^+(a) - t_a = -1 + \sum_{t=1}^{o(a)}\dfrac{1}{\phi(o(a^t))}$. Therefore $[\overrightarrow{e}] = \sum_{[a]\in[G]} d^+([a]) = \sum_{[a]\in[G]}(-1+\sum_{t=1}^{o(a)}\dfrac{1}{\phi(o(a^t))})$. By the fact that $o([a]) = \phi(o(a))$, we have $[e] = [\overrightarrow{e}] = \sum_{a\in G}\dfrac{1}{\phi(o(a))}(-1+\sum_{t=1}^{o(a)}\dfrac{1}{\phi(o(a^t))})$. □

In[20], authors showed that for a finite group $G, \mathscr{G}(G)$ is a tree if and only if G is an elementary abelian 2-group. In the following, we characterize finite groups G, which $\widetilde{\mathscr{G}}(G)$ are isomorphic to trees.

Let $\mathcal{P}$ be the class of the finite groups having all (nontrivial) elements of prime order. Deaconescu in [9], proved the following theorem.

Theorem 3.5. [9, Main Theorem] *Let G be a $\mathcal{P}$-group. Then one the following cases occurs:*
I. G is a p-group of exponent p, p is a prime.
II. G is a group of order $p^m q$ as follows; where p and q are primes.

- *(i)* $|G| = p^m q$, *where* $3 \le p < q, m \ge 3, |\mathcal{F}(G)| = p^{m-1}$ *and* $|G : G'| = p$.
- *(ii)* $|G| = p^m q$, *where* $3 \le q < p, m \ge 1$ *and* $|\mathcal{F}(G)| = |G'| = p^m$.
- *(iii)* $|G| = 2^m p$, *where* $p \ge 3, m \ge 2$ *and* $|\mathcal{F}(G)| = |G'| = 2^m$.
- *(iv)* $|G| = 2p^m$, *where* $p \ge 3, m \ge 1, |\mathcal{F}(G)| = |G'| = p^m$ *and* $\mathcal{F}(G)$ *is elementary abelian.*

III. $G \cong A_5$.

Now we prove the following theorem.

Theorem 3.6. *Let G be a finite group. Then $\widetilde{\mathscr{G}}(G)$ is isomorphic to a tree if and only if G is one of the following groups:*
Case 1) G is a p-group of exponent p, p is a prime.
Case 2) G is a group of order $p^m q$ as follows; where p and q are primes.

- *(i)* $|G| = p^m q$, *where* $3 \le p < q, m \ge 3, |\mathcal{F}(G)| = p^{m-1}$ *and* $|G : G'| = p$.
- *(ii)* $|G| = p^m q$, *where* $3 \le q < p, m \ge 1$ *and* $|\mathcal{F}(G)| = |G'| = p^m$.
- *(iii)* $|G| = 2^m p$, *where* $p \ge 3, m \ge 2$ *and* $|\mathcal{F}(G)| = |G'| = 2^m$.
- *(iv)* $|G| = 2p^m$, *where* $p \ge 3, m \ge 1, |\mathcal{F}(G)| = |G'| = p^m$ *and* $\mathcal{F}(G)$ *is elementary abelian.*

Case 3) $G \cong A_5$.

Proof. Let $\widetilde{\mathscr{G}}(G)$ be a tree. Suppose that there exists an element $a \in G$ such that pq divides $o(a)$, where p and q are prime numbers (not necessarily distinct). Then there exists the cycle: $[1_G], [a], [a^p], [1_G]$ in $\widetilde{\mathscr{G}}(G)$. Thus $\mathbf{g}(\widetilde{\mathscr{G}}(G)) = 3$ and so $\widetilde{\mathscr{G}}(G)$ is not a tree, which is a contradiction. Hence all non-trivial elements of G must be of prime order.

Conversely suppose that all non-trivial elements of G are of prime order. If $\widetilde{\mathscr{G}}(G)$ is not a tree then there exists a cycle in $\widetilde{\mathscr{G}}(G)$ say $[1_G], [a_1], [a_2], \ldots, [a_k]$,
$[1_G]$. Since $\{[a_1], [a_2]\} \in [E]$ we may assume, without loss of generality, that $a_2 = {a_1}^m$ for some positive integer $m > 1$. Since $o(a_1)$ is a prime number, then $[a_1] = [a_2]$, which is a contradiction. Hence $\widetilde{\mathscr{G}}(G)$ is a tree.

Now by using Theorem 3.5, the proof is complete. □

Corollary 3.7. *Let G be a finite group. Then $\widetilde{\mathscr{G}}(G)$ is a tree if and only if $\widetilde{\mathscr{G}}(G)$ is a $(|[V]|-1)$-star.*

Proof. Let $\widetilde{\mathscr{G}}(G)$ be a tree. In the proof of Theorem 3.6, we show that all non-trivial elements of G are of prime order. Now since $[1_G]$ is adjacent with all other elements of $[G]$, then the assertion is established.

The converse is straightforward. □

Suppose $\Gamma = (V, E)$ is a graph. A non-empty subset $X \subseteq V$ is called a *clique* if the induced subgraph on X is a complete graph. The maximum size of a clique in Γ is called *clique number* of Γ and denoted by $\omega(\Gamma)$. The *chromatic number* of Γ is the smallest number of colors needed to color the vertices of Γ so that no two adjacent vertices share the same color. This number is denoted by $\chi(\Gamma)$.

A graph Γ is *perfect* if for every induced subgraph Λ of Γ, the chromatic number of Λ is equal to the size of the largest clique of Λ.

Theorem 3.8. [10, Theorem 3.1] *The power graph of a group is perfect.*

By a similar argument to [10, Theorem 3.1], we immediately conclude the following theorem.

Theorem 3.9. *Suppose G is a finite group. Then the power graph $\widetilde{\mathscr{G}}(G)$ is perfect and so $\omega(\widetilde{\mathscr{G}}(G)) = \chi(\widetilde{\mathscr{G}}(G))$.*

In [8], Chakrabarty et al. proved that for a finite group G, the power graph $\mathscr{G}(G)$ is complete if and only if G is a cyclic group of order 1 or p^m, for some prime number p and some positive integer m. There exist a similar result for quotient power graphs.

Theorem 3.10. *Let G be a finite group. Then $\widetilde{\mathscr{G}}(G)$ is complete if and only if G is of order 1 or p^m for some positive integer m.*

Proof. Since for every element $x \in G$, the induced subgraph of the set $[x]$ of $P(G)$ is complete, then $P(G)$ is complete if and only if $\widetilde{\mathscr{G}}(G)$ is complete. By [8, Theorem 2.12], the proof is complete. □

Mirzargar et al. [19], considered some graph theoretical properties of the power graph $\mathscr{G}(G)$ that can be related to the group theoretical properties of G, such as clique number, independence number and chromatic number. Now we consider the quotient power graph $\widetilde{\mathscr{G}}(G)$ and obtain similar results for $\widetilde{\mathscr{G}}(G)$.

Suppose $D(n)$ denotes the set of all positive divisors of n. It is well-known that $(D(n), |)$ is a distributive lattice. In the following we apply the structure of this lattice to compute the clique and chromatic number of $\widetilde{\mathscr{G}}(\mathbb{Z}_n)$.

Lemma 3.11. *Suppose G is a group and $A \subseteq [G]$. The vertices of A constitute a complete subgraph in $\widetilde{\mathscr{G}}(G)$ if and only if $\{< x > | x \in A\}$ is a chain.*

Proof. By Theorem 3.10, C is a clique in $P(G)$ if and only if $[C]$ is a clique in $\widetilde{\mathscr{G}}(G)$, where $[C]$ is the corresponding subgraph to C in $\widetilde{\mathscr{G}}(G)$. Now by a similar argument to [19, Lemma 1], the sentence is established. □

Theorem 3.12. *Suppose $n = {p_1}^{\alpha_1}{p_2}^{\alpha_2}\cdots{p_r}^{\alpha_r}$, where $p_1 < p_2 < \cdots < p_r$ are prime numbers. Then*

$$\omega(\mathscr{G}(\mathbb{Z}_n)) = \chi(\mathscr{G}(\mathbb{Z}_n)) = 1 + \Sigma_{i=1}^{r}\alpha_i.$$

Proof. The proof is similar to [19, Theorem 2]. □

The *exponent* of a finite group G is defined as the least common multiple of all elements of G, denoted by $Exp(G)$. If G has an element a such that $o(a) = Exp(G)$, then G is called *full exponent.* It is easy to see that the nilpotent groups are full exponent.

Theorem 3.13. *Let G be a full exponent group and $n = Exp(G) = {p_1}^{\beta_1}{p_2}^{\beta_2}\cdots{p_r}^{\beta_r}$, where $p_1 < p_2 < \cdots < p_r$ are prime numbers. If x be an element of order n then*

$$\omega(\mathscr{G}(\mathbb{Z}_n)) = \chi(\mathscr{G}(\mathbb{Z}_n)) = 1 + \Sigma_{i=1}^{r}\beta_i.$$

Proof. The proof is similar to the proof of [19, Theorem 3]. □

Corollary 3.14. *Let G be a finite group. Then the quotient power graph $\widetilde{\mathscr{G}}(G)$ is planar if and only if $\pi_e(G) \subseteq \{1, p, p^2, p^3, pq, p^2q\}$, where p, q are distinct prime numbers.*

Proof. Suppose that $\widetilde{\mathscr{G}}(G)$ is planar and $[x] \in \widetilde{V}$. Then $\widetilde{\mathscr{G}}(G)$ does not have the complete graph K_5 as its induced subgraph and so by Theorem 3.13, $o(x) \in \{1, p, p^2, p^3, pq, p^2q, pqr\}$, where p, q, r are distinct prime numbers. Suppose that $o(x) = pqr$ and $\widetilde{\mathscr{G}}(< x >) = ([V], [E])$. Then $|[V]| = 8$ and $|[E]| = 19$. By [22, 6.1.23.Theorem], $\widetilde{\mathscr{G}}(< x >)$ is not planar and so $\widetilde{\mathscr{G}}(G)$ is not planar. But it is easy to check that in other cases $\widetilde{\mathscr{G}}(G)$ is planar. □

4. The proper quotient power graphs of finite groups

By considering $\widetilde{\mathscr{G}}^*(G)$, we can obtain some information about $\widetilde{\mathscr{G}}(G)$. So in this section we discuss the proper quotient power graphs of finite groups. We classify all groups G where $\widetilde{\mathscr{G}}^*(G)$ is isomorphic to one of trees, paths or bipartite graphs.

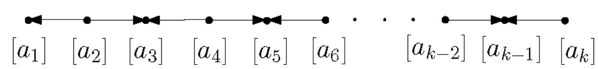
Lemma 4.1. *Let G be a finite group and $\widetilde{\mathscr{G}}^*(G)$ be a path. Then all non-trivial elements of G are of order p, p^2 or pq, for distinct prime numbers p and q.*

Proof. Let $\widetilde{\mathscr{G}}^*(G)$ be a path and $[1_G] \neq [x] \in [G]$. Suppose that $d^+([x]) \geq 1$ and $d^-([x]) \geq 1$. Then there exist two arcs $[y] \to [x]$ and $[x] \to [z]$ in $\overrightarrow{\widetilde{\mathscr{G}}^*(G)}$, where $[y], [z] \in [G] \setminus [1_G]$. So $x = y^m$ and $z = x^n$, for some positive integers m and n. Thus $z = y^{mn}$ and so we have an arc from $[y]$ to $[z]$. Hence $\widetilde{\mathscr{G}}^*(G)$ has a cycle of length three which is a contradiction. Therefore we may distinguish the following cases:
Case 1) $d^+([x]) = 0$ and $d^-([x]) = 0, 1$ or 2. Since $d^+([x]) = 0$, then the order of x must be a prime number.
Case 2) $d^+([x]) = 1$ and $d^-([x]) = 0$. Since $d^+([x]) = 1$, we have $o(x) = p^2$, for some prime number p.
Case 3) $d^+([x]) = 2$ and $d^-([x]) = 0$. Since $d^+([x]) = 2$, then we conclude that $o(x) = pq$, where p and q are prime numbers. Hence the order of each element of G is a prime number or the product of two prime numbers. Suppose that there exist three elements $x', y', z' \in G$ such that $o(x') = p, o(y') = q$ and

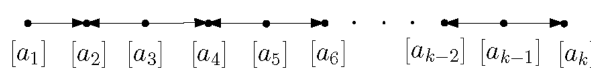
$o(z') = r$, where p, q and r are distinct prime numbers. Then we have $o(x'y'z') = pqr$, a contradiction. Also if there exist two elements $g, h \in G$ such that $o(g) = p^2$ and $o(h) = q^2$, where p and q are distinct prime numbers, then we have $o(gh) = p^2q^2$, a contradiction. Hence all non-trivial elements of G are of order p, p^2 or pq. □

Theorem 4.2. *Let G be a finite group. Then $\widetilde{\mathscr{G}}^*(G)$ is a path if and only if G is isomorphic to one of the groups $\mathbb{Z}_p, \mathbb{Z}_{p^2}$ and $\mathbb{Z}_{pq}$, where p, q are prime numbers.*

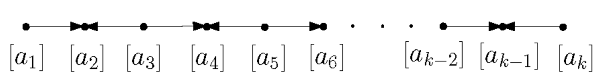
Proof. Let $\widetilde{\mathscr{G}}^*(G)$ be a path. Then by Lemma 4.1, all elements of G has order p, p^2 or pq, for some prime numbers p and q. Suppose that the vertices $[a_1]$ and $[a_k]$ are the ends of this path. If $[a_1] = [a_k]$ then $o(a_1) = p$ and so $G \cong \mathbb{Z}_p$. Suppose that $[a_1] \neq [a_k]$. We consider the following cases:
Case 1) $d^+([a_1]) = 0, d^-([a_1]) = 1, d^+([a_k]) = 1$ and $d^-([a_k]) = 0$. So $\overrightarrow{\mathscr{G}^*}(G)$ is the following path: If



$k = 2$, then $o(a_1) = p, o(a_2) = p^2$ and $a_1 = {a_2}^p$. Thus $G \cong < a_2 > \cong \mathbb{Z}_{p^2}$ and so $\widetilde{\mathscr{G}}^*(G) \cong K_2$. Suppose that $k = 3$. Since $d^-([a_1]) = d^+([a_k]) = 1$ then we have two arcs $a_2 \to a_1$ and $a_3 \to a_2$. So there exist an arc from a_3 to a_1 and we may consider a cycle $[a_1], [a_2], [a_3], [a_1]$ in $\widetilde{\mathscr{G}}^*(G)$, which contradicts the assumption that $\widetilde{\mathscr{G}}^*(G)$ is a path. Suppose that $k \geq 4$. Then $o(a_2) = pq$ and $o(a_k) = p^2$ or q^2. Thus p^2q divides $o(a_2a_k)$ or q^2p divides $o(a_2a_k)$. By Lemma 4.1, this is a contradiction.
Case 2) $d^+([a_1]) = 1, d^-([a_1]) = 0, d^+([a_k]) = 0$ and $d^-([a_k]) = 1$. So $\overrightarrow{\mathscr{G}^*}(G)$ is the following path: By

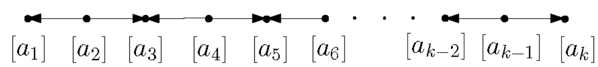


rearranging the indices, we obtain the Case 1).
Case 3) $d^+([a_1]) = 1, d^-([a_1]) = 0, d^+([a_k]) = 1$ and $d^-([a_k]) = 0$. So $\overrightarrow{\mathscr{G}^*}(G)$ is the following path: Suppose that $k = 2$. Then $d^+([a_1]) = d^+([a_2]) = 1$ implies that $[a_1] = [a_2]$, a contradiction. If $k = 3$,



then $o(a_1) = o(a_3) = p^2$ and $o(a_2) = p$. Then $< a_1 >$ and $< a_3 >$ are the only Sylow p-subgroups of G. So the number of Sylow p-subgroups is not of the form $1 + tp$, for some integer t. This is a contradiction. Suppose that $k = 4$. Since $d^+([a_1]) = d^+([a_4]) = 1$ then we have two arcs $a_1 \to a_2$ and $a_4 \to a_3$. On the other hand there exists an arc from a_2 to a_3 or an arc from a_3 to a_2. In first case we have the cycle $[a_1], [a_2], [a_3], [a_1]$ and in second case we have the cycle $[a_4], [a_3], [a_2], [a_4]$ in

$\widetilde{\mathscr{G}}^*(G)$. Since $\widetilde{\mathscr{G}}^*(G)$ is a path, this is a contradiction. Now suppose that $k \geq 5$, then $o(a_1) = p^2$ and $o(a_{k-2}) = pq$. Thus p^2q divides $o(a_1a_{k-2})$, which is, by Lemma 4.1, a contradiction.
Case 4) $d^+([a_1]) = 0, d^-([a_1]) = 1, d^+([a_k]) = 0$ and $d^-([a_k]) = 1$. So $\overrightarrow{\widetilde{\mathscr{G}}^*}(G)$ is the following path:
Suppose that $k = 2$. Then $d^-([a_1]) = d^-([a_2]) = 1$ implies that $[a_1] = [a_2]$, a contradiction. If $k = 3$,



then $o(a_1) = p, o(a_2) = pq$ and $o(a_3) = q$. Also we have $a_1 = a_2^q$ and $a_3 = a_2^p$. Thus $G \cong < a_2 > \cong \mathbb{Z}_{pq}$. In this case $\widetilde{\mathscr{G}}^*(G)$ is a path of length two. Suppose that $k = 4$. Since $d^-([a_1]) = d^-([a_4]) = 1$, we have two arcs $a_2 \to a_1$ and $a_3 \to a_4$. On the other hand there exists either an arc from a_2 to a_3 or an arc from a_3 to a_2. In the first case we have the cycle $[a_2], [a_3], [a_4], [a_2]$ and in the second case we have the cycle $[a_1], [a_2], [a_3], [a_1]$ in $\widetilde{\mathscr{G}}^*(G)$. Since $\widetilde{\mathscr{G}}^*(G)$ is a path, this is a contradiction. Now suppose that $k \geq 5$. Then $o(a_1) = p$ and $o(a_{k-1}) = pq$. Since $< a_1 > \cap < a_{k-1} > = 1_G$, we have $o(a_1a_{k-1}) = p^2q$. This contradicts to Lemma 4.1.

Conversely it is easy to check that the graphs $\widetilde{\mathscr{G}}^*(\mathbb{Z}_p), \widetilde{\mathscr{G}}^*(\mathbb{Z}_{p^2})$ and $\widetilde{\mathscr{G}}^*(\mathbb{Z}_{pq})$, where p and q are prime numbers, are paths of length $0, 1$ and 2, respectively. □

Theorem 4.3. *Let G be a finite group. Then $\widetilde{\mathscr{G}}^*(G)$ is a bipartite graph if and only if $\widetilde{\mathscr{G}}^*(G)$ is connected and the order of each non-trivial element of G is a prime or a product of two primes (not necessary distinct).*

Proof. Let $\widetilde{\mathscr{G}}^*(G)$ be a bipartite graph with two part Γ_1 and Γ_2. We consider the directed power graph $\overrightarrow{\widetilde{\mathscr{G}}^*(G)}$. If there exist arcs $[b] \to [a]$ and $[a] \to [c]$, then there exist the arc $[b] \to [c]$. This implies that $\{[b], [c]\} \in [E_0]$, which is a contradiction. Hence we may assume that all arcs are from Γ_1 to Γ_2. Hence all elements in Γ_2 are of prime order. Let $[a] \in \Gamma_1$. If $o(a) = pqm$, where $m \neq 1$ is a positive number and p and q are prime numbers (not necessary distinct), then we have $\{[a], [a^p]\} \in [E_0]$ and $\{[a], [a^{pq}]\} \in [E_0]$. Thus $[a^p], [a^{pq}] \in \Gamma_2$. But $\{[a^p], [a^{pq}]\} \in [E_0]$, a contradiction. Hence all elements of Γ_1 must be of order p or pq, for some prime numbers p and q, which are not necessary distinct.

Conversely Suppose that $\widetilde{P}_0(G)$ is connected and all elements of G are of order p, p^2 and pq, for some prime numbers p and q. Then all elements of prime order are in one part of $\widetilde{\mathscr{G}}^*(G)$ and the other are in the other part. □

Theorem 4.4. (i) *Suppose G is a p-group. Then $\widetilde{\mathscr{G}}^*(G)$ is bipartite if and only if G is a cyclic or a generalized quaternion group.*
(ii) *Suppose G is a nilpotent group which is satisfy in the Case 2) of Theorem 3.6. Then $\widetilde{\mathscr{G}}^*(G)$ is bipartite.*

Proof. (i) Suppose that G is a p-group. By [20, Theorem 7], $\mathscr{G}^*(G)$ is connected if and only if G is cyclic or a generalized quaternion group. By Lemma 2.2, the number of connected components

of $\mathscr{G}^*(G)$ is equal to the number of connected components of $\widetilde{\mathscr{G}}^*(G)$. Thus by Theorem 4.3, the conclusion is established.
(ii) Suppose G is a nilpotent group which is satisfy in Case 2) of Theorem 3.6. Then by [20, Theorem 10], $\mathscr{G}^*(G)$ is connected and so by Lemma 2.2, $\widetilde{\mathscr{G}}^*(G)$ is connected. Also by [9], all non-trivial elements of G are of prime order. Hence by Theorem 4.3, $\widetilde{\mathscr{G}}^*(G)$ is bipartite. $\square$

Theorem 4.5. *Let G be a finite group. Then $\mathbf{g}(\widetilde{\mathscr{G}}^*(G)) = 3$ if and only if there exist an element $a \in G$ such that pqr divides $o(a)$, where p, q and r are prime numbers (not necessary distinct).*

Proof. Let $\mathbf{g}(\widetilde{\mathscr{G}}^*(G)) = 3$. Then there exists a cycle of length three in $\widetilde{\mathscr{G}}^*(G)$, say $[a], [b], [c], [a]$. We can assume that $b = a^m$ and $c = b^n$, for some positive integers m and n. Thus we have $c = a^{mn}$. Hence this cycle is a subgraph of $\widetilde{\mathscr{G}}^*(< a >)$ and $|V(\widetilde{\mathscr{G}}^*(< a >))| \geq 3$. If $o(a) = p$ or $o(a) = p^2$, for a prime number p, then $V(\widetilde{\mathscr{G}}^*(< a >)) = \{[a]\}$ or $V(\widetilde{\mathscr{G}}^*(< a >)) = \{[a], [a^p]\}$, respectively. This is a contradiction to $|V(\widetilde{\mathscr{G}}^*(< a >))| \geq 3$. If $o(a) = pq$, for some prime numbers p and q, then $\widetilde{\mathscr{G}}^*(< a >)$ is a path of length two, a contradiction. Hence pqr divides $o(a)$, where p, q and r are prime numbers.

Conversely let $a \in G$. Suppose that pqr divides $o(a)$, where p, q and r are prime numbers. Then there exists the cycle $[a], [a^p], [a^{pq}], [a]$ in $\widetilde{\mathscr{G}}^*(G)$. Therefore $\mathbf{g}(\widetilde{\mathscr{G}}^*(G)) = 3$. $\square$

Suppose that $\widetilde{\mathscr{G}}(G)$ is a Hamiltonian graph, with the Hamiltonian cycle C. Obviously the path $P = C \setminus \{[1_G]\}$ is a Hamiltonian path in $\widetilde{\mathscr{G}}^*(G)$. Suppose P is a Hamiltonian path in $\widetilde{\mathscr{G}}^*(G)$. Since $[1_G]$ is adjacent to other vertices in $\widetilde{\mathscr{G}}(G)$ then $C = P \cup \{[1_G]\}$ is a Hamiltonian cycle in $\widetilde{\mathscr{G}}(G)$. Thus we have the following proposition.

Proposition 4.6. *Let G be a finite group. Then $\widetilde{\mathscr{G}}(G)$ is a Hamiltonian graph if and only if $\widetilde{\mathscr{G}}^*(G)$ has a Hamiltonian path.*

Proposition 4.7. *Let G be a finite group. Suppose that $[e_0]$ be the number of edges of $\widetilde{\mathscr{G}}^*(G)$. Then $[e_0] = \sum_{a\in G} \dfrac{1}{\phi(o(a))}(-2 + \sum_{i=1}^{o(a)} \dfrac{1}{\phi(o(a^i))}) + 1.$*

Proof. It follows immediately from Theorem 3.4. $\square$

Proposition 4.8. *Suppose G is a finite group. Then $\omega(\mathscr{G}^*(G)) = \omega(\mathscr{G}(G)) - 1_G$ and $\omega(\widetilde{\mathscr{G}}^*(G)) = \omega(\widetilde{\mathscr{G}}(G)) - 1_G$.*

Proof. Since $1_G([1_G])$ is adjacent to other vertices, the conclusion is established. $\square$

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