www.theoryofgroups.ir

International Journal of Group Theory
ISSN (print): 2251-7650, ISSN (on-line): 2251-7669
Vol. 5 No. 3 (2016), pp. 61-67.

www.ui.ac.ir

GROUPS WHOSE PROPER SUBGROUPS OF INFINITE RANK HAVE POLYCYCLIC-BY-FINITE CONJUGACY CLASSES

MOUNIA BOUCHELAGHEM AND NADIR TRABELSI*

Communicated by Mahmut Kuzucuoğlu

Abstract. A group G is said to be a $(PF)C$-group or to have polycyclic-by-finite conjugacy classes, if $G/C_G(x^G)$ is a polycyclic-by-finite group for all $x \in G$. This is a generalization of the familiar property of being an FC-group. De Falco et al. (respectively, de Giovanni and Trombetti) studied groups whose proper subgroups of infinite rank have finite (respectively, polycyclic) conjugacy classes. Here we consider groups whose proper subgroups of infinite rank are $(PF)C$-groups and we prove that if G is a group of infinite rank having a non-trivial finite or abelian factor group and if all proper subgroups of G of infinite rank are $(PF)C$-groups, then so is G. We prove also that if G is a locally soluble-by-finite group of infinite rank which has no simple homomorphic images of infinite rank and whose proper subgroups of infinite rank are $(PF)C$-groups, then so are all proper subgroups of G.

1. **Introduction and results**

Let $\mathfrak{X}$ be a class of groups. A group G is said to be an $\mathfrak{X}C$-group or to have $\mathfrak{X}$ conjugacy classes if $G/C_G(x^G)$ belongs to $\mathfrak{X}$ for every element x in G. When $\mathfrak{X} = \mathfrak{F}$ is the class of finite groups, then the property $\mathfrak{F}C$ is the familiar property of being an FC-group which has been much studied. Other classes of $\mathfrak{X}C$-groups that have been studied are CC (respectively PC, $(PF)C$, MC and M_rC)-groups which we obtain when we take $\mathfrak{X}$ to be the class of Chernikov (respectively polycyclic, polycyclic-by-finite, soluble-by-finite minimax and reduced minimax) groups (see [14], [10], [11] and [12]).

Also a group G is said to be a minimal non-$\mathfrak{X}$-group if it is not an $\mathfrak{X}$-group but all of whose proper subgroups are $\mathfrak{X}$-groups. Many results have been obtained on minimal non-$\mathfrak{X}$-groups, for various choices of $\mathfrak{X}$ (for minimal non-$\mathfrak{X}C$-groups see [2], [3], [4], [9] and [13]). In particular, in [2] and [3],

MSC(2010): Primary: 20F24; Secondary: 20E07.

Keywords: Polycyclic-by-finite conjugacy classes, minimal non-$(PF)C$-group, minimal non-FC-group, Prüfer rank.

Received: 3 December 2014, Accepted: 9 March 2015.

∗Corresponding author.

Belyaev and Sesekin characterized minimal non-FC-groups when they have a non-trivial finite or abelian factor group. They proved that such minimal non-FC-groups are finite cyclic extensions of divisible abelian p-groups of finite rank, where p is a prime, hence they are Chernikov groups. In [4] it is proved, for groups having a proper subgroup of finite index, that the property of being a minimal non-$(PF)C$-group is equivalent to that of being a minimal non-FC-group. Note that in [4], the property $(PF)C$ was denoted by PC. Since in [9], groups with polycyclic conjugacy classes have been considered and denoted by PC-groups, it is convenient, to denote in this note, groups with polycyclic-by-finite conjugacy classes by $(PF)C$-groups. In [9, Theorem A] it is proved that a non-perfect minimal non-PC-group is a minimal non-FC-group. Here we will generalize this last result to non-perfect minimal non-$(PF)C$-groups. Our first result is the following theorem.

Theorem 1.1. *Let G be a non-perfect group. Then G is a minimal non-$(PF)C$-group if and only if it is a minimal non-FC-group.*

We can deduce by Theorem 1.1 that a minimal non-$(PF)C$-group is non-perfect if and only if it has a proper subgroup of finite index.

It must be pointed out that we were not aware that Theorem 1.1 was already obtained by Artemovych in [1, Theorem 1]. Since our proof is more easier and shorter, we have not removed Theorem 1.1 from our paper.

In recent years, generalized soluble groups G of infinite rank whose proper subgroups of infinite rank belong to a given class $\mathfrak{X}$ have been studied and it was proved that then all proper subgroups of G are $\mathfrak{X}$-groups. In many cases, in particular when minimal non-$\mathfrak{X}$-groups have finite rank, these groups G themselves belong to $\mathfrak{X}$ (see for instance the papers referred to in [8]). In [7] it is proved that if G is a locally soluble-by-finite group of infinite rank without simple factor groups of infinite rank and whose proper subgroups of infinite rank are FC-groups, then all proper subgroups of G are FC-groups. Also in [7] (respectively, [9]) it is proved that a non-perfect group of infinite rank whose proper subgroups of infinite rank are FC-groups (respectively PC-groups) is itself a FC-group (respectively, PC-group). Here we will prove a $(PF)C$ version of the above results.

Theorem 1.2. *Let G be a group of infinite rank having a non-trivial finite or abelian homomorphic image. If all proper subgroups of G of infinite rank are $(PF)C$-groups, then so is G.*

Theorem 1.3. *Let G be a locally soluble-by-finite group of infinite rank which has no simple homomorphic images of infinite rank. If all proper subgroups of G of infinite rank are $(PF)C$-groups, then so are all proper subgroups of G.*

Our notation and terminology are standard and follow [14]. In particular, a group G is said to have *finite (Prüfer) rank r* if every finitely generated subgroup can be generated by at most r elements and r is the least positive integer satisfying this property. If there is no such integer then G is said to have infinite rank.

2. **Proofs of the results**

The following couple of lemmas will be used in the proof of both Theorem 1.1 and Theorem 1.2.

Lemma 2.1. *Let G be a group such that G and G' have no proper subgroups of finite index and G/G' is periodic. If for all $x \in G$, $\langle G', x\rangle$ is a $(PF)C$-group, then G is abelian.*

Proof. Let $x \in G$; since the factor group of a $(PF)C$-group by its centre is residually finite and since G' has no proper subgroup of finite index, we deduce that G' is contained in the centre of $\langle G', x\rangle$. It follows that G' is contained in the centre of G and hence G is nilpotent. Since G/G' is periodic, we deduce that G is periodic. Therefore G is abelian as it has no proper subgroup of finite index [14, Theorem 9.23]. □

In [10, Theorem 2.2] it is proved that G is a $(PF)C$-group if and only if x^G is polycyclic-by-finite for every element x in G. We will use many times this result without further reference.

Lemma 2.2. *Let G be a locally polycyclic-by-finite group and let M and N be normal subgroups of G such that $G = MN$. If for all $x \in G$, $\langle M, x\rangle$ and $\langle N, x\rangle$ are $(PF)C$-groups, then so is G.*

Proof. Let $x \in G$; then $x^G = x^{MN} = (x^M)^N$. Since $\langle M, x\rangle$ is a $(PF)C$-group, x^M is polycyclic-by-finite and hence it is finitely generated. So there exist $x_1, \dots, x_r \in \langle M, x\rangle$ such that $x^M = \langle x_1, \dots, x_r\rangle$ and hence

$$(x^M)^N = \langle x_1, \dots, x_r\rangle^N = \langle x_1^N, \dots, x_r^N\rangle.$$

Since each $\langle N, x_i\rangle$ is a $(PF)C$-group, each x_i^N is finitely generated and hence so is $(x^M)^N$. As G is locally polycyclic-by-finite, we have that x^G is polycyclic-by-finite which gives that G is a $(PF)C$-group. □

The following result is known, we omit its easy proof.

Lemma 2.3. *Let G be a divisible abelian group. Then either G is quasicyclic or there exist two proper subgroups M and N such that $G = MN$ and G/M and G/N are non-cyclic groups of rank one.*

Proof of Theorem 1.1. Let G be a non-perfect group and suppose first that G is a minimal non-$(PF)C$-group. If G is finitely generated, then it has a proper subgroup of finite index, say N, containing G'. So N is a finitely generated $(PF)C$-group and hence it is polycyclic-by-finite, which gives the contradiction that G is polycyclic-by-finite. Therefore G is not finitely generated and hence it is locally polycyclic-by-finite. Assume that G has no proper subgroups of finite index. So G/G' is a divisible abelian group. By Lemma 2.2 and Lemma 2.3, we have that G/G' is quasicyclic and hence Lemma 2.1 gives that G' has a proper subgroup of finite index. Therefore there exists a positive integer n such that $(G')^n \neq G'$. Since $G/(G')^n$ is periodic, its proper subgroups are FC-groups. If $G/(G')^n$ is an FC-group then the factor group of $G/(G')^n$ by its centre is residually finite and hence $G/(G')^n$ is abelian as it has no proper subgroups of finite index. This implies the contradiction that $G' = (G')^n$. We deduce that $G/(G')^n$ is a non-perfect minimal non-FC-group and hence it has a proper subgroup of finite index by [3], which is a contradiction. Consequently G has a proper subgroup of finite index and hence G is a minimal non-FC-group by [4].

Conversely, if G is a non-perfect minimal non-FC-group, then it has a proper subgroup of finite index by [3] and hence G is a minimal non-$(PF)C$-group by [4]. □

The following lemma will reduce the proof of Theorem 1.2 to the case when G' has infinite rank.

Lemma 2.4. *Let G be a group whose proper subgroups of infinite rank are $(PF)C$-groups. If G/G' has infinite rank, then G is a $(PF)C$-group.*

Proof. Assume that G/G' has infinite rank and let H be a proper subgroup of G of finite rank. So HG' is a proper subgroup of G. If HG' has infinite rank , then HG' and hence H is a $(PF)C$-group. Now if HG' has finite rank, then G/HG' is an abelian group of infinite rank and hence it contains a proper subgroup of infinite rank, say N/HG'. So N and hence H is a $(PF)C$-group. Therefore every proper subgroup of G has the property $(PF)C$ and hence G is a $(PF)C$-group since non-perfect minimal non-$(PF)C$-groups have finite rank by Theorem 1.1 and [2]. □

In next two lemmas we generalize some results, which are valid in minimal non-$(PF)C$-groups, to groups whose proper subgroups of infinite rank are $(PF)C$-groups.

Lemma 2.5. *Let G be a non-perfect group of infinite rank whose proper subgroups of infinite rank are $(PF)C$-groups. Then G is locally polycyclic-by-finite.*

Proof. In view of Lemma 2.4, one can assume that G/G' has finite rank, so G' has infinite rank. First consider the case when G/G' is finitely generated. Then G has a maximal subgroup, say M, containing G'. So M is a proper normal subgroup of infinite rank. Hence M is a $(PF)C$-group and G/M is finite. It follows that G is almost locally polycyclic-by-finite and hence locally polycyclic-by-finite. Now assume that G/G' is not finitely generated and let H be a finitely generated subgroup of G. Then HG' is a proper subgroup of G of infinite rank and hence it is a $(PF)C$-group. It follows that H is a $(PF)C$-group and so H is polycyclic-by-finite, which gives that G is locally polycyclic-by-finite. □

Lemma 2.6. *Let G be a group of infinite rank whose proper subgroups of infinite rank are $(PF)C$-groups. If G is not a $(PF)C$-group, then there is a prime p such that every finite homomorphic image of G is a cyclic p-group, and for all positive integer n, G has at most one subgroup of index p^n.*

Proof. Since G is not a $(PF)C$-group, there is $x \in G$ such that x^G is not polycyclic-by-finite. Let H be a normal subgroup of finite index in G and assume for a contradiction that $\langle H, x\rangle$ is a proper subgroup of G. Therefore H has infinite rank, so that x^H is polycyclic-by-finite. Let $\{g_1, \ldots, g_k\}$ be a right transversal to H in G, then $x^G = \langle (x^H)^{g_1}, \ldots, (x^H)^{g_k}\rangle$. Since $(x^H)^{g_i}$ is polycyclic-by-finite, it is finitely generated and hence so is x^G. Since $x^G H/H \simeq x^G/H \cap x^G$ is finite, we deduce that $x^G \cap H$ is a finitely generated $(PF)C$-group and hence it is polycyclic-by-finite. It follows that x^G is polycyclic-by-finite, a contradiction that gives that $G = \langle H, x\rangle$ and hence G/H is cyclic. If $|G/H| = mn$ where m and n are co-prime positive integers, then $\langle x^m, H\rangle$ and $\langle x^n, H\rangle$ are proper subgroups of infinite rank and hence they are $(PF)C$-groups. It follows that $(x^m)^G = (x^m)^H$ and $(x^n)^G = (x^n)^H$ are polycyclic-by-finite and hence so is $x^G = (x^m)^G(x^n)^G$, a contradiction that implies that G/H has a prime power order. Note that if H_1 and H_2 are two proper normal subgroups of finite index, then so is $H_1 \cap H_2$. It follows that every finite homomorphic image of G is a power of the same prime, say p, and for all positive integer n, G has at most one subgroup of index p^n. □

Next lemma reduce the proof of Theorem 1.2 to the case when the finite residual subgroup has finite index.

Lemma 2.7. *Let G be a group of infinite rank whose proper subgroups of infinite rank are $(PF)C$-groups. If $G^\star$, the finite residual subgroup of G, has infinite index in G, then G is a $(PF)C$-group.*

Proof. Assume that G is not a $(PF)C$-group and let H be a proper normal subgroup of G of finite index. Then H has infinite rank and hence it is a $(PF)C$-group. By Lemma 2.6 there exists a prime p such that G/H is cyclic of p-power order. It follows that $G/G^\star$ is abelian and hence $G' \leq G^\star$. Since $G/G^\star$ is infinite and since for all positive integer n, G has a unique subgroup of index p^n, there is an infinite descending chain of normal subgroups of G,

$$G = G_0 > G_1 > \cdots > G_n > \cdots ,$$

such that for all positive integer n, G/G_n is of order p^n and $G^\star$ is the intersection of all such G_n. There is an $x \in G$ such that $G/G_1 = \langle xG_1 \rangle$ and $x^p \in G_1$. It is easy to prove by induction on n that $G/G_n = \langle xG_n \rangle$ and hence for all positive integer n we have $G = \langle G_n, x \rangle$. We show that $G = \langle G^\star, x \rangle$. Assume for a contradiction that $G \neq \langle G^\star, x \rangle$. If $G/\langle G^\star, x \rangle$ has a proper subgroup of finite index, say $H/\langle G^\star, x \rangle$, then $H = G_n$ for some positive integer n. This gives that $x \in G_n$ and hence $G = G_n$ which is a contradiction. So that $G/\langle G^\star, x \rangle$ has no proper subgroups of finite index and hence it is divisible abelian. Note that if $xG^\star$ is of finite order, then its order is a power of p as $G/G^\star$ is a residually finite p-group. Hence for some integer $s > 0$, $x^{p^s} \in G_n$ for all integer $n > 0$, a contradiction. So $xG^\star$ is of infinite order. Since $(G/G^\star)/\langle xG^\star \rangle \simeq G/\langle G^\star, x \rangle$, it is divisible abelian. Suppose that $(G/G^\star)/\langle xG^\star \rangle$ is a quasicyclic r-group for some prime r and let q be a prime such that $q \neq p$ and $q \neq r$. If $G/G^\star \neq G^qG^\star/G^\star$ then $(G/G^\star)/(G^qG^\star/G^\star)$ is an elementary abelian q-group, so it has a proper subgroup of index q, a contradiction as $q \neq p$. Hence $G/G^\star = G^qG^\star/G^\star$ and so $xG^\star = (yG^\star)^q$ for some $y \in G$. Note that $y \notin G^\star$ since $x \notin G^\star$. Hence $(yG^\star\langle xG^\star \rangle)^q = \langle xG^\star \rangle$ so $yG^\star \in \langle xG^\star \rangle$ as $r \neq q$ and $(G/G^\star)/\langle xG^\star \rangle$ is a r-group. Consequently $yG^\star = (xG^\star)^i$ for some $i > 0$, so $xG^\star = (yG^\star)^q = (xG^\star)^{iq}$. Since $xG^\star$ is of infinite order, we deduce that $iq = 1$, a contradiction. Therefore $(G/G^\star)/\langle xG^\star \rangle$ is not quasicyclic and hence by Lemma 2.3, there are proper normal subgroups M and N of G such that $G = MN$ and G/M, G/N are non-cyclic groups of finite rank. It follows that for all $g \in G$, $\langle M, g \rangle$ and $\langle N, g \rangle$ are proper subgroups of G of infinite rank and hence there are $(PF)C$-groups. This gives that G is a $(PF)C$-group by Lemma 2.2 and Lemma 2.5, a contradiction. Therefore $G = \langle G^\star, x \rangle$. Consequently $G/G^\star$ is an infinite cyclic group and hence for all integer $n > 0$, G has a proper subgroup of index n, a final contradiction. Therefore G is a $(PF)C$-group. □

Proof of Theorem 1.2. (i) Let G be a group as stated and first consider the case when G has a proper subgroup of finite index. By Lemma 2.7 one can assume that $G/G^\star$ is finite, where $G^\star$ is the finite residual subgroup of G. Therefore $G^\star$ has infinite rank and has no proper subgroups of finite index and $G/G^\star$ is cyclic by Lemma 2.6. It follows that $G^\star$ is a $(PF)C$-group and hence it is abelian, so that G is soluble. Consequently G is a non-perfect group whose proper subgroups of infinite rank are PC-groups and hence so is G by [9, Theorem B].

(ii) Now consider the case when G is not perfect and suppose for a contradiction that G is not a $(PF)C$-group. By (i) we deduce that G has no proper subgroups of finite index and hence G/G' is a divisible abelian group which is of finite rank by Lemma 2.4. Therefore G' has infinite rank and hence G/G' is quasicyclic by Lemma 2.2, Lemma 2.3 and Lemma 2.5. So Lemma 2.1 gives that G' has a proper subgroup of finite index. It follows that there exists a positive integer n such that $(G')^n$ is a proper subgroup of G'. So $G/(G')^n$ is a periodic group and hence its proper subgroups of infinite rank are FC-groups. We deduce by [7, Lemma 11] that $G/(G')^n$ is either a FC-group or it has finite rank. If $G/(G')^n$ has the property FC, then it is abelian as it has no proper subgroups of finite index, which gives the contradiction that $(G')^n = G'$. Consequently $G/(G')^n$ has finite rank and hence $(G')^n$ has infinite rank. So proper subgroups of $G/(G')^n$ are FC-groups and hence $G/(G')^n$ is a non-perfect minimal non-FC-group. We deduce by [3] that $G/(G')^n$ has a proper subgroup of finite index which is a contradiction. □

Proof of Theorem 1.3. By Theorem 1.2, one can assume that G is perfect and has no proper subgroups of finite index. First assume that all proper normal subgroups of G have finite rank. It follows that every homomorphic image of G has infinite rank and hence it is not simple. We deduce by [6, Theorem 2] that G has finite rank, a contradiction. Therefore G has a proper normal subgroup of infinite rank, say N. If G/N has finite rank, then it is almost locally soluble [5] and hence locally soluble as G has no proper subgroups of finite index. We deduce by [14, Lemma 10.39] that there exists a non-negative integer n such that $(G/N)^{(n)}$ is hypercentral, a contradiction as G is perfect. Therefore G/N has infinite rank and hence if H is a proper subgroup of G of finite rank, then HN is a proper subgroup of G of infinite rank. So HN and a fortiori H is a $(PF)C$-group, as claimed. □

Acknowledgments

Both authors are grateful to the referee for carefull reading.

References

[1] O. D. Artemovych, Minimal non-PC-groups, *Algebra Discrete Math.*, **18** (2014) 1–7.

[2] V. V. Beljaev and N. F. Sesekin, Infinite groups of Miller-Moreno type, *Acta Math. Acad. Sci. Hungar.*, **26** (1975) 369–376.

[3] V. V. Beljaev, *Minimal non-FC-groups*, Sixth All-Union Symposium on Group Theory (Čerkassy, 1978) (Russian), **221**, "Naukova Dumka", Kiev, 1980 97–102.

[4] M. Bouchelaghem and N. Trabelsi, On minimal non-MrC-groups, *Ric. Mat.*, **62** (2013) 97–105.

[5] N. S. Chernikov, A theorem on groups of finite special rank, *Ukrainian Math. J.*, **42** (1990) 855–861.

[6] M. R. Dixon, M. J. Evans and H. Smith, Goups with all proper subgroups nilpotent-by-finite rank, *Arch. Math.*, **75** (2000) 81–91.

[7] M. De Falco, F. de Giovanni, C. Musella and N. Trabelsi, Groups whose proper subgroups of infinite rank have finite conjugacy classes, *Bull. Aust. Math. Soc.*, **89** (2014) 41–48.

[8] F. de Giovanni, Infinite groups with rank restrictions on subgroups, *Problems in the theory of representations of algebras and groups*, 3139, Part 25, Zap. Nauchn. Sem. POMI, 414, POMI, St. Petersburg (2013).

[9] F. de Giovanni and M. Trombetti, Groups whose proper subgroups of infinite rank have polycyclic conjugacy classes, (to appear).

[10] S. Franciosi, F. de Giovanni and M. J. Tomkinson, Groups with polycyclic-by-finite conjugacy classes, *Boll. Un. Mat. Ital. B (7)*, **4** (1990) 35–55.

[11] L. A. Kurdachenko, On groups with minimax conjugacy classes, In : Infinite groups and adjoining algebraic structures, (Naukova Dumka, Kiev), 1999 160–177.

[12] L. A. Kurdachenko and J. Otal, *Frattini properties of groups with minimax conjugacy classes*, Topics in Infinite Groups, Topics in infinite groups, Quad. Mat., **8**, Dept. Math., Seconda Univ. Napoli, Caserta, 2001 221–237.

[13] J. Otal and J. M. Peña, Minimal non-CC-groups, *Comm. Algebra*, **16** (1988) 1231–1242.

[14] D. J. S. Robinson, *Finiteness conditions and generalized soluble groups*, Springer Verlag, New York-Berlin, 1972.

Mounia Bouchelaghem

Department of Mathematics, University Ferhat Abbas, Setif 1, Setif, Algeria

Email: bouchelaghem_math@yahoo.fr

Nadir Trabelsi

Department of Mathematics, University Ferhat Abbas, Setif 1, Setif, Algeria

Email: nadir_trabelsi@yahoo.fr