www.theoryofgroups.ir

International Journal of Group Theory
ISSN (print): 2251-7650, ISSN (on-line): 2251-7669
Vol. 5 No.4 (2016), pp. 17-22.

www.ui.ac.ir

A NOTE ON THE COPRIME GRAPH OF A GROUP

HAMID REZA DORBIDI

Communicated by Alireza Abdollahi

Abstract. In this paper we study the coprime graph of a group G. The coprime graph of a group G, denoted by Γ_G, is a graph whose vertices are elements of G and two distinct vertices x and y are adjacent if and only if $(|x|,|y|)=1$. In this paper, we show that $\chi(\Gamma_G)=\omega(\Gamma_G)$. We classify all the groups which Γ_G is a complete $r-$partite graph or a planar graph. Also we study the automorphism group of Γ_G.

1. **Introduction**

The coprime graph of a group is defined in [9]. The coprime graph of a group G, denoted by Γ_G, is a graph whose vertices are elements of G and two elements $x \neq y$ are adjacent iff $(|x|,|y|)=1$. In this paper we generalize the results in [9]. Also, we give a complete answer to the following question which is asked in [9]:

Is it possible to characterize all finite groups G having the property that $Aut(\Gamma_G) \cong G$?.

First we recall some facts and notations related to this paper. Throughout this paper G denotes a nontrivial finite group. The centralizer of $H \leq G$ is denoted by $C_G(H)$. The normalizer of a subgroup H is denoted by $N_G(H)$. Also $Z(G)$ denotes the center of G. The symmetric group on n letters is denoted by S_n.

Let $\pi(n)$ be the set of prime divisors of n. For a natural number $n=p_1^{n_1}\cdots p_k^{n_k}$ set $r(n)=p_1\cdots p_k$. Also $\phi(n)=p_1^{n_1-1}\cdots p_k^{n_k-1}(p_1-1)\cdots(p_k-1)$ is the number of integers less than n which are coprime to n.

Let Γ be a simple graph. The *degree* of $v \in V(\Gamma)$ is denoted by $d(v)$. The set of vertices which are adjacent to v is denoted by $N_\Gamma(v)$. A clique is a complete subgraph of Γ. The *clique number* of

MSC(2010):Primary:05C25; Secondary:20F65.

Keywords: Coprime graph, Planar graph, Graph automorphism.

Received: 31 October 2014, Accepted: 13 April 2015.

Γ, denoted by $\omega(\Gamma)$, is the suprimum of orders of cliques in Γ. The *chromatic number* of Γ, $\chi(\Gamma)$, is the minimum k for which there is an assignment of k colors, $1, \ldots, k$, to the vertices of Γ such that adjacent vertices have different colors.

We denote $\chi(\Gamma_G), \omega(\Gamma_G)$ by $\chi(G), \omega(G)$ respectively. Also denote $\pi(|G|)$ by $\pi(G)$.

Remark 1.1. *If S is a commutative semigroup with a zero element then the zero-divisor graph of S is a graph whose vertices are elements of S and two distinct elements x, y are adjacent iff $xy = 0$. Let G be a finite group and $\pi : G \to A = \{|g| : g \in G\}$ be the order map. Also let $f : A \to G$ be a map such that $\pi o f = 1_A$. For $g, h \in G$ set $g * h = f((|g|, |h|))$. Then $(G, *)$ is a semi group whose zero-divisor graph is the coprime graph of G.*

2. Preliminaries

Before proving the main theorems, we need the following lemmas and theorems from group theory and number theory. The following lemma and theorems are well known.

Lemma 2.1.
(1) *Let H be a subgroup of G. Then $N_G(H)/C_G(H)$ is isomorphic to a subgroup of $Aut(H)$.*
(2) *Assume that $K \trianglelefteq G$ and $|K| = p$ where p is the least prime divisor of $|G|$. Then $K \subseteq Z(G)$.*

Theorem 2.2. *Let $a \in G$ be an element of order n such that $\langle a \rangle \trianglelefteq G$. If $bab^{-1} = a^i$ then $(i, n) = 1, ba^kb^{-1} = a^{ik}$ and $b^kab^{-k} = a^{i^k}$. In particular, $b^k \in C_G(a)$ iff $n | i^k - 1$.*

Theorem 2.3. *Let G be a finite group.*
(1) *G is a nilpotent group iff $G/Z(G)$ is a nilpotent group.*
(2) *G is a nilpotent group iff every Sylow subgroup of G is a normal subgroup. So G is isomorphic to direct product of its Sylow subgroups.*
(3) *Let G be a nilpotent group. If p is a prime divisor of $|G|$ then p divides $|Z(G)|$.*

Lemma 2.4. *If $n = p_1^{n_1} \cdots p_k^{n_k}$ then $\sum_{d|n} d = \frac{(p_1^{n_1+1}-1)}{p_1-1} \cdots \frac{(p_k^{n_k+1}-1)}{p_k-1}$.*

Lemma 2.5. *Set $f(n) = \sum \phi(d)$, where $r(n)|d|n$. If $n = p_1^{n_1} \cdots p_k^{n_k}$ then $f(n) = (p_1^{n_1} - 1) \cdots (p_k^{n_k} - 1)$.*

Proof. Set $a = (1 - \frac{1}{p_1}) \cdots (1 - \frac{1}{p_k})$. It is clear that $\phi(n) = na$. If $r(n)|d|n$ then set $d' = \frac{d}{r(n)}$. It is clear that $r(n)|d|n$ iff $d'|n'$. So $\phi(d) = da = r(n)ad'$. Hence $f(n) = \sum \phi(d) = \Sigma_{d'|n'} r(n)ad' = r(n)a\Sigma_{d'|n'} d' = (p_1^{n_1} - 1) \cdots (p_k^{n_k} - 1)$ by Lemma 2.4. □

3. Some properties of coprime graph

The following theorem is a special property of Γ_G. For other graphs such as non-commuting graph or zero divisor graph the clique number and chromatic number aren't equal in general. But for coprime graph the clique number and chromatic number are equal.

Theorem 3.1. $\chi(G) = \omega(G) = |\pi(G)| + 1$.

Proof. Let $|G| = p_1^{n_1} \cdots p_k^{n_k}$. Assume g_i is an element of G of order p_i for each $1 \leq i \leq k$. Define $f : G \to \{1, \dots, k+1\}$ as follows: $f(1) = k+1$ and for $g \neq 1$, $f(g) = i$ where i is the first index which $p_i \big| |g|$. It is clear that f is a coloring of Γ_G. Also $\{g_1, \dots, g_k, 1\}$ is a clique in Γ_G. Hence $k+1 \leq \omega(G) \leq \chi(G) \leq k+1$ which implies that $k+1 = \omega(G) = \chi(G)$. □

The next theorem is a generalization of Propositions in [9, Propositions 2.6, 2.7, 2.8].

Theorem 3.2. *Γ_G is a complete $r-$partite graph iff the order of every non-identity element of G is a prime power and $r = |\pi(G)| + 1$.*

Proof. First, assume that the order of every non-identity element of G is a prime power and $|G| = p_1^{n_1} \cdots p_k^{n_k}$. Let $V_i = \{g \in G : g \neq 1, |g| \big| p_i^{n_i}\}$ and $V_{k+1} = \{1\}$. Then it is clear that $G = \bigcup_{i=1}^{r} V_i$. Also if $i \neq j$ then each vertex in V_i is adjacent to vertices in V_j. So Γ_G is a complete $r-$partite graph. Conversely, assume that Γ_G is a complete $r-$partite graph. Assume $p_i p_j \big| |g|$. Let g_i, g_j be two elements of G of order p_i, p_j respectively. Then g_i, g_j lie in different parts of Γ_G. But g is not adjacent to g_i, g_j. So g must be in both parts which is a contradiction. So the order of every non-identity element is a prime power. Also $\omega(G) = r = |\pi(G)| + 1$. □

Example 3.3. *It is easily seen that every element of A_5 has prime power order. According to Theorem 3.2 , Γ_G is a complete 4-partite group. In particular , $\Gamma_G \cong K_{1,15,20,24}$.*

The following Theorem of *G. Higman* classifies all the solvable groups G in which every element has prime power order[7, p 213].

Theorem 3.4. *Let G be a solvable group in which every element has prime power order. Then G is one of the following groups:*

(1) *A Frobenius group $G = FH$, where F is an abelian $p-$group $(p > 2)$ and H is a generalized quaternion group.*
(2) *G has a normal series $P \trianglelefteq PQ \trianglelefteq G$ where G/P and PQ are Frobenius groups, P and G/PQ are $p-$groups, PQ/P is a $q-$group, PQ/P and G/PQ are cyclic(Here $p|q-1$).*

Corollary 3.5. *If $|\pi(G)| \geq 3$ and G is a solvable group i.e $|G|$ is odd then Γ_G is not a complete $r-$partite graph.*

Proof. It follows from Theorems 3.2 and 3.4. □

The following theorem improves [9, proposition 2.10].

Theorem 3.6. *Γ_G is a planar graph iff G is a $p-$group or $G \cong \mathbb{Z}_2 \times Q$ where Q is a $q-$group.*

Proof. First assume that Γ_G is a planar graph. So Γ_G doesn't contain $K_{3,3}$(see [4]). Assume that G is not a $p-$group and $p^a q^b \big| |G|$ where $p < q$ are distinct primes and $p^{a+1}, q^{b+1} \nmid |G|$. Let P be the union of Sylow $p-$subgroups of G and Q be the union of Sylow $q-$subgroups of G. Thus $|Q| \geq 3$. If $|P| - 1 \geq 3$ then every vertex in $P - \{1\}$ is adjacent to every vertex of Q. So Γ_G contains $K_{3,3}$ which is a contradiction. Hence $p^a - 1 \leq |P| - 1 < 3$. If $p^a = 3$ then $|Q| \geq 5$ and every vertex in P

is adjacent to every vertex of $Q - \{1\}$. So Γ_G contains $K_{3,3}$ which is a contradiction. Hence $p^a = 2$. This implies that $|G| = 2q^b$. If $|P| = 3$ and $|Q| \geq 4$ then every vertex in P is adjacent to every vertex of $Q - \{1\}$. So Γ_G contains $K_{3,3}$ which is a contradiction. Thus $|Q| = 3$. This implies that $|G| = 6$. Because $\Gamma_{S_3} \cong K_{1,2,3}$ is not planar so $G \cong \mathbb{Z}_6 \cong \mathbb{Z}_2 \times \mathbb{Z}_3$. Assume $|P| = 2$ which implies that the Sylow $2-$subgroup of G is a normal subgroup of G. So $P \subseteq Z(G)$ by lemma 2.1. Hence $G/Z(G)$ is a $q-$group. So G is a nilpotent group. Hence $G \cong \mathbb{Z}_2 \times Q$. The converse is clear. □

Remark 3.7. *In* [9, Remark 2.12] *it is claimed that for* $G = \mathbb{Z}_4 \times \mathbb{Z}_3 \times \mathbb{Z}_3$, Γ_G *is a planar graph. But the elements* $(1,0,0),(2,0,0),(3,0,0)$ *and* $(0,1,0),(0,2,0),(0,0,1)$ *constitute a* $K_{3,3}$ *in* Γ_G*. So* Γ_G *is not a planar graph.*

The following theorem is a generalization of [9, Proposition 3.3].

Theorem 3.8. *If* G *is a nilpotent group of order* n *then* Γ_G *has* $f(n)$ *end vertices.*

Proof. G is isomorphic to direct product of its Sylow subgroups. Assume $G = P_1 \times \cdots \times P_k$. Then $(P_1 \backslash \{1\}) \times \cdots \times (P_k \backslash \{1\})$ is the set of end vertices which has cardinality $f(n)$. □

4. Automorphism group of coprime graph

In this section, we study the Automorphism group of coprime graph. First we recall the following theorem about automorphisms of a simple graph.

Theorem 4.1. *Let* V *be a simple graph. Let* $\sigma : V \to V$ *be a bijection such that for all* $v \in V$*, we have* $N_V(v) = N_V(\sigma(v))$*. Then* σ *is a graph automorphism.*

Proof. Let a, b be two adjacent vertices. Then $a \in N_V(b) = N_V(\sigma(b))$. So $\sigma(b) \in N_V(a) = N_V(\sigma(a))$. Hence $\sigma(a), \sigma(b)$ are adjacent. So σ is a graph automorphism. □

Define an equivalence relation on G as follows:

$a \sim b$ iff $r(|a|) = r(|b|)$. It is clear that $a \sim b$ iff $N_\Gamma(a) = N_\Gamma(b)$. Let $[a]$ be the equivalence class of a. It is clear that $\{b \in G : |a| = |b|\} \subseteq [a]$. In particular, $Cl(a) \bigcup \{a^k : (k, |a|) = 1\} \subseteq [a]$. So $[G : C_G(a)] = |Cl(a)| \leq |[a]|$ and $\phi(|a|) \leq f(|a|) \leq |[a]|$. Assume that $G/\sim = \{[a_i]\}$ and $|[a_i]| = n_i,\ |G/\sim| = t$

Theorem 4.2.
1. $S_{n_1} \times \cdots \times S_{n_t} \subseteq Aut(\Gamma_G)$.
2. *If* $a \in P$ *where* P *is Sylow* $p-$*subgroup then* $[a] = \bigcup_{g \in G} gPg^{-1} - \{1\}$
3. *If* $p^n \big| |G|$ *then* $S_{p^n - 1} \subseteq Aut(\Gamma_G)$.
4. *If* $|a| = m$ *then* $f(m) \leq |[a]|$ *and* $S_{f(m)} \subseteq Aut(\Gamma_G)$.
5. *If* G *contains* m *end vertices then* $S_m \subseteq Aut(\Gamma_G)$*. In particular, for a nilpotent group* G *of order* n*,* $S_{f(n)} \subseteq Aut(\Gamma_G)$.

Proof. (1) Let $\sigma = (\sigma_1, \ldots, \sigma_t) \in S_{n_1} \times \cdots \times S_{n_t}$. If σ_i acts on $[a_i]$ then σ is an automorphism by Theorem 4.1.

(2) It is clear that $r(|a|) = p$. So the order of every element in $[a]$ is a p-power. Hence by Sylow's theorem on conjugacy of Sylow subgroups the proof is complete.
(3) Let a be a $p-$element. Then $p^n - 1 \leq |[a]|$ by part (2). Hence $S_{p^n-1} \subseteq Aut(\Gamma_G)$ by part (1).
(4) The subgroup $\langle a \rangle$ contains $f(m)$ elements g which $r(m)\big||g|\big|m$. Hence $r(m) = r(|g|)$. So these elements are in $[a]$. So $S_{f(m)} \subseteq Aut(\Gamma_G)$.
(5) The end vertices constitute a class.

□

Theorem 4.3. *$Aut(G) = Aut(\Gamma_G)$ iff $G \cong \mathbb{Z}_3, \mathbb{Z}_2 \bigoplus \mathbb{Z}_2$.*

Proof. If $Aut(G) = Aut(\Gamma_G)$ then every graph automorphism is a group automorphism. Assume there are $a \neq b \in G$ of the same order and $x \in G\backslash\{1, a, b, a^{-1}b\}$. Define $\sigma : \Gamma_G \to \Gamma_G$ by $\sigma(a) = b, \sigma(b) = a, \forall g \in G\backslash\{a, b\}$ $\sigma(g) = g$. So σ is a graph automorphism by Theorem 4.1. Hence it is a group automorphism. Hence $ax = \sigma(ax) = \sigma(a)\sigma(x) = bx$ which is a contradiction. This implies that $|G| \leq 4$. So $G \cong \mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_2 \bigoplus \mathbb{Z}_2$. We have $Aut(\mathbb{Z}_2) = \{1\}, Aut(\mathbb{Z}_3) = \mathbb{Z}_2, Aut(\mathbb{Z}_4) = \mathbb{Z}_2, Aut(\mathbb{Z}_2 \bigoplus \mathbb{Z}_2) = S_3$. Also $Aut(\Gamma_{\mathbb{Z}_2}) = \mathbb{Z}_2, Aut(\Gamma_{\mathbb{Z}_3}) = \mathbb{Z}_2, Aut(\Gamma_{\mathbb{Z}_4}) = S_3, Aut(\Gamma_{\mathbb{Z}_2 \bigoplus \mathbb{Z}_2}) = S_3$. So the proof is complete. □

The next theorem is a generalization of [9, Theorem 4.3].

Theorem 4.4. *Let G be a nontrivial group. $Aut(\Gamma_G)$ is a solvable group iff $G \cong \mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_2 \bigoplus \mathbb{Z}_2, \mathbb{Z}_5, \mathbb{Z}_6, \mathbb{Z}_{10}, S_3$.*

Proof. We know that S_n is a solvable group iff $n \leq 4$. So every equivalence class has size less than 4. In particular, if G has an element of order m then $f(m) \leq 4$. Let $|G| = n = p_1^{n_1} \cdots p_k^{n_k}$. Let P_i be the union of Sylow p_i-subgroups and $p_i^{n_i} \leq |P_i| = t_i$. According to part $(1), (2)$ of Theorem 4.2, $S_{t_i-1} \subseteq Aut(\Gamma_G)$. Hence $t_i - 1 \leq 4$. So $p_i \in \{2, 3, 5\}$ and $t_i \in \{2, 3, 4, 5\}$. Assume $p_i^{n_i} \in \{3, 4, 5\}$. We claim that Sylow p_i-subgroup is normal. Else there are at least three Sylow p_i-subgroup which implies that $|P_i| \geq 6$ which is a contradiction. If $4\big||G|$ then G is a nilpotent group because its Sylow subgroups are normal. So $S_{f(n)} \subseteq Aut(\Gamma_G)$. Hence $f(n) \leq 4$. So $n = 4$ and $G \cong \mathbb{Z}_4, \mathbb{Z}_2 \bigoplus \mathbb{Z}_2$. If $|G|$ be an odd number then G is a nilpotent group because its Sylow subgroups are normal. So $S_{f(n)} \subseteq Aut(\Gamma_G)$. Hence $f(n) \leq 4$. So $n = 3, 5$ and $G \cong \mathbb{Z}_3, \mathbb{Z}_5$. If $15\big||G|$ then G contains a subgroup of order 15. But every group of order 15 is a cyclic group. Hence the equivalence class of its generator has $f(15) = 8$ elements which is a contradiction. So $|G| = 6$ or $|G| = 10$. If $|G| = 6$ then $G \cong \mathbb{Z}_6, S_3$. Because D_{10} contains 5 elements of order two so G has an equivalence class of size 5 which is a contradiction. So, if $|G| = 10$ then $G \cong \mathbb{Z}_{10}$. Conversely, by drawing graphs,we have $Aut(\Gamma_{\mathbb{Z}_2}) \cong Aut(\Gamma_{\mathbb{Z}_3}) \cong \mathbb{Z}_2, Aut(\Gamma_{\mathbb{Z}_2 \bigoplus \mathbb{Z}_2}) \cong Aut(\Gamma_{\mathbb{Z}_4}) \cong S_3, Aut(\Gamma_{\mathbb{Z}_5}) \cong S_4, Aut(\Gamma_{\mathbb{Z}_6}) \cong S_2 \times S_2, Aut(\Gamma_{S_3}) \cong S_2 \times S_3, Aut(\Gamma_{\mathbb{Z}_{10}}) \cong S_4 \times S_4,$ □

The following theorem gives a complete answer to in [9, Question 4.5].

Theorem 4.5. *$G \cong Aut(\Gamma_G)$ iff $G \cong \mathbb{Z}_2$.*

Proof. Let $|G| = n = p_1^{n_1} \cdots p_k^{n_k}$ and $p_1^{n_1} < \cdots < p_k^{n_k}$. According to parts $(1), (2)$ of Theorem 4.2, $(p_1^{n_1} - 1)! \cdots (p_k^{n_k} - 1)!\big||Aut(G)| = |G|$. So $\frac{(p_1^{n_1}-1)!}{p_1^{n_1}} \cdots \frac{(p_k^{n_k}-1)!}{p_k^{n_k}} \leq 1$. Note that $\frac{(m-1)!}{m}$ is an increasing function. If $k \geq 3$ then $p_3^{n_3} \geq 5$. So $\frac{(p_1^{n_1}-1)!}{p_1^{n_1}} \cdots \frac{(p_k^{n_k}-1)!}{p_k^{n_k}} \geq \frac{1!}{2}\frac{2!}{3}\frac{4!}{5} > 1$ which is a contradiction. Hence $k \leq 2$. If $p_2^{n_2} \geq 5$ then $\frac{(p_1^{n_1}-1)!(p_2^{n_2}-1)!}{p_1^{n_1}p_2^{n_2}} \geq \frac{1!}{2}\frac{4!}{5} > 1$ which is a contradiction. Thus $k \leq 2$ and $p_2^{n_2} \leq 4$. So $|G| = 2, 3, 4, 6, 12$. By drawing coprime graphs of these groups we conclude that $G \cong \mathbb{Z}_2$. □

Remark 4.6. *In* [9] *it is claimed that* $Aut(\Gamma_{S_3}) \cong S_3$*. But* $Aut(\Gamma_{S_3}) \cong S_2 \times S_3$

Acknowledgments

The author would like to express his deep gratitude to the referee for a very careful reading of the paper, and many valuable comments, which have greatly improved the presentation of the paper.

References

[1] S. Akbari and A. Mohammadian, On zero-divisor graphs of finite rings, *J. Algebra*, **314** (2007) 168–184.

[2] D. F. Anderson and P. S. Livingston, The zero-divisor graph of a commutative ring, *J. Algebra*, **217** (1999) 434–447.

[3] I. Beck, Coloring of commutative rings, *J. Algebra*, **116** (1988) 208–226.

[4] J. A. Bondy and U. S. R. Murty, *Graph theory with applications*, American Elsevier Publishing Co, Inc., New York, 1976.

[5] Y. Chen, On Thompson's conjecture, *J. Algebra*, **185** (1996) 184–193.

[6] J. A. Gallian, *Contemporary Abstract Algebra*, D. C. Heath and company, 1994.

[7] B. Huppert, *Character Theory of Finite Groups*, De Gruyter Expositions in Mathematics, **25**, Walter de Gruyter Co., Berlin, 1998.

[8] D. Lu and T. Wu, On bipartite zero-divisor graphs, *Discrete Math.*, **309** (2009) 755–762.

[9] X. Ma, H. Wei and L. Yang, The coprime graph of a group, *Int. J. Group Theory*, **3** no. 3 (2014) 13–23.

[10] S. P. Redmond, On zero-divisor graphs of small finite commutative rings, *Discrete Math.*, **307** (2007) 1155–1166.

[11] S. Spiroff and C. Wickham, A zero divisor graph determined by equivalence classes of zero divisors, *Comm. Algebra*, **39** (2011) 2338–2348.

[12] J. S. Williams, Prime graph components of finite groups, *J. Algebra*, **69** (1981) 487–513.

[13] T. Wu, Q. Liu and L. Chen, Zero-divisor semigroups and refinements of a star graph, *Discrete Math.*, **309** (2009) 2510–2518.

Hamid Reza Dorbidi

Department of Basic Sciences, University of Jiroft, P. O. Box 78671-61167, Jiroft, Kerman, Iran

Email: hr_dorbidi@ujiroft.ac.ir