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CONJUGACY SEPARABILITY OF CERTAIN HNN EXTENSIONS WITH NORMAL ASSOCIATED SUBGROUPS

KOK BIN WONG* AND PENG CHOON WONG

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Abstract. In this paper, we will give necessary and sufficient conditions for certain HNN extensions of subgroup separable groups with normal associated subgroup to be conjugacy separable. In fact, we will show that these HNN extensions are conjugacy separable if and only if the normalizer of one of its associated subgroup is conjugacy separable.

1. Introduction

Blackburn [5] first proved that the finitely generated nilpotent groups are conjugacy separable. Later, Formanek [10] and Remeslennikov [29] independently extended this result by proving that polycyclic-by-finite groups are conjugacy separable. Dyer in [8] proved that free-by-finite groups are conjugacy separable. Fuchsian groups are proved to be conjugacy separable by Fine and Rosenberger [9].

At present, the conjugacy separability of HNN extensions is difficult to determine. Indeed one of the simplest type of HNN extensions, the Baumslag-Solitar group, $\langle h, t; t^{-1}h^2t = h^3\rangle$ is not even residually finite (see [4]). However, Kim and Tang had characterized the conjugacy separability of HNN extensions of finitely generated abelian groups with cyclic associated subgroups in [18] and they gave a criterion for the conjugacy separability in HNN extensions with cyclic associated subgroups in [20] (see also [23] for a more recent result). Raptis, Talelli and Varsos [28] proved the equivalence of residual finiteness and conjugacy separability in certain HNN extensions of finitely generated abelian groups. Recently, Wong and Wong [34] extended the result of Raptis, Talelli and Varsos by proving the equivalence of residual finiteness and conjugacy separability in certain HNN extensions of polycyclic-by-finite groups.

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*Corresponding author.

Grossman [11] showed that if all the class-preserving automorphisms of a finitely generated conjugacy separable group G are inner, then the outer automorphism group of G is residually finite. This criterion has been used by many authors to show that certain outer automorphism groups are residually finite (see [2, 3, 7, 16, 22, 34, 38]). Recently, Zhou and Kim [40, 41] have studied the class-preserving automorphisms of certain groups.

Cyclic subgroup separability is a property that is used to show that certain generalized free products are conjugacy separable (see [17, 19, 31, 32]). Cyclic subgroup separability was introduced by Stebe [30]. Kim [14, 15] gave useful criteria for certain generalized free products and HNN extensions to be cyclic subgroup separable. By using Kim's criterion for HNN extensions, Wong and Wong [37] have given a characterization for certain HNN extensions with central associated subgroups to be cyclic subgroup separable (see also [1, 13, 21, 33, 35, 36] for some recent results on cyclic subgroup separability).

Subgroup separability is a property stronger than cyclic subgroup separability. It is well known that polycyclic groups and free groups are subgroup separable (Hall [12], Mal'cev [25]). Since a finite extension of a subgroup separable group is again subgroup separable, polycyclic-by-finite groups and free-by-finite groups are subgroup separable. Metaftsis and Raptis [27] gave a sufficient and necessary condition for certain HNN extensions to be subgroup separable. By applying their result, Wong and Wong [39] showed that subgroup separability and conjugacy separability are equivalent in certain HNN extensions.

In this paper, we will give sufficient and necessary conditions for certain HNN extensions of subgroup separable groups with normal associated subgroup to be conjugacy separable. In fact, we will show that these HNN extensions are conjugacy separable if and only if the normalizer of an associated subgroup is conjugacy separable. This will be done in Section 3 (Theorem 3.7, Theorem 3.8 and Corollary 3.9).

2. **Preliminaries.**

The notation used here is standard. In addition, the following will be used for any group G:

(i) $N \triangleleft_f G$ means N is a normal subgroup of finite index in G;
(ii) $x \sim_G y$ means x and y are conjugate in G;
(iii) $\{y\}^G = \{g^{-1}yg \ ; \ g \in G\}$ denotes the conjugacy class of y in G;
(iv) Let X be a subset of a group G. The subgroup generated by X is denoted by $\mathrm{gp}\{X\}$;
(v) $G = \langle t, A; t^{-1}Ht = K, \varphi\rangle$ shall denote an HNN extension where A is the base group, H, K are the associated subgroups and φ is the associated isomorphism $\varphi : H \longrightarrow K$;
(vi) $\|g\|$ denotes the length of the element g in $G = \langle t, A; t^{-1}Ht = K, \varphi\rangle$;
(vi) $[X, Y]$ denotes the set $\{[x, y] = x^{-1}y^{-1}xy \ : \ x \in X, y \in Y\}$.

Definition 2.1. *A group G is said to be* conjugacy separable *if for each pair of elements $x, y \in G$ such that x and y are not conjugate in G, there exists $N \triangleleft_f G$, such that Nx and Ny are not conjugate in G/N. Equivalently, G is conjugacy separable if for each pair $x, y \in G$ and $x \notin \{y\}^G$, there exists $N \triangleleft_f G$ such that $x \notin \{y\}^G N$.*

Definition 2.2. *A group G is called H-*separable *for the subgroup H if for each $x \in G \backslash H$, there exists $N \lhd_f G$ such that $x \notin HN$.*

G is termed subgroup separable *if G is H-separable for every finitely generated subgroup H.*

G is termed residually finite *if G is 1-separable.*

*G is (H,K)-*double coset separable *for the subgroups H, K if $g_1 \notin Hg_2K$ for some $g_1, g_2 \in G$, there exists $N \lhd_f G$ such that $g_1 \notin Hg_2KN$.*

Definition 2.3. [24, p. 181] *Let $G = \langle t, A; t^{-1}Ht = K, \varphi\rangle$ be an HNN extension. A word $w = w_0t^{\epsilon_1}w_1t^{\epsilon_2}w_2 \cdots t^{\epsilon_n}w_n$, $(n \geq 0)$, where $w_i \in A$ and $\epsilon_i = \pm 1$ for all i, is said to be* reduced *if there is no j for which*

(a) *$\epsilon_{j+1} = -\epsilon_j = 1$ and $w_j \in H$, or*
(b) *$\epsilon_{j+1} = -\epsilon_j = -1$ and $w_j \in K$.*

Definition 2.4. [24, p. 185] *Let $G = \langle t, A; t^{-1}Ht = K, \varphi\rangle$ be an HNN extension. A word $w = w_0t^{\epsilon_1}w_1t^{\epsilon_2}w_2 \cdots t^{\epsilon_n}$, $(n \geq 0)$, where $w_i \in A$ and $\epsilon_i = \pm 1$ for all i, is said to be* cyclically reduced *if all cyclic permutations of w is reduced.*

It is not hard to see that every element in G is conjugate to a cyclically reduced element. The following theorem will be used throughout this note.

Theorem 2.5. [6] *Let $G = \langle t, A; t^{-1}Ht = K, \varphi\rangle$ be an HNN extension. Suppose $x, y \in G$ and x, y are cyclically reduced. If $x \sim_G y$, then $\|x\| = \|y\|$ and one of the following holds;*

(a) *$\|x\| = \|y\| = 0$ and there exists a finite sequence $z_1, z_2, \ldots, z_n$ of elements in $H \cup K$ such that $x \sim_A z_1 \sim_{A,t^*} z_2 \sim_{A,t^*} \sim_{A,t*} z_n \sim_A y$ where $u \sim_{A,t^*} v$ means one of: $(u \sim_A v)$ or ($u \in H$ and $v = t^{-1}ut \in K$) or ($u \in K$ and $v = tut^{-1} \in H$).*
(b) *$\|x\| = \|y\| = n \geq 1$ and $x = c^{-1}y^*c$ where y^* is a cyclic permutation of y and $c \in X$ with $X = H$ if $\epsilon_n = -1$ and $X = K$ if $\epsilon_n = 1$, where ϵ_n is the power of t in the last term of x expressed in cyclically reduced form, $x = x_0t^{\epsilon_1} \cdots x_{n-1}t^{\epsilon_n}$.*

3. The sufficient and necessary conditions.

Let H be a finitely generated group and $S \lhd_f H$. If S is a characteristic subgroup of H, then we set $f_H(S) = S$. Suppose S is not a characteristic subgroup of H. Let $[H : S] = m$ where m is a positive integer. Since H is finitely generated, the number of subgroups of index m in H is finite. Let N be the intersection of all these subgroups. Then N is a characteristic subgroup of finite index in H and $N \subseteq S$. We set $f_H(S) = N$ (see [33, Lemma 3.1]).

The following lemma is a special case of [36, Lemma 3.1]

Lemma 3.1. *Let A be a subgroup separable group, H, K be finitely generated normal subgroups of A, and $H \cap K = 1$. If $S_1 \lhd_f H, S_2 \lhd_f K$, then there exists $N \lhd_f A$ such that $N \cap H = f_H(S_1)$, $N \cap K = f_K(S_2)$ and $NH \cap NK = N$.*

Lemma 3.2. *Let $G = \langle t, A; t^{-1}Ht = K, \varphi\rangle$ be an HNN extension where A is subgroup separable, H, K are finitely generated normal subgroups of A and $H \cap K = 1$. Then for any $N_H \lhd_f H$ there exists $N \lhd_f A$ such that $\varphi(N \cap H) = N \cap K$, $N \cap H \subseteq N_H$ and $NH \cap NK = N$.*

Proof. Let $N_H \lhd_f H$ be given. Note that $S_2 = \varphi(f_H(N_H))$ is a characteristic subgroup of K, for φ is an isomorphism. By Lemma 3.1, there exists $N \lhd_f A$ such that $N \cap H = f_H(N_H)$, $N \cap K = f_K(S_2) = S_2$ and $NH \cap NK = N$. Note that N satisfies the conclusions of the lemma. □

Lemma 3.3. *Let $G = \langle t, A; t^{-1}Ht = K, \varphi\rangle$ be an HNN extension where H, K are non-trivial normal subgroups of A and $H \cap K = 1$. Let G_H be the normalizer of H in G. Let g be any element in G. Then the following hold:*

(a) *If $g^{-1}hg \in H$ for some $1 \neq h \in H$, then $g \in A$ or $g = a_0t^{\epsilon_1}a_1 \cdots t^{\epsilon_n}a_n$ in reduced form with $n = 2m \geq 2$ and $\epsilon_i = (-1)^{i+1}$,*

(b) *$g \in G_H$ if and only if $g \in A$ or $g = a_0t^{\epsilon_1}a_1 \cdots t^{\epsilon_n}a_n$ in reduced form with $n = 2m \geq 2$ and $\epsilon_i = (-1)^{i+1}$,*

(c) *$G_H = \mathrm{gp}\{A \cup tAt^{-1}\}$.*

Proof. (a) Suppose $g^{-1}hg \in H$ for some $1 \neq h \in H$. If $g \in A$, we are done. Suppose $\|g\| = 1$ and $g = a_0t^{\epsilon_1}a_1$ in reduced form. Then $g^{-1}hg = a_1^{-1}t^{-\epsilon_1}a_0^{-1}ha_0t^{\epsilon_1}a_1 = a_1^{-1}t^{-\epsilon_1}h_0t^{\epsilon_1}a_1$ where $h_0 = a_0^{-1}ha_0 \in H$. Since $g^{-1}hg \in H$, we have $\epsilon_1 = 1$, for otherwise, $g^{-1}hg$ has length 2. Thus $a_1^{-1}t^{-1}h_0ta_1 = a_1^{-1}k_0a_1 \in K$ where $k_0 = t^{-1}h_0t \in K$. But then $g^{-1}hg \in H \cap K = 1$ and hence $h = 1$, a contradiction. Thus $\|g\| \geq 2$.

Suppose $\|g\| = 2$ and $g = a_0t^{\epsilon_1}a_1t^{\epsilon_2}a_2$ in reduced form. Then we see that

$$g^{-1}hg = a_2^{-1}t^{-\epsilon_2}a_1^{-1}t^{-\epsilon_1}a_0^{-1}ha_0t^{\epsilon_1}a_1t^{\epsilon_2}a_2 = a_2^{-1}t^{-\epsilon_2}a_1^{-1}t^{-\epsilon_1}h_0t^{\epsilon_1}a_1t^{\epsilon_2}a_2,$$

where $h_0 = a_0^{-1}ha_0 \in H$. As above, we have $\epsilon_1 = 1$. Thus $g^{-1}hg = a_2^{-1}t^{-\epsilon_2}a_1^{-1}k_0a_1t^{\epsilon_2}a_2$ where $k_0 = t^{-1}h_0t \in K$. Since K is normal, we have $a_2^{-1}t^{-\epsilon_2}k_1t^{\epsilon_2}a_2 \in H$ where $k_1 = a_1^{-1}k_0a_1 \in K$. This implies that $\epsilon_2 = -1$, for otherwise, $g^{-1}hg$ has length 2. Thus $g = a_0(ta_1t^{-1})a_2$.

Let $\|g\| = n \geq 3$ and assume that if $g \in G$, $1 \leq \|g\| = k \leq n-1$ and $g^{-1}hg \in H$ for some $1 \neq h \in H$, then $g = a_0t^{\epsilon_1}a_1 \cdots t^{\epsilon_k}a_k$ in reduced form with $k = 2m \geq 2$ and $\epsilon_i = (-1)^{i+1}$. Let $g = a_0t^{\epsilon_1}a_1t^{\epsilon_2}a_2t^{\epsilon_3}a_3 \cdots t^{\epsilon_n}a_n$ in reduced form. Arguing as above, we have $\epsilon_1 = 1$ and $\epsilon_2 = -1$. Therefore $g = a_0ta_1t^{-1}a_2t^{\epsilon_3}a_3 \cdots t^{\epsilon_n}a_n$. Let $g_1 = a_0ta_1t^{-1}a_2$ and $g_2 = t^{\epsilon_3}a_3 \cdots t^{\epsilon_n}a_n$. Then $g = g_1g_2$. Furthermore $g_2^{-1}h_2g_2 \in H$ where $1 \neq h_2 = g_1^{-1}hg_1 \in H$. Since $1 \leq \|g_2\| = n-2 \leq n-1$, by the induction hypothesis $n-2 = 2m \geq 2$ and $\epsilon_i = (-1)^{i+1}$ for $3 \leq i \leq n$. Thus $g = a_0t^{\epsilon_1}a_1 \cdots t^{\epsilon_n}a_n$ in reduced form with $n = 2m+2 \geq 2$ and $\epsilon_i = (-1)^{i+1}$.

(b) Clearly, if $g \in A$, then $g \in G_H$. Let $g = a_0t^{\epsilon_1}a_1 \cdots t^{\epsilon_n}a_n$ in reduced form with $n = 2m \geq 2$ and $\epsilon_i = (-1)^{i+1}$. Then $g = a_0ta_1t^{-1}a_2 \cdots a_{2m-2}ta_{2m-1}t^{-1}a_{2m}$ can be written as $g = a_0g_1a_2 \cdots a_{2m-2}g_{2m-1}a_{2m}$ where $g_j = ta_jt^{-1}$, $j = 1, 3, \ldots, 2m-1$. Since $tat^{-1}, a \in G_H$ for $a \in A$, we conclude that $g \in G_H$.

If $g \in G_H$ then $g^{-1}Hg = H$. Since H is non-trivial, there exists $1 \neq h \in H$ such that $g^{-1}hg \in H$. By part (a) of this lemma, the result follows.

(c) Clearly, $\mathrm{gp}\{A \cup tAt^{-1}\} \subseteq G_H$. Let $g \in G_H$. If $g \in A$, then $g \in \mathrm{gp}\{A \cup tAt^{-1}\}$. Suppose $\|g\| \geq 1$. Then by part (b) of this lemma, $g = a_0 t^{\epsilon_1} a_1, \ldots, t^{\epsilon_n} a_n$ in reduced form with $n = 2m \geq 2$ and $\epsilon_i = (-1)^{i+1}$. We can write $g = a_0 g_1 a_2 \cdots a_{2m-2} g_{2m-1} a_{2m}$ where $g_j = t a_j t^{-1}$, $j = 1, 3, \ldots, 2m-1$. This means that $g \in \mathrm{gp}\{A \cup tAt^{-1}\}$. The proof of this lemma is completed. □

Lemma 3.4. *Let $G = \langle t, A; t^{-1}Ht = K, \varphi \rangle$ be an HNN extension where H, K are non-trivial normal subgroups of A and $H \cap K = 1$. Let G_H and G_K be the normalizers of H and K, respectively in G. Then $G_K = t^{-1} G_H t$.*

Proof. Let $g \in G_H$. Then $t^{-1}g^{-1}tKt^{-1}gt = t^{-1}g^{-1}Hgt = t^{-1}Ht = K$. Thus $t^{-1}G_H t \subseteq G_K$. Let $g \in G_K$. Then $tg^{-1}t^{-1}Htgt^{-1} = tg^{-1}Kgt^{-1} = tKt^{-1} = H$. Hence $tG_K t^{-1} \subseteq G_H$ and $G_K = t^{-1}G_H t$. □

Lemma 3.5. *Let $G = \langle t, A; t^{-1}Ht = K, \varphi \rangle$ be an HNN extension where H, K are non-trivial normal subgroups of A and $H \cap K = 1$. Let G_H be the normalizer of H in G. Then $G_H \cong A \underset{H = tKt^{-1}}{*} tAt^{-1}$.*

Proof. Let $\psi_1 : A \to G_H$ be the homomorphism defined by $\psi_1(a) = a$ for all $a \in A$, and let $\psi_2 : tAt^{-1} \to G_H$ be the homomorphism defined by $\psi_2(tat^{-1}) = tat^{-1}$. Then $\psi_1(h) = h = t\varphi(h)t^{-1} = \psi_2(t\varphi(h)t^{-1})$ in G_H. Therefore ψ_1 and ψ_2 can be extended to become a homomorphism $\psi : A \underset{H = tKt^{-1}}{*} tAt^{-1} \to G_H$. Furthermore, ψ is an epimorphism, for $G_H = \mathrm{gp}\{A \cup tAt^{-1}\}$ by part (c) of Lemma 3.3. Let $x \in A \underset{H = tKt^{-1}}{*} tAt^{-1}$. Then $x = a_1(tb_1t^{-1})a_2(tb_2t^{-1}) \cdots a_n(tb_nt^{-1})$ where $a_i \notin H, b_i \notin K$. So $\psi(x) = x$. Since G_H is a subgroup of $G = \langle t, A; t^{-1}Ht = K, \varphi \rangle$, $\psi(x) \in G$. Note that $\psi(x)$ is reduced, since it contains no consecutive subsequence $t^{-1}ht$ with $h \in H$ and subsequence tkt^{-1} with $k \in K$. Hence $\psi(x) \neq 1$ and ψ is an isomorphism. □

Lemma 3.6. *Let $G = \langle t, A; t^{-1}Ht = K, \varphi \rangle$ be an HNN extension where A is subgroup separable, H, K are finitely generated normal subgroups of A and $H \cap K = 1$. Then G is (X, Y)-double coset separable where $X, Y \in \{H, K\}$.*

Proof. Let $p, q \in G$ and $p \notin XqY$. We may assume that p, q are in reduced forms. It is sufficient to find an image $\overline{G}$ of G such that $\overline{G}$ is residually finite, $\overline{X}\overline{q}\overline{Y}$ is finite, and $\overline{p} \notin \overline{X}\overline{q}\overline{Y}$. If such image can be found, then there exists $\overline{N} \triangleleft_f \overline{G}$ such that $\overline{p} \notin \overline{X}\overline{q}\overline{Y}\overline{N}$. Let N be the preimage of $\overline{N}$. Then $p \notin XqYN$.

Case 1. $\|p\| \neq \|q\|$. Let $p = p_0 t^{\epsilon_1} p_1 \cdots t^{\epsilon_n} p_n$ and $q = q_0 t^{\epsilon'_1} q_1 \cdots t^{\epsilon'_m} q_m$ be reduced where $n \neq m$ and $n, m \geq 1$. Since A is H, K-separable, there exists $M_A \triangleleft_f A$ such that $p_i, q_j \notin HM_A$ for $p_i, q_j \notin H$, and $p_i, q_j \notin KM_A$ for $p_i, q_j \notin K$. Let $N_H = M_A \cap H \cap \varphi^{-1}(M_A \cap K)$. By Lemma 3.2, there exists $N_A \triangleleft_f A$ such that $N_A \cap H \subseteq N_H$, $N_A H \cap N_A K = N_A$ and $\varphi(N_A \cap H) = N_A \cap K$. Let $R_A = N_A \cap M_A$. Then $R_A \triangleleft_f A$. We claim that $R_A H \cap R_A K = R_A$ and $\varphi(R_A \cap H) = R_A \cap K$.

First we show that $R_A H \cap R_A K = R_A$. Clearly, $R_A \subseteq R_A H \cap R_A K$. Let $y \in R_A H \cap R_A K$. Then $y = r_1 h = r_2 k$ where $r_1, r_2 \in R_A$ and $h \in H$ and $k \in K$. This implies that $h = r_1^{-1} r_2 k \in R_A K \cap H \subseteq N_A K \cap H \subseteq N_A$, whence $h \in N_A \cap H \subseteq N_H \subseteq M_A$ and $h \in N_A \cap M_A = R_A$. Hence $y \in R_A$ and $R_A H \cap R_A K = R_A$.

Next we show that $\varphi(R_A \cap H) = R_A \cap K$. Note that $R_A \cap H = N_A \cap M_A \cap H = N_A \cap H \cap M_A = N_A \cap H$, for $N_A \cap H \subseteq N_H \subseteq M_A$. Also $R_A \cap K = N_A \cap M_A \cap K = N_A \cap K \cap M_A = N_A \cap K$, for $N_A \cap K \subseteq \varphi(N_H) \subseteq M_A$. Hence $\varphi(R_A \cap H) = R_A \cap K$.

Let $\overline{G} = \langle t, \overline{A}; t^{-1}\overline{H}t = \overline{K}, \overline{\varphi}\rangle$ where $\overline{A} = A/R_A$, $\overline{H} = HR_A/R_A$, $\overline{K} = KR_A/R_A$ and $\overline{\varphi}$ is the homomorphism induced by φ. Clearly, $\overline{G}$ is a homomorphic image of G. By the choice of M_A, $\overline{p}$, $\overline{q}$ are reduced, and $\|\overline{p}\| = \|p\|, \|\overline{q}\| = \|q\|$. Note that $\|\overline{xqy}\| = \|\overline{q}\|$ for all $x \in X, y \in Y$. So $\overline{p} \notin \overline{X}\overline{q}\overline{Y}$, for $\overline{p}$ has length n while the reduced elements in $\overline{X}\overline{q}\overline{Y}$ have length m, and $m \neq n$.

By [8, Theorem 13], $\overline{G}$ is conjugacy separable, and thus residually finite. Since $\overline{X}\overline{q}\overline{Y}$ is finite and $\overline{p} \notin \overline{X}\overline{q}\overline{Y}$, we are done.

Case 2. $\|p\| = \|q\| = n \geq 0$. Suppose $n = 0$. Then $p, q \in A$. Note that $q^{-1}p \notin q^{-1}XqY = XY$, for $X, Y \triangleleft A$. Since A is XY-separable, there exists $M_A \triangleleft_f A$ such that $q^{-1}p \notin XYM_A$. It follows that $p \notin XqYM_A$. Let M_A, N_H, N_A, R_A and $\overline{G}$ be defined as in Case 1. Suppose $\overline{p} \in \overline{X}\overline{q}\overline{Y}$. Then $\overline{p} = \overline{xqy}$ for some $x \in X, y \in Y$. This implies that $p \in XqYR_A \subseteq XqYM_A$, a contradiction. Thus $\overline{p} \notin \overline{X}\overline{q}\overline{Y}$. As in Case 1, $\overline{G}$ is residually finite and we are done.

Suppose $n \geq 1$. Let $p = p_0t^{\epsilon_1}p_1\cdots t^{\epsilon_n}p_n$ and $q = q_0t^{\epsilon'_1}q_1\cdots t^{\epsilon'_n}q_n$ be in reduced forms. Since A is H, K-separable, there exists $M_A \triangleleft_f A$ such that $p_i, q_j \notin HM_A$ for $p_i, q_j \notin H$, and $p_i, q_j \notin KM_A$ for $p_i, q_j \notin K$. Let M_A, N_H, N_A, R_A and $\overline{G}$ be defined as in Case 1. Then $\overline{p}$ and $\overline{q}$ are reduced and $\|\overline{p}\| = n = \|\overline{q}\|$. Furthermore, $\overline{H} \cap \overline{K} = 1$ and $\overline{G}$ is residually finite.

If $\epsilon_i \neq \epsilon'_i$ for some i, then $\overline{p} \notin \overline{X}\overline{q}\overline{Y}$ and we are done, so assume that $\epsilon_i = \epsilon'_i$ for all i. Note that $p \in XqY$ if and only if the following system of equations (3.1) holds:

$$
\begin{aligned}
p_0 &= v_0^{-1}q_0u_1\\
p_1 &= v_1^{-1}q_1u_2\\
&\vdots\\
p_{n-1} &= v_{n-1}^{-1}q_{n-1}u_n\\
p_n &= v_n^{-1}q_nu_{n+1}.
\end{aligned}
\tag{3.1}
$$

where $v_0 \in X$, $u_{n+1} \in Y$, $u_i \in H$, $v_i \in K$ if $\epsilon_i = 1$, or $u_i \in K$, $v_i \in H$ if $\epsilon_i = -1$, and $t^{-\epsilon_i}u_it^{\epsilon_i} = v_i$ for $i = 1, \ldots, n$.

For ease of exposition, let $U_i = H$ if $\epsilon_i = 1$ or $U_i = K$ if $\epsilon_i = -1$ and $t^{-\epsilon_i}U_it^{\epsilon_i} = V_i$ for $i = 1, \ldots, n$. Furthermore, let $V_0 = X$ and $U_{n+1} = Y$. By using this notation, $u_i \in U_i$ and $v_i \in V_i$ for all i.

SubCase 2.1. Suppose $p_i \notin V_iq_iU_{i+1}$ for some $0 \leq i \leq n$. Then $q_i^{-1}p_i \notin q_i^{-1}V_iq_iU_{i+1} = V_iU_{i+1}$. Since $V_iU_{i+1} = H$ or K or HK, the subgroup M_A above can be chosen with the additional property that $q_i^{-1}p_i \notin V_iU_{i+1}M_A$. Thus $p_i \notin V_iq_iU_{i+1}M_A$. Note that $\overline{p}_i \notin \overline{V}_i\overline{q}_i\overline{U}_{i+1}$ for this choice of M_A. Hence $\overline{p} \notin \overline{X}\overline{q}\overline{Y}$ and we are done.

SubCase 2.2. So, we may assume that $p_i \in V_iq_iU_{i+1}$ for all $0 \leq i \leq n$.

We claim that $V_i \neq U_{i+1}$ for some $0 \leq i \leq n$. Suppose $V_i = U_{i+1}$ for all i. Let $p_0 = v_0^{-1}q_0u_1$ where $v_0 \in V_0$, $u_1 \in U_1$. We shall show that by "matching forward", we will make equations (3.1) hold. What we meant by matching forward is that given $p_i = v_i^{-1} q_i u_{i+1}$ for some $0 \le i < n$, we are able to find $v_{i+1} \in V_{i+1}$ and $u_{i+2} \in U_{i+2}$ such that $p_{i+1} = v_{i+1}^{-1} q_{i+1} u_{i+2}$ and $t^{-\epsilon_{i+1}} u_{i+1} t^{\epsilon_{i+1}} = v_{i+1}$.

Let $p_1 = {v'_1}^{-1} q_1 u'_2$ where $v'_1 \in V_1$, $u'_2 \in U_2$. Suppose $t^{-\epsilon_1} u_1 t^{\epsilon_1} = v_1$. Note that $p_1 = v_1^{-1} v_1 {v'_1}^{-1} q_1 u'_2 = v_1^{-1} q_1 (q_1^{-1} (v_1 {v'_1}^{-1} q_1 u'_2))$. Since $V_1 = U_2 = H$ or K, we have $V_1 = U_2 \triangleleft A$, and thus $(q_1^{-1} v_1 {v'_1}^{-1} q_1 u'_2) \in U_2$. Let $u_2 = (q_1^{-1} v_1 {v'_1}^{-1} q_1 u'_2)$. Then we have $p_0 = v_0^{-1} q_0 u_1$ and $p_1 = v_1^{-1} q_1 u_2$ with $t^{-\epsilon_1} u_1 t^{\epsilon_1} = v_1$.

By continuing to match forward in this way, we have $p_i = v_i^{-1} q_i u_{i+1}$ and $t^{-\epsilon_i} u_i t^{\epsilon_i} = v_i$ for $0 \le i \le n$. So the equations (3.1) hold, a contradiction. Hence $V_i \neq U_{i+1}$ for some $0 \le i \le n$.

Let r be the least non-negative integer such that $V_r \neq U_{r+1}$. We claim that $r \neq n$. Suppose $r = n$. Then $V_i = U_{i+1}$ for all $0 \le i \le n-1$.

Let $p_n = v_n^{-1} q_n u_{n+1}$ where $v_n \in V_n$, $u_{n+1} \in U_{n+1}$. We shall show that by "matching backward", we will make equations (3.1) hold. What we meant by matching backward is that given $p_i = v_i^{-1} q_i u_{i+1}$ for some $0 < i \le n$, we are able to find $v_{i-1} \in V_{i-1}$ and $u_i \in U_i$ such that $p_{i-1} = v_{i-1}^{-1} q_{i-1} u_i$ and $t^{-\epsilon_i} u_i t^{\epsilon_i} = v_i$.

Let $p_{n-1} = {v'}_{n-1}^{-1} q_{n-1} u'_n$ where $v'_{n-1} \in V_{n-1}, u'_n \in U_n$. Let $t^{\epsilon_n} v_n t^{-\epsilon_n} = u_n$. Note that $p_{n-1} = {v'}_{n-1}^{-1} q_{n-1} u'_n u_n^{-1} u_n = ({v'}_{n-1}^{-1} q_{n-1} u'_n u_n^{-1} q_{n-1}^{-1}) q_{n-1} u_n$. Since $V_{n-1} = U_n = H$ or K, we have $V_{n-1} = U_n \triangleleft A$, and thus $({v'}_{n-1}^{-1} q_{n-1} u'_n u_n^{-1} q_{n-1}^{-1}) \in V_{n-1}$. Let $v_{n-1}^{-1} = ({v'}_{n-1}^{-1} q_{n-1} u'_n u_n^{-1} q_{n-1}^{-1})$. Then $p_n = v_n^{-1} q_n u_{n+1}$ and $p_{n-1} = v_{n-1}^{-1} q_{n-1} u_n$ with $t^{-\epsilon_n} u_n t^{\epsilon_n} = v_n$.

By continuing to match backward in this way, we have $p_i = v_i^{-1} q_i u_{i+1}$ and $t^{-\epsilon_i} u_i t^{\epsilon_i} = v_i$ for $0 \le i \le n$. So equation (3.1) holds, a contradiction. Hence $r \neq n$.

By the choice of r, we have $V_i = U_{i+1}$ for $0 \le i \le r-1$. Note that r could be 0. Let $p_r = v_r^{-1} q_r u_{r+1}$ where $v_r \in V_r$ and $u_{r+1} \in U_{r+1}$. Now by matching backward, we have

$$\begin{aligned} p_0 &= v_0^{-1} q_0 u_1 \\ &\vdots \\ p_{r-1} &= v_{r-1}^{-1} q_{r-1} u_r \\ p_r &= v_r^{-1} q_r u_{r+1}. \end{aligned} \tag{3.2}$$

where $v_i \in V_i$, $u_{i+1} \in U_{i+1}$ for $i = 0, \ldots, r$ and $t^{-\epsilon_i} u_i t^{\epsilon_i} = v_i$ for $i = 1, \ldots, r$.

Since equation (3.1) does not hold, there exists a s, where $r+1 \le s \le n$ such that

$$\begin{aligned} p_r &= v_r^{-1} q_r u_{r+1} \\ &\vdots \\ p_{s-1} &= v_{s-1}^{-1} q_{s-1} u_s \\ p_s &= {v'}_s^{-1} q_s u'_{s+1}. \end{aligned} \tag{3.3}$$

where $v_i \in V_i$, $u_{i+1} \in U_{i+1}$ for $i = r, \ldots, s$ and $t^{-\epsilon_i} u_i t^{\epsilon_i} = v_i$ for $i = r+1, \ldots, s-1$ but $t^{-\epsilon_s} u_s t^{\epsilon_s} \neq v'_s$. We may also assume that s is the largest integer such that equation (3.3) holds.

If $V_s = U_{s+1}$, then from $p_{s-1} = v_{s-1}^{-1} q_{s-1} u_s$, and by matching forward we would have $p_s = v_s^{-1} q_s u_{s+1}$ with $t^{-\epsilon_s} u_s t^{\epsilon_s} = v_s$, but this contradicts the choice of s as the largest integer such that equation (3.3) holds. Hence $V_s \neq U_{s+1}$.

Now we have $V_r \neq U_{r+1}$ and $V_s \neq U_{s+1}$ and (3) holds. The M_A above can be chosen with the additional property that ${v'}_s^{-1} t^{-\epsilon_s} u_s t^{\epsilon_s} \notin M_A$.

Suppose $\overline{p} \in \overline{X}\overline{q}\overline{Y}$. Then the following system of equations (3.4) holds:

$$
\begin{aligned}
\overline{p}_0 &= \overline{z}_0^{-1} \overline{q}_0 \overline{w}_1 \\
\overline{p}_1 &= \overline{z}_1^{-1} \overline{q}_1 \overline{w}_2 \\
&\vdots \\
\overline{p}_{n-1} &= \overline{z}_{n-1}^{-1} \overline{q}_{n-1} \overline{w}_n \\
\overline{p}_n &= \overline{z}_n^{-1} \overline{q}_n \overline{w}_{n+1}.
\end{aligned} \tag{3.4}
$$

where $z_i \in V_i$, $w_{i+1} \in U_{i+1}$ for $i = 0, \ldots, n$ and $\overline{z}_i = t^{-\epsilon_i} \overline{w}_i t^{\epsilon_i}$ for $i = 1, \ldots, n$.

From (3.3), we have the following system of equations:

$$
\begin{aligned}
\overline{p}_r &= \overline{v}_r^{-1} \overline{q}_r \overline{u}_{r+1} \\
&\vdots \\
\overline{p}_{s-1} &= \overline{v}_{s-1}^{-1} \overline{q}_{s-1} \overline{u}_s \\
\overline{p}_s &= \overline{v'}_s^{-1} \overline{q}_s \overline{u'}_{s+1}.
\end{aligned} \tag{3.5}
$$

From equations (3.4) and (3.5), we have $\overline{p}_r = \overline{v}_r^{-1} \overline{q}_r \overline{u}_{r+1}$ and $\overline{p}_r = \overline{z}_r^{-1} \overline{q}_r \overline{w}_{r+1}$. Thus $\overline{z}_r \overline{v}_r^{-1} = \overline{q}_r \overline{w}_{r+1} \overline{u}_{r+1}^{-1} \overline{q}_r^{-1} \in \overline{V}_r \cap \overline{U}_{r+1} = \overline{H} \cap \overline{K} = 1$. This implies that $\overline{z}_r = \overline{v}_r$, $\overline{w}_{r+1} = \overline{u}_{r+1}$ and $\overline{z}_{r+1} = t^{-\epsilon_{r+1}} \overline{w}_{r+1} t^{\epsilon_{r+1}} = t^{-\epsilon_{r+1}} \overline{u}_{r+1} t^{\epsilon_{r+1}} = \overline{v}_{r+1}$.

Again from equations (3.4) and (3.5), we have $\overline{p}_{r+1} = \overline{v}_{r+1}^{-1} \overline{q}_{r+1} \overline{u}_{r+2}$, $\overline{p}_{r+1} = \overline{z}_{r+1}^{-1} \overline{q}_{r+1} \overline{w}_{r+2}$. Since $\overline{z}_{r+1} = \overline{v}_{r+1}$, we have $\overline{u}_{r+2} = \overline{w}_{r+2}$. By continuing this way, we see that $\overline{u}_j = \overline{w}_j$ for all $r+1 \leq j \leq s$. From $\overline{p}_s = \overline{v'}_s^{-1} \overline{q}_s \overline{u'}_{s+1}$ and $\overline{p}_s = \overline{z}_s^{-1} \overline{q}_s \overline{w}_{s+1}$, we have $\overline{z}_s \overline{v'}_s^{-1} = \overline{q}_s \overline{w}_{s+1} \overline{u'}_{s+1}^{-1} \overline{q}_s^{-1} \in \overline{V}_s \cap \overline{U}_{s+1} = \overline{H} \cap \overline{K} = 1$. This implies that $\overline{z}_s = \overline{v'}_s$, but then $t^{-\epsilon_s} \overline{u}_s t^{\epsilon_s} = t^{-\epsilon_s} \overline{w}_s t^{\epsilon_s} = \overline{z}_s = \overline{v'}_s$, which implies that ${v'}_s^{-1} t^{-\epsilon_s} u_s t^{\epsilon_s} \in M_A$, contradicting the choice of M_A. Hence $\overline{p} \notin \overline{X}\overline{q}\overline{Y}$ and we are done.

The proof of the lemma is now complete. □

Theorem 3.7. *Let $G = \langle t, A; t^{-1}Ht = K, \varphi \rangle$ be an HNN extension where A is subgroup separable, H, K are non-trivial finitely generated normal subgroups of A and $H \cap K = 1$. Let G_H be the normalizer of H in G. If G_H is conjugacy separable then G is conjugacy separable.*

Proof. Let $x, y \in G$ and $x \notin \{y\}^G$. We may assume that x and y are of minimal length in $\{x\}^G$ and $\{y\}^G$, respectively. It is sufficient to find an image $\overline{G}$ of G such that $\overline{G}$ is conjugacy separable and $\overline{x} \notin \{\overline{y}\}^{\overline{G}}$. If such image can be found, then there exists $\overline{N} \triangleleft_f \overline{G}$ such that $\overline{x} \notin \{\overline{y}\}^{\overline{G}}\overline{N}$. Let N be the preimage of $\overline{N}$. Then $x \notin \{y\}^G N$.

Case 1. Suppose $\|x\| \neq \|y\|$. Since x, y are elements of minimal length in $\{x\}^G$, $\{y\}^G$ respectively, x, y are cyclically reduced. Let $x = x_0 t^{\epsilon_1} x_1 t^{\epsilon_2} \cdots x_{n-1} t^{\epsilon_n}$ and $y = y_0 t^{\epsilon'_1} y_1 t^{\epsilon'_2} \cdots y_{m-1} t^{\epsilon'_m}$ be reduced where $x_i, y_j \in A$, $\epsilon_i = \pm 1$, $\epsilon'_j = \pm 1$ for all i, j, and $n \neq m$. Then there exists $M_A \triangleleft_f A$ such that $x_i, y_j \notin HM_A$ for $x_i, y_j \notin H$ and $x_i, y_j \notin KM_A$ for $x_i, y_j \notin K$.

Let $N_H = M_A \cap H \cap \varphi^{-1}(M_A \cap K)$. By Lemma 3.2, there exists $N_A \triangleleft_f A$ such that $N_A \cap H \subseteq N_H$, $N_A H \cap N_A K = N_A$ and $\varphi(N_A \cap H) = N_A \cap K$. Let $R_A = N_A \cap M_A$. Then $R_A H \cap R_A K = R_A$ and $\varphi(R_A \cap H) = R_A \cap K$.

Let $\overline{G} = \langle t, \overline{A}; t^{-1}\overline{H}t = \overline{K}, \overline{\varphi}\rangle$ where $\overline{A} = A/R_A$, $\overline{H} = HR_A/R_A$, $\overline{K} = KR_A/R_A$ and $\overline{\varphi}$ is the homomorphism induced by φ. Note that $\overline{H} \cap \overline{K} = 1$, and both $\overline{H}$ and $\overline{K}$ are non-trivial, for $\overline{x} \neq 1$.

Note that $\overline{x}, \overline{y}$ are cyclically reduced and $\|\overline{x}\| = n$, $\|\overline{y}\| = m$. Furthermore, $\overline{x} \notin \{\overline{y}\}^{\overline{G}}$ because $\overline{x}$ has length n while the cyclically reduced elements in $\{\overline{y}\}^{\overline{G}}$ have length m, and $m \neq n$. By [8, Theorem 13], $\overline{G}$ is conjugacy separable, we are done.

Case 2. Suppose $\|x\| = \|y\| = 0$. Then $x, y \in A$.

Subcase 2.1. Suppose $x, y \in H$ and $x \neq 1$. Then $x \notin \{y\}^{G_H}$. Since G_H is conjugacy separable, there exists $M \triangleleft_f G_H$ such that $x \notin \{y\}^{G_H} M$ and $x \notin M$. Let $M_A = M \cap A$. Let N_H, N_A, R_A and $\overline{G}$ be as in Case 1. Again, $\overline{H} \cap \overline{K} = 1$, and both $\overline{H}$ and $\overline{K}$ are non-trivial, for $\overline{x} \neq 1$.

Claim 1. Let Ψ be the epimorphism of G onto $\overline{G}$. We claim that $G_H \cap Ker\Psi \subseteq M$.

Proof of Claim 1. Now, by Lemma 3.5, $G_H = A \underset{H = tKt^{-1}}{*} tAt^{-1}$ is a generalised free product. Let $\| \; \|_{G_H}$ denote the free product length of an element in G_H.

Let $w \in G_H \cap Ker\Psi$ and $w \in A$ or $w \in tAt^{-1}$. Then $\Psi(w) = 1$. Since $\Psi(w) = \overline{w}$, we have $\overline{w} = 1$, and this implies that $w \in R_A$ or $w = tw't^{-1}$ and $w' \in R_A$. In either case, $w \in M$.

Next, we assume that if $g \in G_H \cap Ker\Psi$ and $\|g\|_{G_H} \leq n-1$, then $g \in M$.

Let $w \in G_H \cap Ker\Psi$ and $\|w\|_{G_H} = n \geq 2$. If $n = 2r+1$, then $w = a_1(tb_1t^{-1}) \cdots a_r(tb_rt^{-1})a_{r+1}$ or $w = (tb_1t^{-1})a_2(tb_2t^{-1}) \cdots a_{r+1}(tb_{r+1}t^{-1})$. If $n = 2r$, then $w = a_1(tb_1t^{-1}) \cdots a_r(tb_rt^{-1})$ or $w = (tb_1t^{-1})a_2(tb_2t^{-1}) \cdots (tb_rt^{-1})a_{r+1}$.

We shall only consider the case $w = a_1(tb_1t^{-1}) \cdots a_r(tb_rt^{-1})$ (other cases are similar). Since $\overline{w} = 1$ in $\overline{G}$, either $\overline{a}_i \in \overline{H}$ or $\overline{b}_i \in \overline{K}$ for some i.

Suppose $\overline{a}_i \in \overline{H}$. Then $a_i = hm$ for some $h \in H, m \in R_A \subseteq M$. Therefore

$$w = a_1(tb_1t^{-1}) \cdots (tb_{i-1}t^{-1})hm(tb_it^{-1}) \cdots a_n(tb_nt^{-1}).$$

Let $g_1 = a_1(tb_1t^{-1}) \cdots (tb_{i-1}t^{-1})$ and $g_2 = (tb_it^{-1}) \cdots a_n(tb_nt^{-1})$. Then $g_1, g_2 \in G_H$. Let $w_1 = g_1hg_2$ and $w_2 = {g_2}^{-1}mg_2$. Then $w = w_1w_2$ where $w_1 \in G_H$ and $w_2 \in M$, for $M \triangleleft G_H$. Furthermore, $\|w_1\|_{G_H} \leq n-1$ (in G_H), because the term $t^{-1}ht$ is an element in K and it appears in w_1. Therefore

w_1 is of the form

$$a_1(tb_1t^{-1})\cdots a_{i-2}(tb_{i-2}t^{-1})a_{i-1}(tb_{i-1}kb_it^{-1})a_{i+1}(tb_{i+1}t^{-1})\cdots a_n(tb_nt^{-1}),$$

where $k = t^{-1}ht \in K$. Since $w_2 \in Ker\Psi$ $(m \in M)$ and $w \in Ker\Psi$, we have $w_1 \in Ker\Psi$, and thus by assumption, $w_1 \in M$ because $\|w_1\|_{G_H} \leq n-1$. Now $w_1, w_2 \in M$ implies that $w \in M$. The proof of the case $\overline{b}_i \in \overline{K}$ is similar. This establishes Claim 1. □

We can now continue with the proof of Subcase 2.1.

Suppose $\overline{x} \in \{\overline{y}\}^{\overline{G}}$. Then $\overline{x} = \overline{g}^{-1}\overline{yg}$. If $\overline{y} = 1$, then $\overline{x} = 1$ and $x \in R_A \subseteq M$, a contradiction. Hence $\overline{y} \neq 1$. Since $\overline{x}, \overline{y} \in \overline{H}$, by Lemma 3.3, $\overline{g} \in \overline{G}_{\overline{H}}$ where $\overline{G}_{\overline{H}}$ is the normalizer of $\overline{H}$ in $\overline{G}$. Again, by Lemma 3.3, $\overline{G}_{\overline{H}} = gp\{\overline{A}, t\overline{A}t^{-1}\} = \overline{G_H}$. Therefore $\overline{g} \in \overline{G_H}$, and by Claim 1, $x \in \{y\}^{G_H}Ker\Psi \subseteq \{y\}^{G_H}M$, a contradiction. Hence $\overline{x} \notin \{\overline{y}\}^{\overline{G}}$. By [8, Theorem 13], $\overline{G}$ is conjugacy separable, we are done.

Subcase 2.2. Suppose $x \in K, y \in H$. Let $x' = txt^{-1}$. Then $x' \in H$. As in Subcase 2.1, one can find an image $\overline{G}$ of G such that $\overline{G}$ is conjugacy separable and $\overline{x}' \notin \{\overline{y}\}^{\overline{G}}$. Clearly, $\overline{x} \notin \{\overline{y}\}^{\overline{G}}$, we are done.

Subcase 2.3. Suppose $x \in A-(H \cup K), y \in H$. Since A is H,K-separable, there exists $M_A \triangleleft_f A$ such that $x \notin HM_A \cup KM_A$. Let N_H, N_A, R_A and $\overline{G}$ be as in Case 1. Again, $\overline{H} \cap \overline{K} = 1$, and both $\overline{H}$ and $\overline{K}$ are non-trivial, for $\overline{x} \neq 1$.

Suppose $\overline{x} \in \{\overline{y}\}^{\overline{G}}$. Let $\overline{x} = \overline{g}^{-1}\overline{yg}$ where $\overline{g} \in \overline{G}$. If $\overline{g} \in \overline{A}$, we would have $\overline{x} \in \overline{H}$, a contradiction. Let $\overline{g} = \overline{a}_0\overline{t}^{\epsilon_1}\overline{a}_1\cdots\overline{t}^{\epsilon_n}\overline{a}_n$ be in reduced form where $n \geq 1$. Then $\overline{gxg}^{-1} = \overline{y}$. This implies that $\overline{a}_n\overline{xa}_n^{-1} \in \overline{H} \cup \overline{K}$, for otherwise, $\overline{gxg}^{-1}$ has length $2n \geq 2$. So $\overline{x} \in \overline{H} \cup \overline{K}$, a contradiction. Hence $\overline{x} \notin \{\overline{y}\}^{\overline{G}}$, and we are done.

Subcase 2.4. Suppose $x \in A, y \in K$. Let $y' = tyt^{-1}$. Then $y' \in H$ and this is similar to Subcase 2.1, 2.2 or 2.3.

Subcase 2.5. Suppose $x \in A, y \in A-(H \cup K)$. If $x \in H$, then it is similar to Subcase 2.3. If $x \in K$, we let $x' = txt^{-1}$, then it is similar to Subcase 2.3. So we just need to consider the case $x, y \in A-(H \cup K)$.

Since A is H,K-separable, there exists $M'_A \triangleleft_f A$ such that $x, y \notin HM'_A \cup KM'_A$. Since $x \notin \{y\}^{G_H}$, there exists $M \triangleleft_f G_H$ such that $x \notin \{y\}^{G_H}M$. Note that $x \notin \{y\}^A(M \cap A)$ for $A \subseteq G_H$.

Let $M_A = M'_A \cap M$. Let N_H, N_A, R_A and $\overline{G}$ be as in Case 1. Again, $\overline{H} \cap \overline{K} = 1$, and both $\overline{H}$ and $\overline{K}$ are non-trivial, for $\overline{x} \neq 1$.

Suppose $\overline{x} \in \{\overline{y}\}^{\overline{G}}$. Let $\overline{x} = \overline{g}^{-1}\overline{yg}$ where $\overline{g} \in \overline{G}$. If $\overline{g} \in \overline{A}$, then $\overline{x} \in \{\overline{y}\}^{\overline{A}}$ and $x \in \{y\}^AR_A \subseteq \{y\}^A(M \cap A)$, a contradiction. Let $\overline{g} = \overline{a}_0\overline{t}^{\epsilon_1}\overline{a}_1\cdots\overline{t}^{\epsilon_n}\overline{a}_n$ be in reduced form where $n \geq 1$. Then $\overline{gxg}^{-1} = y$. This implies that $\overline{a}_n\overline{xa}_n^{-1} \in \overline{H} \cup \overline{K}$, for otherwise, $\overline{gxg}^{-1}$ has length $2n \geq 2$. So $\overline{x} \in \overline{H} \cup \overline{K}$, a contradiction. Hence $\overline{x} \notin \{\overline{y}\}^{\overline{G}}$, and we are done.

Case 3. Suppose $\|x\| = \|y\| = n \geq 1$. Let $x = x_0t^{\epsilon_1}x_1t^{\epsilon_2}\cdots x_{n-1}t^{\epsilon_n}$ and $y = y_0t^{\epsilon'_1}y_1t^{\epsilon'_2}\cdots y_{n-1}t^{\epsilon'_n}$.

Note that by Theorem 2.5, $x \in \{y\}^G$ if and only if $x \in \{y^{(j)}\}^X$ for some $0 \leq j \leq n-1$ where $y^{(j)} = y_jt^{\epsilon_{j+1}}y_{j+1}\cdots t^{\epsilon_{n+j-1}}y_{n+j-1}t^{\epsilon_{n+j}}$ and the subscripts are taken modulo n and $X = H$ if $\epsilon_n = -1$ and $X = K$ if $\epsilon_n = 1$. Let j be fixed.

Claim 2. We claim that if $x \notin \{y^{(j)}\}^X$, then there exists $N_j \triangleleft_f G$ such that $x \notin \{y^{(j)}\}^XN_j$.

Proof of Claim 2. For simplicity, we shall only show the case $j = 0$, that is $y^{(j)} = y^{(0)} = y$.

Suppose $x \notin XyX$. By Lemma 3.6, there exists $N \lhd_f G$ such that $x \notin XyXN$. Hence $x \notin \{y\}^X N$, and we are done.

So we may assume that $x \in XyX$. This implies that $\epsilon_i = \epsilon_i'$ for all $1 \le i \le n$. Furthermore, the following system of equations holds:

$$\begin{aligned} x_0 &= v_n^{-1} y_0 u_1 \\ x_1 &= v_1^{-1} y_1 u_2 \\ &\vdots \\ x_{n-2} &= v_{n-2}^{-1} y_{n-2} u_{n-1} \\ x_{n-1} &= v_{n-1}^{-1} y_{n-1} u_n \end{aligned} \tag{3.6}$$

where $u_i \in H$, $v_i \in K$ if $\epsilon_i = 1$ or $u_i \in K$, $v_i \in H$ if $\epsilon_i = -1$, and $v_i = t^{-\epsilon_i} u_i t^{\epsilon_i}$ for $i = 1, \ldots, n-1$. Note that $v_n, t^{-\epsilon_n} u_n t^{\epsilon_n} \in X$ and $v_n \neq t^{-\epsilon_n} u_n t^{\epsilon_n}$ for $x \notin \{y\}^X$.

For ease of exposition, let $U_i = H$ if $\epsilon_i = 1$ or $U_i = K$ if $\epsilon_i = -1$ and $V_i = t^{-\epsilon_i} U_i t^{\epsilon_i}$ for $i = 1, \ldots, n$. By using this notation, $u_i \in U_i$, $v_i \in V_i$ and $X = V_n$. So we can write $x \notin \{y\}^{V_n}$.

Subcase C1. Suppose $\epsilon_1 = \epsilon_n$. Then $V_n \neq U_1$ and $V_n \cap U_1 = H \cap K = 1$.

There exists $M_A' \lhd_f A$ such that $x_i, y_j \notin HM_A'$ for $x_i, y_j \notin H$ and $x_i, y_j \notin KM_A'$ for $x_i, y_j \notin K$. Furthermore, we may choose M_A' so that $t^{-\epsilon_n} u_n t^{\epsilon_n} v_n^{-1} \notin M_A'$. Let $N_H = M_A' \cap H \cap \varphi^{-1}(M_A' \cap K)$. Let N_A, R_A and $\overline{G}$ be as in Case 1. Again, $\overline{H} \cap \overline{K} = 1$, and both $\overline{H}$ and $\overline{K}$ are non-trivial, for $\overline{x} \neq 1$. Furthermore, $\overline{x}, \overline{y}$ are cyclically reduced and $\|\overline{x}\| = n = \|\overline{y}\|$.

Suppose $\overline{x} \in \{\overline{y}\}^{\overline{X}}$. Since $\overline{x} = \overline{x}_0 t^{\epsilon_1} \overline{x}_1 \cdots t^{\epsilon_{n-1}} \overline{x}_{n-1} t^{\epsilon_n}$ and $\overline{y} = \overline{y}_0 t^{\epsilon_1} \overline{y}_1 \cdots t^{\epsilon_{n-1}} \overline{y}_{n-1} t^{\epsilon_n}$, the following system of equations holds:

$$\begin{aligned} \overline{x}_0 &= \overline{z}_n^{-1} \overline{y}_0 \overline{w}_1 \\ \overline{x}_1 &= \overline{z}_1^{-1} \overline{y}_1 \overline{w}_2 \\ &\vdots \\ \overline{x}_{n-2} &= \overline{z}_{n-2}^{-1} \overline{y}_{n-2} \overline{w}_{n-1} \\ \overline{x}_{n-1} &= \overline{z}_{n-1}^{-1} \overline{y}_{n-1} \overline{w}_n \end{aligned} \tag{3.7}$$

where $w_i \in U_i$, $z_i \in V_i$, and $\overline{z}_i = t^{-\epsilon_i} \overline{w}_i t^{\epsilon_i}$ for $i = 1, \ldots, n$.

From (3.6) we have the following system of equations:

$$
\begin{aligned}
\overline{x}_0 &= \overline{v}_n^{-1}\overline{y}_0\overline{u}_1 \\
\overline{x}_1 &= \overline{v}_1^{-1}\overline{y}_1\overline{u}_2 \\
&\vdots \\
\overline{x}_{n-2} &= \overline{v}_{n-2}^{-1}\overline{y}_{n-2}\overline{u}_{n-1} \\
\overline{x}_{n-1} &= \overline{v}_{n-1}^{-1}\overline{y}_{n-1}\overline{u}_n.
\end{aligned} \tag{3.8}
$$

From (3.7) and (3.8), we have $\overline{z}_n^{-1}\overline{y}_0\overline{w}_1 = \overline{v}_n^{-1}\overline{y}_0\overline{u}_1$, i.e., $\overline{v}_n\overline{z}_n^{-1} = \overline{y}_0\overline{u}_1\overline{w}_1^{-1}\overline{y}_0^{-1} \in \overline{H} \cap \overline{K} = 1$. Therefore $\overline{v}_n = \overline{z}_n$ and $\overline{u}_1 = \overline{w}_1$. This implies that $\overline{v}_1 = t^{-\epsilon_1}\overline{u}_1 t^{\epsilon_1} = t^{-\epsilon_1}\overline{w}_1 t^{\epsilon_1} = \overline{z}_1$. Continuing this way, we see that $\overline{u}_i = \overline{w}_i$ and $\overline{v}_i = \overline{z}_i$ for $i = 1, \ldots, n$. So $\overline{v}_n = \overline{z}_n = t^{-\epsilon_n}\overline{w}_n t^{\epsilon_n} = t^{-\epsilon_n}\overline{u}_n t^{\epsilon_n}$, i.e., $t^{-\epsilon_n}u_n t^{\epsilon_n} v_n^{-1} \in R_A \subseteq M'_A$, a contradiction. Hence $\overline{x} \notin \{\overline{y}\}^{\overline{X}}$.

Subcase C2. Suppose $\epsilon_i = \epsilon_{i+1}$ for some i. Then $V_i \neq U_{i+1}$ and $V_i \cap U_{i+1} = H \cap K = 1$. This is similar to Subcase 3.1.

Subcase C3. Suppose $\epsilon_1 \neq \epsilon_n$ and $\epsilon_i \neq \epsilon_{i+1}$ for $i = 1, \ldots, n-1$. We shall show that n is even. Suppose n is odd, say $n = 2p+1$. Since $\epsilon_i = 1$ or -1, we have $\epsilon_1 = \epsilon_3 = \epsilon_5 = \cdots = \epsilon_{2p+1}$, but then $\epsilon_1 = \epsilon_{2p+1} = \epsilon_n$, a contradiction, for $\epsilon_1 \neq \epsilon_n$.

Let $n = 2p$.

Subcase C3.1. Suppose $\epsilon_1 = 1$. Then $\epsilon_i = (-1)^{i+1}$ for $1 \leq i \leq 2p$. In particular, $\epsilon_{2p} = -1$ and $X = H$. In this case $x = x_0(tx_1t^{-1})\cdots x_{2p-2}(tx_{2p-1}t^{-1})$ and $y = y_0(ty_1t^{-1})\cdots y_{2p-2}(ty_{2p-1}t^{-1})$. By part (b) of Lemma 3.3, $x, y \in G_H$. Since G_H is conjugacy separable, there exists $M \lhd_f G_H$ such that $x \notin \{y\}^{G_H}M$. So $x \notin \{y\}^H M$.

There exists $M'_A \lhd_f A$ such that $x_i, y_j \notin HM'_A$ for $x_i, y_j \notin H$ and $x_i, y_j \notin KM'_A$ for $x_i, y_j \notin K$. Let $N_H = M'_A \cap M \cap H \cap \varphi^{-1}(M'_A \cap M \cap K)$. Let N_A, R_A and $\overline{G}$ be as in Case 1. Again, $\overline{H} \cap \overline{K} = 1$, and both $\overline{H}$ and $\overline{K}$ are non-trivial, for $\overline{x} \neq 1$. Furthermore, $\overline{x}, \overline{y}$ are cyclically reduced and $\|\overline{x}\| = n = \|\overline{y}\|$.

As in Claim 1, let Ψ be the epimorphism of G onto $\overline{G}$. Then $Ker\Psi \cap G_H \subseteq M$. If $\overline{x} \in \{\overline{y}\}^{\overline{H}}$, we have $x \in \{y\}^H(Ker\Psi \cap G_H) \subseteq \{y\}^H M$, a contradiction. Hence $\overline{x} \notin \{\overline{y}\}^{\overline{H}}$.

Subcase C3.2. Suppose $\epsilon_1 = -1$. Then $\epsilon_i = (-1)^i$ for $1 \leq i \leq 2p$. In particular, $\epsilon_{2p} = 1$ and $X = K$. Then $x = x_0(t^{-1}x_1t)\cdots x_{2p-2}(t^{-1}x_{2p-1}t)$ and $y = y_0(t^{-1}y_1t)\cdots y_{2p-2}(t^{-1}y_{2p-1}t)$. Let $x' = txt^{-1} = (tx_0t^{-1})x_1(tx_2t^{-1})\cdots(tx_{2p-2}t^{-1})x_{2p-1}$. By part (b) of Lemma 3.3, $x' \in G_H$. Then by Lemma 3.4, $x \in G_K$. Let $y' = tyt^{-1}$. Again by Lemma 3.3 and Lemma 3.4, $y \in G_K$. Note that $x \notin \{y\}^K$ if and only if $x' \notin \{y'\}^H$. As in Subcase C3.1, there is a epimorphic image of G, say $\overline{G}$, for which $\overline{x} \notin \{\overline{y}\}^{\overline{K}}$.

Note that in each of these cases, there is a epimorphic image of G, say $\overline{G}$, for which $\overline{x} \notin \{\overline{y}\}^{\overline{X}}$. Since $\{\overline{y}\}^{\overline{X}}$ is finite and $\overline{G}$ is conjugacy separable by [8, Theorem 13] (and thus $\overline{G}$ is residually finite), there exists $\overline{N} \lhd_f \overline{G}$ such that $\overline{x} \notin \{\overline{y}\}^{\overline{X}}\overline{N}$. Let N be the preimage of $\overline{N}$ in G. Then $x \notin \{y\}^X N$. The proof of the Claim 2 is complete. $\square$

We can now continue with the proof of Case 3.

By Claim 2, for each j, there exists $N_j \triangleleft_f G$ such that $x \notin \{y^{(j)}\}^X N_j$ for all j. Let $N = \bigcap_j N_j$. There exists $M'_A \triangleleft_f A$ such that $x_i, y_j \notin HM'_A$ for $x_i, y_j \notin H$ and $x_i, y_j \notin KM'_A$ for $x_i, y_j \notin K$. Let $N_H = M'_A \cap N \cap H \cap \varphi^{-1}(M'_A \cap N \cap K)$. Let N_A, R_A and $\overline{G}$ be as in Case 1. Again, $\overline{H} \cap \overline{K} = 1$, and both $\overline{H}$ and $\overline{K}$ are non-trivial, for $\overline{x} \neq 1$. Furthermore, $\overline{x}, \overline{y}$ are cyclically reduced and $\|\overline{x}\| = n = \|\overline{y}\|$. Then $\overline{x}$ and $\overline{y}$ have length n and $\overline{x} \notin \{\overline{y}\}^{\overline{G}}$. By [8, Theorem 13], $\overline{G}$ is conjugacy separable, we are done.

The proof of this theorem is now complete. □

Theorem 3.8. *Let $G = \langle t, A; t^{-1}Ht = K, \varphi\rangle$ be an HNN extension where A is subgroup separable, H, K are non-trivial finitely generated normal subgroups of A and $H \cap K = 1$. Let G_H be the normalizer of H in G. Suppose A is conjugacy separable. Then G_H is conjugacy separable if and only if G is conjugacy separable.*

Proof. By Theorem 3.7, it is sufficient to show that if G is conjugacy separable, then G_H is conjugacy separable.

Let $x, y \in G_H$ and $x \notin \{y\}^{G_H}$. Since G is residually finite, G_H is residually finite. So, we may assume that $x, y \neq 1$.

Suppose $x \notin \{y\}^G$. Since G is conjugacy separable, there exists $N_G \triangleleft_f G$ such that $x \notin \{y\}^G N_G$. Let $N = N_G \cap G_H$. Then $x \notin \{y\}^{G_H} N$ and we are done.

Suppose $x \in \{y\}^G$. Then the conclusions of Theorem 2.5 hold.

Case 1. Suppose $\|x\| = \|y\| = n \geq 1$. Since $x, y \in G_H$, by Lemma 3.3, we have $x = a_0(ta_1t^{-1}) \cdots a_{2s-2}(ta_{2s-1}t^{-1})$ and $y = b_0(tb_1t^{-1}) \cdots b_{2s-2}(tb_{2s-1}t^{-1})$ in reduced form in the HNN extension G.

By Lemma 3.5, $G_H = A \underset{H = tKt^{-1}}{*} tAt^{-1}$ is a generalised free product. Let $y_{2k} = b_{2k}$, $y_{2k+1} = tb_{2k+1}t^{-1}$, $x_{2k} = a_{2k}$ and $x_{2k+1} = ta_{2k+1}t^{-1}$ for $0 \leq k \leq s-1$. Then $x = x_0 \cdots x_{n-1}$ and $y = y_0 \cdots y_{n-1}$ in reduced form in the generalised free product G_H.

Note that by [26, Theorem 4.6 on p. 212], $x \in \{y\}^{G_H}$ if and only if $x \in \{y^{(j)}\}^H$ for some $j = 0, 1, \ldots, n-1$ where $y^{(j)} = y_j y_{j+1} \cdots y_{j+n-1}$.

Now $x \in \{y\}^G$ implies that $x = c^{-1}y'c$, where y' is a cyclic permutation of y in G and $c \in X$ with $X = H$ if $\epsilon_n = -1$ and $X = K$ if $\epsilon_n = 1$, where ϵ_n is the power of t appearing in the last term of x. In this case, $\epsilon_n = -1$. So $X = H$. Furthermore $y' = (tb_jt^{-1})b_{j+1} \cdots (tb_{j+2s-2}t^{-1})b_{j+2s-1}$ if j is odd or $y' = b_j(tb_{j+1}t^{-1}) \cdots b_{j+2s-2}(tb_{j+2s-1}t^{-1})$ if j is even. Here the subscripts are taken modulo $2s$ (see Theorem 2.3). So $y' = y^{(j)}$ and $x \in \{y^{(j)}\}^H$. This implies that $x \in \{y\}^{G_H}$, a contradiction. Hence Case 1 does not occur.

Case 2. Suppose $\|x\| = \|y\| = 0$. Then $x, y \in H \cup K$, for $H, K \triangleleft A$.

Subcase 2.1. Suppose $x, y \in H$. Let $x = g^{-1}yg$ for some $g \in G$. Then by Lemma 3.3, $g \in G_H$ and $x \in \{y\}^{G_H}$, a contradiction.

Subcase 2.2. Suppose $x, y \in K$. Since A is H-separable and $x, y \notin H$, there exists $M'_A \lhd_f A$ such that $x, y \notin HM'_A$. Note that $x \notin \{y\}^A$, for $A \subseteq G_H$. Since A is conjugacy separable, there exists $M_A \lhd_f A$ such that $x \notin \{y\}^A M_A$.

Let $N_H = M_A \cap M'_A \cap H \cap \varphi^{-1}(M_A \cap M'_A \cap K)$. By Lemma 3.2, there exists $N_A \lhd_f A$ such that $N_A \cap H \subseteq N_H$, $N_A H \cap N_A K = N_A$ and $\varphi(N_A \cap H) = N_A \cap K$. Let $R_A = N_A \cap M_A \cap M'_A$. Then $R_A H \cap R_A K = R_A$ and $\varphi(R_A \cap H) = R_A \cap K$. So $t^{-1}(R_A \cap H)t = R_A \cap K$ and $R_A \cap H = t(R_A \cap K)t^{-1} = tKt^{-1} \cap tR_A t^{-1}$. Therefore $HR_A/R_A \cong H/(H \cap R_A) \cong tKt^{-1}/(tKt^{-1} \cap tR_A t^{-1}) \cong (tKt^{-1})(tR_A t^{-1})/tR_A t^{-1}$. Note that $tR_A t^{-1} \lhd_f tAt^{-1}$ and $tKt^{-1} \lhd tAt^{-1}$. So we may form an epimorphic image of G_H, which is $\overline{G}_H = \overline{A}_{\overline{H} \overset{*}{=} t\overline{K}t^{-1}} t\overline{A}t^{-1}$, where $\overline{A} = A/R_A$, $t\overline{A}t^{-1} = tAt^{-1}/tR_A t^{-1}$, $\overline{H} = HR_A/R_A$ and $t\overline{K}t^{-1} = (tKt^{-1})(tR_A t^{-1})/tR_A t^{-1}$. Note that $\overline{H} \cap \overline{K} = 1$. Now, in $\overline{G_H}$, we have $\overline{x}, \overline{y} \notin \overline{H}$ and $\overline{x} \notin \{\overline{y}\}^{\overline{A}}$. By [26, Theorem 4.6 on p. 212], $\overline{x} \notin \{\overline{y}\}^{\overline{G_H}}$. By [8, Theorem 4], $\overline{G_H}$ is conjugacy separable. So there exists $\overline{N} \lhd_f \overline{G_H}$ such that $\overline{x} \notin \{\overline{y}\}^{\overline{G_H}} \overline{N}$. Let N be the preimage of $\overline{N}$ in G_H. Then $x \notin \{y\}^{G_H} N$ and we are done.

Subcase 2.3. Suppose $x \in H$ and $y \in K$, or $y \in H$ and $x \in K$. We shall assume that $x \in H$ and $y \in K$.

Since A is H-separable and $y \notin H$, there exists $M'_A \lhd_f A$ such that $y \notin HM'_A$. Let $N_H = M'_A \cap H \cap \varphi^{-1}(M'_A \cap K)$. Let N_A, R_A and $\overline{G}_H$ be as in Subcase 1.2. Note that $\overline{H} \cap \overline{K} = 1$. Now, in $\overline{G_H}$, we have $\overline{x} \notin \overline{H}$, $\overline{y} \notin \overline{H}$ and $\overline{H} \lhd \overline{A}$. So, by [26, Theorem 4.6 on p. 212], $\overline{x} \notin \{\overline{y}\}^{\overline{G_H}}$. By [8, Theorem 4], $\overline{G_H}$ is conjugacy separable. So there exists $\overline{N} \lhd_f \overline{G_H}$ such that $\overline{x} \notin \{\overline{y}\}^{\overline{G_H}} \overline{N}$. Let N be the preimage of $\overline{N}$ in G_H. Then $x \notin \{y\}^{G_H} N$ and we are done.

The proof of this theorem is now complete. □

Corollary 3.9. *Let $G = \langle t, A; t^{-1}Ht = K, \varphi\rangle$ be an HNN extension where A is a polycyclic-by-finite group, H, K are non-trivial normal subgroups of A, and $H \cap K = 1$. Let G_H be the normalizer of H in G. Then G_H is conjugacy separable if and only if G is conjugacy separable.*

Proof. Follows from Theorem 3.8. □

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Kok Bin Wong

Institute of Mathematical Sciences, University of Malaya, 50603 Kuala Lumpur, Malaysia

Email: kbwong@um.edu.my

Peng Choon Wong

Institute of Mathematical Sciences, University of Malaya, 50603 Kuala Lumpur, Malaysia

Email: wongpc@um.edu.my