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A NOTE ON GROUPS WITH MANY LOCALLY SUPERSOLUBLE SUBGROUPS

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ABSTRACT. It is proved here that if G is a locally graded group satisfying the minimal condition on subgroups which are not locally supersoluble, then G is either locally supersoluble or a Černikov group. The same conclusion holds for locally finite groups satisfying the weak minimal condition on non-(locally supersoluble) subgroups. As a consequence, it is shown that any infinite locally graded group whose non-(locally supersoluble) subgroups lie into finitely many conjugacy classes must be locally supersoluble.

1. Introduction

If $\mathfrak{X}$ is a class of groups, a group G is said to be *minimal non-$\mathfrak{X}$* if G is not an $\mathfrak{X}$-group, but all its proper subgroups belong to $\mathfrak{X}$. Of course, *Tarski groups*, i.e. infinite simple groups whose proper non-trivial subgroups have prime order, are minimal non-abelian; the existence of Tarski groups has been proved by A. Y. Ol'shanskiĭ [10], and in order to avoid these and other similar pathologies, the attention is often restricted to the universe of locally graded groups. Recall here that a group G is *locally graded* if every finitely generated non-trivial subgroup of G contains a proper subgroup of finite index; locally graded groups form a large group class, containing every locally (soluble-by-finite) group. The structure of minimal non-$\mathfrak{X}$ groups has been investigated for several different choices of the group class $\mathfrak{X}$. In particular, finite minimal non-supersoluble groups were described by K. Doerk [4], and

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from his results it follows that any infinite locally graded group whose proper subgroups are locally supersoluble is itself locally supersoluble (see [6]).

The aim of this paper is to obtain further informations on groups such that the set of subgroups which are not locally supersoluble is small in some sense. Our first main result deals with groups for which this set satisfies the minimal condition, i.e. groups admitting no infinite descending chains consisting of subgroups that are not locally supersoluble. It turns out that only the extreme cases can occur.

Theorem A. *Let G be a locally graded group satisfying the minimal condition on subgroups which are not locally supersoluble. Then G is either locally supersoluble or a Černikov group.*

Note that the consideration of the standard wreath product of a group of order 2 by an infinite cyclic group shows that the imposition of the maximal condition on subgroups which are not locally supersoluble does not give rise to a statement similar to Theorem A.

Let $\mathfrak{X}$ be any class of groups. A group G is said to satisfy the *weak minimal condition* on $\mathfrak{X}$-subgroups if there are no infinite descending chains

$$X_1 > X_2 > \cdots > X_n > X_{n+1} > \cdots$$

of $\mathfrak{X}$-subgroups of G such that the index $|X_n : X_{n+1}|$ is infinite for each positive integer n. It was proved by D. I. Zaicev [13] that a locally soluble group satisfies the weak minimal condition on all subgroups if and only if it is minimax. Recall that a group is *minimax* if it admits a series of finite length whose factors satisfy either the minimal or the maximal condition on subgroups; in particular, any minimax locally finite group is Černikov. In contrast to this result, it can be proved that there exists a metabelian group satisfying the weak minimal condition on non-(locally supersoluble) subgroups, which neither is locally supersoluble nor satisfies the weak minimal condition on subgroups. On the other hand, a sharp result can be proved in the case of periodic groups.

Theorem B. *Let G be a locally finite group satisfying the weak minimal condition on subgroups which are not locally supersoluble. Then G is either locally supersoluble or a Černikov group.*

As a consequence of Theorem A, it can also be proved that if G is an infinite locally graded group which is not locally supersoluble, then the non-(locally supersoluble) subgroups of G fall into infinitely many conjugacy classes.

Theorem C. *Let G be a locally graded group with only finitely many conjugacy classes of subgroups which are not locally supersoluble. Then G is either finite or locally supersoluble.*

Observe finally that the structure of groups satisfying the minimal condition on non-supersoluble subgroups and that of groups with finitely many conjugacy classes of non-supersoluble subgroups were described in the locally graded case by M. De Falco [3].

Most of our notation is standard and can be found in [11].

2. **Proofs and Examples**

Our first lemma is a combination of results by F. Napolitani and E. Pegoraro [9] and by A. O. Asar [2]. It shows that within the universe of locally graded groups there are no minimal non-(nilpotent-by-Černikov) groups (a group G is called *nilpotent-by-Černikov* if it contains a nilpotent normal subgroup N such that G/N is a Černikov group).

Lemma 2.1. *Let G be a locally graded group whose proper subgroups are nilpotent-by-Černikov. Then G is nilpotent-by-Černikov.*

Of course, every group whose proper subgroups either are locally supersoluble or have the minimal condition on subgroups, satisfies the minimal condition on non-(locally supersoluble) groups, and the above lemma can be used to show that simple locally graded groups with this property must be finite.

Lemma 2.2. *Let G be an infinite locally graded group whose proper subgroups either are locally supersoluble or satisfy the minimal condition. Then G cannot be simple.*

Proof. Assume for a contradiction that the group G is simple. In particular, G is not locally supersoluble, and so it contains a finitely generated non-supersoluble subgroup E. Moreover, G cannot be finitely generated, because it is infinite and locally graded, and so $\langle E, X\rangle$ is properly contained in G for each finitely generated subgroup X. It follows that $\langle E, X\rangle$ satisfies the minimal condition, and hence it is finite. Therefore G is a locally finite group whose proper subgroups are either locally supersoluble or Černikov groups. Since every proper subgroup of G is (locally soluble)-by-finite, it is known that G is isomorphic either to the projective group $PSL(2, F)$ or to the Suzuki group $Sz(F)$ for a suitable infinite field F (see [8]). On the other hand, every locally supersoluble linear group is nilpotent-by-finite (see [12], Theorem 11.21), and so all proper subgroups of G are nilpotent-by-finite. Then G is nilpotent-by-Černikov by Lemma 2.1, and this contradiction proves the statement. □

Proof of Theorem A – Assume for a contradiction that the statement is false, and in the counterexample G choose a subgroup H which is minimal with respect to the condition of being neither locally supersoluble nor a Černikov group. Clearly, G can be replaced by H, and so we may suppose without loss of generality that each proper subgroup of G is either locally supersoluble or a Černikov group. Consider now a minimal element E of the set of all subgroups of G which are not locally supersoluble. Then E is a minimal non-(locally supersoluble) group, so that it is finite (see [6]).

Suppose now that

$$X_1 > X_2 > \cdots > X_n > \cdots$$

is an infinite descending sequence of normal subgroups of G. For each positive integer n, the subgroup X_nE neither is locally supersoluble nor a Černikov group, so that $X_nE = G$ and hence the index $|G : X_n|$ is bounded by the size of E. This is clearly impossible, and so G satisfies the minimal condition on normal subgroups. It follows from a result of J. S. Wilson that every subgroup of finite index of G likewise satisfies the minimal condition on normal subgroups (see [11] Part 1, Theorem 5.21). Since any locally supersoluble group with the minimal condition on normal subgroups is a Černikov group (see [11] Part 1, p.156), we obtain that G cannot contain proper subgroups of finite index. In particular, G is not finitely generated. If G contains an element a of infinite order, the non-periodic subgroup $\langle a, E\rangle$ is not supersoluble, and so $G = \langle a, E\rangle$. This contradiction shows that G is periodic, and so even locally finite.

Let N be any proper normal subgroup of G. As the index $|G : N|$ is infinite, the product NE is a proper subgroup of G and so N is a Černikov group. It follows that also $G/C_G(N)$ is a Černikov group (see [11] Part 1, Theorem 3.29), and hence the normal subgroup $C_G(N)$ is not a Černikov group. Thus $C_G(N) = G$ and N is contained in the centre $Z(G)$. Therefore the factor group $G/Z(G)$ is simple, so that it must be finite by Lemma 2.2, and this last contradiction completes the proof of the theorem. □

It is known that a locally finite group satisfying the weak minimal condition on subgroups is a Černikov group ([13]; see also [7], Corollary 5.9), so that in particular any infinite locally finite group which is also residually finite cannot satisfy the weak minimal condition on subgroups. Here we need the following slight extension of this result.

Lemma 2.3. *Let G be an infinite locally finite group, and let E be a finite subgroup of G. If G is residually finite, then there exists an infinite descending chain $(X_n)_{n\in\mathbb{N}}$ of subgroups of G containing E such that the index $|X_n : X_{n+1}|$ is infinite for each positive integer n.*

Proof. Put $E_1 = E$. Since G is residually finite, it contains a normal subgroup K_1 of finite index such that $E_1 \cap K_1 = \{1\}$. Choose a finite non-trivial subgroup E_2 of K_1, and let K_2 be a normal subgroup of finite index of G such that $\langle E_1, E_2\rangle \cap K_2 = \{1\}$. Iterating this procedure we can construct a sequence $(E_n)_{n\in\mathbb{N}}$ of finite non-trivial subgroups of G and a sequence $(K_n)_{n\in\mathbb{N}}$ of normal subgroups of finite index of G such that E_{n+1} is contained in K_n and

$$\langle E_1, E_2, \dots, E_{n+1}\rangle \cap K_{n+1} = \{1\}$$

for all n. Put $H(\emptyset) = \{1\}$ and

$$H(I) = \langle E_i \mid i \in I\rangle$$

for every non-empty subset I of $\mathbb{N}$. If I and J are finite subsets of $\mathbb{N}$ such that $I \cap J = \emptyset$, it can be easily proved by induction on $|I| + |J|$ that $H(I) \cap H(J) = \{1\}$, this equality being obvious if one of the sets I and J is empty. Suppose that both I and J are non-empty, and assume that the largest element m of I is greater than the largest element of J. Put

$$L = H(I \setminus \{m\}) = \langle E_i \mid i \in I, i < m\rangle,$$

so that $H(I) = \langle L, E_m\rangle$; since E_m is contained in K_{m-1} and

$$\langle H(J), L\rangle \cap K_{m-1} = \{1\},$$

we obtain that

$$H(I) \cap H(J) = L \cap H(J) = \{1\}.$$

It follows that $H(I) \cap H(J) = \{1\}$ for each pair (I, J) of disjoint subsets of $\mathbb{N}$. Consider now a descending sequence

$$\mathbb{N} = I_1 \supset I_2 \supset \cdots \supset I_n \supset I_{n+1} \supset \cdots$$

of infinite subsets of $\mathbb{N}$ containing 1 and such that the difference $I_n \setminus I_{n+1}$ is infinite for each n. Put $X_n = H(I_n)$, so that

$$X_1 > X_2 > \cdots > X_n > X_{n+1} > \cdots$$

is an infinite sequence of subgroups of G containing E, and the index $|X_n : X_{n+1}|$ is infinite for all n, because

$$H(I_{n+1}) \cap H(I_n \setminus I_{n+1}) = \{1\}.$$

The lemma is proved. □

Proof of Theorem B – Assume for a contradiction that the statement is false, and let G be a counterexample. It follows from Theorem A that G cannot satisfy the minimal condition on subgroups which are not locally supersoluble, and so there exists in G an infinite descending sequence

$$X_1 > X_2 > \cdots > X_n > X_{n+1} > \cdots$$

consisting of subgroups which are not locally supersoluble. Since G satisfies the weak minimal condition on non-(locally supersoluble) subgroups, there is a positive integer k such that the index $|X_n : X_{n+1}|$ is finite for each $n \geq k$. Let J be the finite residual of $Y = X_k$, and consider a finite non-supersoluble subgroup E of Y. Clearly, Y/J is an infinite residually finite group, and hence Lemma 2.3 yields that there exists an infinite descending chain $(Y_n/J)_{n\in\mathbb{N}}$ of subgroups of Y/J containing EJ/J such that the index $|Y_n : Y_{n+1}|$ is infinite for each positive integer n. Then Y_n is not locally supersoluble for each n, contradicting the weak minimality assumption on the group G. The statement is proved. □

Consider the abelian group

$$A = \langle a\rangle \times \langle b\rangle \times \langle c_1\rangle \times \langle c_2\rangle \times \cdots \times \langle c_n\rangle \times \langle c_{n+1}\rangle \times \cdots ,$$

where a and b have order 2 and c_n has order 3 for each positive integer n, and let x be the automorphism of A defined by putting $a^x = b$, $b^x = ab$ and $c_n^x = c_1c_2\cdots c_n$ for all n. Then x has infinite order, and the semidirect product

$$G = \langle x\rangle \ltimes A$$

is a metabelian group, which of course neither is locally supersoluble nor minimax. It is easy to see that A does not contain infinite proper $\langle x^k\rangle$-invariant subgroups, for each positive integer k which is not divisible by 3; on the other hand, the subgroup $\langle x^3, A\rangle$ is obviously hypercentral, and so in particular locally supersoluble. It follows that G satisfies the weak minimal condition on subgroups which are not locally supersoluble, and hence this example shows that Theorem B cannot be extended to the case of soluble non-periodic groups.

In order to prove Theorem C, we need the following result due to D. I. Zaicev, for a proof of which we refer to [1], Lemma 4.6.3.

Lemma 2.4. *Let G be a group locally satisfying the maximal condition on subgroups. If X is a subgroup of G and $X^g \leq X$ for some element g of G, then $X^g = X$.*

Proof of Theorem C – Clearly, the subgroups of G which do not locally satisfy the maximal condition fall into finitely many conjugacy classes, and hence the group G itself locally satisfies the maximal condition on subgroups (see [5], Proposition 3.3). Then it follows from Lemma 2.4 that G satisfies both the minimal and the maximal condition on subgroups which are not locally supersoluble. Suppose that G is not locally supersoluble, so that it is a Černikov group by Theorem A. Assume for a contradiction that the largest divisible subgroup J of G is non-trivial, and for every positive integer n let J_n be the n-th term of the upper socle series of J. If E is any finite subgroup of G, there is a positive integer k such that $E \cap J$ is contained in J_k. Then $EJ_n \neq EJ_{n+1}$ for every $n \geq k$, and so

$$EJ_k < EJ_{k+1} < \cdots < EJ_n < EJ_{n+1} < \cdots$$

is an infinite ascending sequence of subgroups of G. It follows that EJ_h is supersoluble for some $h \geq k$, so that in particular E is supersoluble, and G is locally supersoluble. This contradiction proves that $J = \{1\}$, and hence G is finite. □

Let G be a group satisfying the maximal condition on subgroups which are not locally supersoluble, and suppose that G is not locally supersoluble. If E is a finitely generated non-supersoluble subgroup of G, the set of all subgroups of G containing E satisfies the maximal condition, and hence G itself is finitely generated. In particular, any locally finite group satisfying the maximal condition on subgroups which are not locally supersoluble is either finite or locally supersoluble. On the other hand, the standard wreath product $\mathbb{Z}_2 \wr \mathbb{Z}$ of a group of order 2 by an infinite cyclic group is a metabelian group of infinite rank which satisfies the maximal condition on non-abelian subgroups (and so also on subgroups which are not locally supersoluble). This is a special case of the following result.

Lemma 2.5. *Let G be a finitely generated abelian-by-cyclic group. Then G satisfies the maximal condition on non-abelian subgroups.*

Proof. Let A be an abelian normal subgroup of G such that the factor group G/A is cyclic. Assume for a contradiction that

$$X_1 < X_2 < \cdots < X_n < X_{n+1} < \cdots$$

is a strictly ascending chain of non-abelian subgroups of G. Clearly, X_1 is not contained in A, and so it has an element of the form ag^k, where a belongs to A, the coset gA is a generator of G/A and k is a positive integer. Then the intersection $A_n = A \cap X_n$ is a normal subgroup of $H = \langle A, g^k \rangle$ for each positive integer n. As the group G satisfies the maximal condition on normal subgroups (see [11] Part 1, Theorem 5.34), and the index $|G : H|$ is finite, also H satisfies the maximal condition on normal subgroups (see [11] Part 1, Theorem 5.31) and hence there exists a positive integer m such that $A_n = A_m$ for each $n \geq m$. Moreover, m can obviously be chosen in such a way that also the equality $X_n A = X_m A$ holds for $n \geq m$. It follows that

$$X_n = X_n \cap X_m A = X_m(A \cap X_n) = X_m(A \cap X_m) = X_m$$

for all $n \geq m$, and this contradiction completes the proof of the lemma. □

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