www.theoryofgroups.ir

International Journal of Group Theory
ISSN (print): 2251-7650, ISSN (on-line): 2251-7669
Vol. 4 No. 2 (2015), pp. 25-48.

www.ui.ac.ir

ON DOUBLE COSETS WITH THE TRIVIAL INTERSECTION PROPERTY AND KAZHDAN-LUSZTIG CELLS IN S_n

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Communicated by Patrizia Longobardi

Abstract. For a composition λ of n our aim is to obtain reduced forms for all the elements in the Kazhdan-Lusztig (right) cell containing $w_{J(\lambda)}$, the longest element of the standard parabolic subgroup of S_n corresponding to λ. We investigate how far this is possible to achieve by looking at elements of the form $w_{J(\lambda)}d$, where d is a prefix of an element of minimum length in a $(W_{J(\lambda)}, B)$ double coset with the trivial intersection property, B being a parabolic subgroup of S_n whose type is 'dual' to that of $W_{J(\lambda)}$.

1. Introduction

In [13], when investigating the representations of a Coxeter group and its associated Hecke algebra, Kazhdan and Lusztig introduced three partitionings of the Coxeter group, the parts of which they called *left cells*, *right cells* and *two-sided cells*. There is a very simple connection between the left cells and the right cells; namely, the mapping $x \mapsto x^{-1}$ maps a left cell to a right cell and vice-versa. In the case of the symmetric group S_n, they showed that the equivalence relation whose equivalence classes are the left cells is essentially the same as an equivalence relation defined by Knuth in [14]. The cell to which an element of S_n belongs can be determined by examining the tableaux resulting from an application of the *Robinson-Schensted process* to that element. Also, the elements of a cell can be computed by applying the reverse of the Robinson-Schensted process to a suitable selection of tableaux pairs.

The Robinson-Schensted process however, does not provide a straightforward way of obtaining reduced forms for the elements of these cells. This paper is mainly concerned with the problem of

MSC(2010): Primary: 20C08; Secondary: 05E10, 20C30, 20F55.

Keywords: symmetric group, Hecke algebra, Kazhdan-Lusztig cell, generalized tableau, parabolic subgroup.

Received: 24 January 2015, Accepted: 3 June 2015.

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finding a direct way to describe reduced forms for the elements in certain cells, namely the (right) cells for which the unique involution they contain is in fact an element of longest length in some standard parabolic subgroup of S_n. (It is easy to observe from the Robinson-Schensted process that each right cell in S_n contains a unique involution.)

Let us introduce some notation at this point. For a composition λ of n, let $W_{J(\lambda)}$ be the standard parabolic subgroup of S_n corresponding to λ and let $w_{J(\lambda)}$ be the longest element of $W_{J(\lambda)}$. Also let $\mathcal{X}_{J(\lambda)}$ be a complete set of distinguished right coset representatives of $W_{J(\lambda)}$ in S_n. It is well known that the right cell containing $w_{J(\lambda)}$ has form $w_{J(\lambda)}Z(\lambda)$ for some subset $Z(\lambda)$ of $\mathcal{X}_{J(\lambda)}$.

In the special case that λ is a partition the elements of $Z(\lambda)$ are precisely the prefixes of the element of minimum length in the unique $(W_{J(\lambda)}, W_{J(\lambda')})$ double coset with the trivial intersection property (see for example [18, Lemma 3.3]).

Below let λ be an arbitrary composition of n. In the present paper we observe that every element of $\mathcal{X}_{J(\lambda)}$ occurs as the element of minimum length in a $(W_{J(\lambda)}, B)$ double coset with the trivial intersection property for some standard parabolic subgroup B of S_n. Moreover, we generalize our algorithm in [19] so that it gives a reduced form for the element of minimum length (and its prefixes) in an (A, B) double coset with the trivial intersection property where now A and B are arbitrary standard parabolic subgroups of S_n. For the investigations of the present paper we concentrate in the special case A and B are of 'dual' type, that is, they correspond respectively to compositions λ, μ such that the conjugate composition λ' is a rearrangement of μ. Setting $\hat{Z}(\lambda) = \{u \in \mathcal{X}_{J(\lambda)}$: there exists $\mu \vDash n$ with $\mu'' = \lambda'$ such that u is a prefix of the element of minimum length in the unique $(W_{J(\lambda)}, W_{J(\mu)})$ double coset with the trivial intersection property$\}$, it is then easy to observe that $\hat{Z}(\lambda) \subseteq Z(\lambda)$.

In Propositions 5.2 and 5.3 we give some examples of compositions λ for which $\hat{Z}(\lambda) = Z(\lambda)$. However there are examples of compositions for which $\hat{Z}(\lambda)$ is properly contained in $Z(\lambda)$. We investigate how far these examples can be dealt with by looking at $(W_{J(\lambda)}, e^{-1}W_{J(\lambda')}e)$ double cosets with the trivial intersection property, where $e \in S_n$. (Recall that subgroups $W_{J(\mu)}$ and $W_{J(\lambda')}$ are conjugate in S_n whenever $\mu'' = \lambda'$.) In Theorem 5.4 we show that any element of minimum length in such a double coset (and hence each of its prefixes) belongs to $Z(\lambda)$. For various examples of λ this allows us to obtain reduced forms for the elements of a subset of $Z(\lambda)$ which properly contains $\hat{Z}(\lambda)$.

Finally, in Theorems 5.6 and 5.8 (which are generalizations of Theorem 5.4), we obtain sufficient conditions for the element $d \in \mathcal{X}_{J(\lambda)}$ to belong to $Z(\lambda)$ which now depend on double cosets with the trivial intersection property of the form $W_{J(\lambda)}w(e^{-1}W_{J(\lambda')}e)$ where $e \in S_n$ and $w \in \mathcal{X}_{J(\lambda)}$ with $w \leq d$ in the strong Bruhat order.

2. **Preliminaries and generalities**

Let (W, S) be a Coxeter system corresponding to a Weyl group W and let l be the associated length function. We recall some basic notions concerning Weyl groups and the associated Hecke algebras. Where appropriate, we will give references to these notions in [11] or [13]. Every result involving a 'left-oriented' object connected with a Weyl group or Hecke algebra, e.g. a left transversal, a relation defined in terms of multiplication on the left or a left module, has an analogous result involving the corresponding 'right-oriented' object. We shall freely translate results from the literature involving one orientation to results involving the other.

For each element $w \in W$, the *left descent set*, $L(w)$, and the *right descent set*, $R(w)$, are defined by $L(w) := \{s \in S\colon l(sw) < l(w)\}$ and $R(w) := \{s \in S\colon l(ws) < l(w)\}$. For each subset $J \subseteq S$, the subgroup W_J generated by J is called a *standard parabolic subgroup* of W. It has a Coxeter system (W_J, J). Its length function l_J is that induced from l. It has a unique longest element w_J. By tradition, w_0 is written for w_S. Let $x, y \in W$. We say that x is a *prefix* of y if $y = s_1 s_2 \cdots s_p$ where $s_i \in S$ for $i = 1, \ldots, p$, $p = l(y)$ and $x = s_1 s_2 \cdots s_r$, for some $r \leq p$. The prefix relation corresponds to the *weak Bruhat order* in [8]. We use $\leq$ to denote the strong Bruhat order on W and we write $x < y$ if $x \leq y$ and $x \neq y$.

Result 1 ([11, Propositions 2.1.1 and 2.1.7 and Lemma 2.2.1])**.** *(i) There is a special set of right coset representatives $\mathcal{X}_J$ associated with each parabolic subgroup W_J. An element of $\mathcal{X}_J$ is the unique element of minimum length in its coset. Moreover, if $w = vx$ where $v \in W_J$ and $x \in \mathcal{X}_J$ then $l(w) = l(v) + l(x)$. Also, $\mathcal{X}_J = \{w \in W\colon L(w) \subseteq S - J\}$ and, if d_J denotes the longest element in $\mathcal{X}_J$, then $\mathcal{X}_J$ is the set of prefixes of d_J.*

(ii) If $J, K \subseteq S$ and $\mathcal{X}_{J,K}$ is defined to be $\{d \in \mathcal{X}_J\colon d^{-1} \in \mathcal{X}_K\}$, then $\mathcal{X}_{J,K}$ is a complete set of representatives of the (W_J, W_K) double cosets in W and, for any $w \in W$, there are $u \in W_J$ and $v \in W_K$, and a unique $d \in \mathcal{X}_{J,K}$ such that $w = udv$ and $l(w) = l(u) + l(d) + l(v)$.

The Hecke algebra $\mathcal{H}$ corresponding to (W, S) and defined over the ring $A = \mathbf{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$, where q is an indeterminate, has a free A-basis $\{T_w\colon w \in W\}$ and multiplication defined by the rules

$$\text{(2.1)}\qquad \begin{array}{rl} \text{(i)} & T_w T_{w'} = T_{ww'} \text{ if } l(ww') = l(w) + l(w') \text{ and} \\ \text{(ii)} & (T_s + 1)(T_s - q) = 0 \text{ if } s \in S. \end{array}$$

The basis $\{T_w\colon w \in W\}$ is called the *T-basis* of $\mathcal{H}$. (See [13]).

Result 2 ([13, Theorem 1.1])**.** *$\mathcal{H}$ has a basis $\{C_w\colon w \in W\}$, the C-basis, whose terms have the form $C_y = \sum_{x \leq y} (-1)^{l(y)-l(x)} q^{\frac{1}{2}l(y)-l(x)} P_{x,y}(q^{-1}) T_x$, where $P_{x,y}(q)$ is a polynomial in q with integer coefficients of degree $\leq \frac{1}{2}\left(l(y) - l(x) - 1\right)$ if $x < y$ and $P_{y,y} = 1$.*

If the degree of $P_{x,y}(q)$ is exactly $\frac{1}{2}\left(l(y) - l(x) - 1\right)$, we write $\mu(x, y)$, and $\mu(y, x)$, for its leading coefficient, which is a nonzero integer. For all other pairs $x, y \in W$, we set $\mu(x, y) = 0$.

There is an automorphism $\jmath$ of $\mathcal{H}$ defined by $\left(\sum_{y \in W} a_y T_y\right)\jmath = \sum_{y \in W} \overline{a}_y \left(-q^{-1}\right)^{l(y)} T_y$, where $a \mapsto \overline{a}$ is the automorphism of A defined by $q^{\frac{1}{2}} \mapsto q^{-\frac{1}{2}}$ (see [13, p.166]). This automorphism is used to relate the C-basis of $\mathcal{H}$ to another basis $\{C'_w\colon w \in W\}$ known as the C'-basis, which may be defined by $C'_w = (-1)^{l(w)} C_w \jmath$.

The multiplication of C-basis elements by T_s, $s \in S$, is described in [13] and is as follows,

Result 3 ([13, 2.3ac])**.**

$$sx < x \Rightarrow T_sC_x = -C_x \text{ and } x < sx \Rightarrow T_sC_x = qC_x + q^{\frac{1}{2}}C_{sx} + \sum_{z<x,\ sz<z} \mu(z,x)C_z.$$

$$xs < x \Rightarrow C_xT_s = -C_x \text{ and } x < xs \Rightarrow C_xT_s = qC_x + q^{\frac{1}{2}}C_{xs} + \sum_{z<x,\ zs<z} \mu(x,z)C_z.$$

There are two reflexive transitive relations (preorders), $\leq_L$ and $\leq_R$, defined on W using the C-basis. The preorder $\leq_L$ is generated by all statements of the form: $x \leq_L y$ if C_x occurs with nonzero coefficient in the expression of T_sC_y in the C-basis, for some $s \in S$. The preorder $\leq_R$ is defined similarly, taking C_yT_s instead of T_sC_y in the preceding sentence.

A third preorder $\leq_{LR}$ is defined using the previous two preorders: $x \leq_{LR} y$ if there is a sequence of elements $x_0 = x$, $x_1,\ldots,$ $x_r = y$ of W such that for each integer i, $0 \leq i \leq r-1$, either $x_i \leq_L x_{i+1}$ or $x_i \leq_R x_{i+1}$.

$\sim_L$, $\sim_R$ and $\sim_{LR}$ are the equivalence relations generated by $\leq_L$, $\leq_R$ and $\leq_{LR}$, respectively. Their equivalence classes are called *left cells*, *right cells* and *two-sided cells*, respectively. It is immediate that two-sided cells are unions of left-cells which are also unions of right cells.

We write $x <_L y$ if $x \leq_L y$ and $x \not\sim_L y$. The relations $<_R$ and $<_{LR}$ are defined similarly. (See [13]).

Result 4 ([16, 5.26.1])**.** Let $\mathcal{Y}_J = w_J\mathcal{X}_J$. If $x \in W$ and $x \leq_R y$ for some $y \in \mathcal{Y}_J$ then $x \in \mathcal{Y}_J$. Moreover, $\mathcal{Y}_J = \{w \in W \colon w \leq_R w_J\}$ is a union of right cells.

For any subset $J \subseteq S$, let $\mathcal{H}_J$ denote the Hecke algebra corresponding to (W_J, J). From [13, Theorem 1.1 and Lemma 2.6(vi)], we see that $C_{w_J} = \left(-q^{\frac{1}{2}}\right)^{l(w_J)} \sum_{y \leq w_J} (-q)^{-l(y)}T_y$. The right $\mathcal{H}_J$-module $C_{w_J}\mathcal{H}_J$ has rank 1, since $C_{w_J}T_s = -C_{w_J}$ for all $s \in J$. The corresponding representation is described as the *alternating representation* in [8, §3] and as the *sign representation* in [5, §67]. The right $\mathcal{H}$-module $C_{w_J}\mathcal{H}$ is isomorphic to the module induced from $C_{w_J}\mathcal{H}_J$. We refer to any $\mathcal{H}$-module of the form $C_{w_J}\mathcal{H}$, and any module arising from it by extending the scalars, as a *monomial* module. Note that in [4, page 314] an induced monomial representation for a group is defined as any induced representation from a one-dimensional representation of a subgroup.

It is clear that $C_{w_J}\mathcal{H}$ is spanned by elements of the form $C_{w_J}T_d$, $d \in \mathcal{X}_J$ over A. Since their 'leading terms' in the T-basis of $\mathcal{H}$ have the form $a_dT_{w_Jd}$, where a_d is invertible in A, they are independent over A. Thus,

Result 5. *The module $C_{w_J}\mathcal{H}$ has an A-basis $\{C_{w_J}T_d \colon d \in \mathcal{X}_J\}$ and the module $C'_{w_J}\mathcal{H}$ has an A-basis $\{C'_{w_J}T_d \colon d \in \mathcal{X}_J\}$.*

The second part comes from applying the automorphism $\jmath$. We will refer to the first of these bases as the *T-basis* of $C_{w_J}\mathcal{H}$ and the second as the *T-basis* of $C'_{w_J}\mathcal{H}$.

In [22, Corollary 1.19], Xi obtains an A-basis for a module similar to the monomial module $C_{w_J}\mathcal{H}$. His result is contained in the following result—though the reader should note that Xi uses the term C-basis for a basis which is different from the Kazhdan-Lusztig C-basis in [13]. The second part arises from the first by applying the automorphism $\jmath$.

Result 6 ([18, Lemma 2.11]). *The module $C_{w_J}\mathcal{H}$ has an A-basis $\{C_y : y \leq_R w_J\} = \{C_{w_J d} : d \in \mathcal{X}_J\}$. Similarly, the module $C'_{w_J}\mathcal{H}$ has an A-basis $\{C'_y : y \leq_R w_J\} = \{C'_{w_J d} : d \in \mathcal{X}_J\}$.*

We will refer to the first of these bases as the C-*basis* of $C_{w_J}\mathcal{H}$ and the second as the C'-*basis* of $C'_{w_J}\mathcal{H}$. Morever, if $x \leq_L w_J$, then $C'_x \in \mathcal{H}C'_{w_J}$ so $C'_x\mathcal{H}$ is a homomorphic image of $C'_{w_J}\mathcal{H}$, a fact that we will need later on.

We now see that the change-of-basis matrix associated with the transition from the C-basis of $C_{w_J}\mathcal{H}$ to the T-basis is, for a suitable ordering of the elements of the basis, a triangular matrix over A which is invertible over A. The polynomials $g^J_{e,d}$, which appear in the proof below, first appeared in work of Deodhar [6]. See in particular [6, Proposition 3.4]. The following result is a small extension of [18, Proposition 2.13].

Result 7 ([18, Proposition 2.13]). *For each $e \in \mathcal{X}_J$,*

$$C_{w_J}T_e = \sum_{d\in\mathcal{X}_J,\ d\leq e} g^J_{e,d} C_{w_J d}, \tag{2.2}$$

where $g^J_{e,d}$ denotes an element of A and $g^J_{e,e}$ is a power of $q^{\frac{1}{2}}$, and

$$C_{w_J e} = \sum_{d\in\mathcal{X}_J,\ d\leq e} \hat{g}^J_{e,d} C_{w_J} T_d, \tag{2.3}$$

where $\hat{g}^J_{e,d}$ denotes an element of A and $\hat{g}^J_{e,e}$ is a power of $q^{\frac{1}{2}}$.

In fact, $\hat{g}^J_{e,d} = (-1)^{l(e)-l(d)} q^{\frac{1}{2}l(e)-l(d)} P_{w_J d, w_J e}(q^{-1})$, if $d \leq e$. So, $\hat{g}^J_{e,e} = q^{-\frac{1}{2}l(e)}$ since $P_{w_J e, w_J e}(q^{-1}) = 1$. Also, $g^J_{e,e} = q^{\frac{1}{2}l(e)}$.

Applying $\jmath$ to the equations (2.3) and (2.2) in Result 7 and simplifying, we get the following:

Result 8. *For each $e \in \mathcal{X}_J$,*

$$C'_{w_J e} = \sum_{d\in\mathcal{X}_J,\ d\leq e} (-1)^{l(d)} \overline{\hat{g}^J_{e,d}} C'_{w_J} T_d, \tag{2.4}$$

where $\overline{\hat{g}^J_{e,d}}$ denotes an element of A and $\overline{\hat{g}^J_{e,e}}$ is a power of $q^{\frac{1}{2}}$, and

$$C'_{w_J} T_e = \sum_{d\in\mathcal{X}_J,\ d\leq e} (-1)^{l(d)} \overline{g^J_{e,d}} C'_{w_J d}, \tag{2.5}$$

where $\overline{g^J_{e,d}}$ denotes an element of A and $\overline{g^J_{e,e}}$ is a power of $q^{\frac{1}{2}}$.

Of course, the preceding results could be stated slightly differently using the fact that $\{w : w \leq_R w_J\} = \{w_J d : d \in \mathcal{X}_J\}$.

For $w \in W$, let M_w and $\hat{M}_w$ denote the $\mathcal{H}$-modules with A-bases $\{C_y : y \leq_R w\}$ and $\{C_y : y <_R w\}$, respectively, and let $S_w = M_w/\hat{M}_w$. Then S_w is a Kazhdan-Lusztig cell module and affords the cell representation corresponding to the right cell containing w. If $\mathfrak{C}$ denotes the cell containing w, we also write $S_{\mathfrak{C}}$ for S_w. Note that $C_w\mathcal{H}$ is a submodule of M_w. We see from Result 6 that, if $w = w_J$ for some $J \subseteq S$, then $C_w\mathcal{H} = M_w$. In this case, we say call the cell module a *parabolic* cell module and, if $\lambda = \lambda(J)$ we write $\mathfrak{C}(\lambda)$ for the cell (see Section 3 for the definition of $\lambda(J)$).

Next, we establish that certain pairs of the $\mathcal{H}$-modules M_w are isomorphic. We say that two subsets J and K of S are in the same *Coxeter class* if $K = w^{-1}Jw$ for some $w \in W$.

Proposition 2.1. *Let J and K be subsets of S. If J and K are in the same Coxeter class then (i) w_J and w_K are in the same two-sided cell of W and (ii) $M_{w_J} \cong M_{w_K}$ as $\mathcal{H}$-modules.*

Proof. Of all elements w satisfying $w^{-1}Jw = K$, choose one d of minimum length. Then $d \in \mathcal{X}_{J,K}$ and $d^{-1}W_J d = W_K$. Let $u \in W_J$ and $v \in W_K$ satisfy $ud = dv$. By Result 1, $l(u) + l(d) = l(ud) = l(dv) = l(v) + l(d)$. So $l(u) = l(v)$. Hence, $C_{w_J}T_d = T_d C_{w_K}$. Statement (i) follows immediately from this.

We know that $M_{w_J} = C_{w_J}\mathcal{H}$ and $M_{w_K} = C_{w_K}\mathcal{H}$. Hence, we may define a mapping $\theta\colon M_{w_J} \to M_{w_K}$ by $(C_{w_J}h)\theta = T_d^{-1}C_{w_J}h$ for all $h \in \mathcal{H}$. This is clearly a $\mathcal{H}$-module isomorphism and establishes (ii). □

Now clearly $\hat{M}_{w\jmath}$ is an $\mathcal{H}$-submodule of $M_{w\jmath}$. Define $S^\bullet_w = M_{w\jmath}/\hat{M}_{w\jmath}$. Then $S^\bullet_w$ has A-basis $\{C'_z + \hat{M}_{w\jmath}\colon z \sim_R w\}$. As above, we also write $S^\bullet_{\mathfrak{C}}$ for $S^\bullet_w$ if w is in the right cell $\mathfrak{C}$.

It will be convenient on occasion to extend the scalars of the algebras under consideration. Let R be any commutative ring with 1 and let $A \to R$ be a ring homomorphism. With each A-module L, we have an associated R-module $R \otimes_A L$, which we will denote briefly as L_R. In particular, we obtain an R-algebra $\mathcal{H}_R$, and $\mathcal{H}_R$-modules $M_{R,w} = R \otimes M_w$, $\hat{M}_{R,w} = R \otimes \hat{M}_w$, and Kazhdan-Lusztig cell modules $S_{R,w} = M_{R,w}/\hat{M}_{R,w} \cong R \otimes S_w$. Since $\jmath$ can be extended easily and uniquely to an automorphism of $\mathcal{H}_R$, we see that the $\mathcal{H}_R$-module $S^\bullet_{R,w}$ is isomorphic to $M_{R,w\jmath}/\hat{M}_{R,w\jmath}$. In particular, we will use F to denote any field containing the field of fractions $\mathbf{Q}(q^{\frac{1}{2}})$ of A, and assume that the homomorphism $A \to F$ is inclusion. The module $S_{F,w}$ has an F-basis $\{C_z + \hat{M}_{F,w}\colon z \sim_R w\}$ and the module $S^\bullet_{F,w}$ has F-basis $\{C'_z + \hat{M}_{F,w\jmath}\colon z \sim_R w\}$. Also, $C'_w\mathcal{H}_F$ is a submodule of $M_{F,w\jmath}$, $C'_{w_J}\mathcal{H}_F = M_{F,w_J\jmath}$, and if $w' <_R w$ then $C'_{w'}\mathcal{H}_F$ is a submodule of $\hat{M}_{F,w\jmath}$.

Result 9 ([18, Proposition 2.15])**.** *For each $w \in W$, $S^\bullet_{F,w}$ and S_{F,w_0w} are isomorphic $\mathcal{H}_F$-modules.*

We conclude this section with an elementary proposition concerning homomorphisms between principal ideals in the Hecke algebra $\mathcal{H}_F$ of an arbitrary finite Coxeter group.

Proposition 2.2. *Let $e, f \in \mathcal{H}_F$ where $e^2 = ke$ for some $k \in F \setminus \{0\}$. Then $\mathrm{Hom}_{\mathcal{H}_F}(e\mathcal{H}_F, f\mathcal{H}_F) \cong f\mathcal{H}_F e$ as F-spaces.*

Proof. For $h \in \mathcal{H}_F$, let $\varphi_h \in \mathrm{Hom}_{\mathcal{H}_F}(e\mathcal{H}_F, f\mathcal{H}_F)$ be the map given by left multiplication with fhe. The required isomorphism is given by $\theta\colon f\mathcal{H}_F e \to \mathrm{Hom}_{\mathcal{H}_F}(e\mathcal{H}_F, f\mathcal{H}_F)\colon fhe \mapsto \varphi_h$ $(h \in \mathcal{H}_F)$. □

3. Basic combinatorics of the symmetric group

In this section, we collect various basic definitions and results concerning S_n, considering it both as a permutation group in its natural form and as a Coxeter group. We refer to James and Kerber [12], Sagan [20], Dipper and James [8], and Geck and Pfeiffer [11] for the basic theory.

The symmetric group S_n is a Coxeter group with Coxeter system (W, S) where $W = S_n$, $S = \{s_1, \ldots, s_{n-1}\}$, and s_i is the transposition $(i, i+1)$. We will describe an element w of W in different forms: as a word in the generators $s_1, \ldots, s_{n-1}$, as products of disjoint cycles on $1, \ldots, n$, and in *row-form* $[w_1, \ldots, w_n]$ where $w_i = iw$ for $i = 1, \ldots, n$. The Coxeter *length* $l(w)$ of the element $w \in W$, that is the shortest length of a word in the elements of S representing w, has an easy combinatorial description; $l(w)$ is the number of pairs (w_i, w_j) with $i < j$ and $w_i > w_j$. If $x, y \in W$ and $l(x^{-1}y) = l(y) - l(x)$, then x is a *prefix* of y; in this case, y has a *reduced form*, that is a word in the elements of S representing it of length $l(y)$, beginning with a reduced form for x followed by a reduced form for $x^{-1}y$.

In the case of a symmetric group, a standard parabolic subgroup is also known as a *Young subgroup*. The longest element w_0 in W is the permutation defined by $i \mapsto n + 1 - i$.

If $\lambda = (\lambda_1, \ldots, \lambda_r)$ is a *composition* of n with r *parts*, that is, $\lambda_1, \ldots, \lambda_r$ are non-negative integers whose sum is n, we define the subset $J(\lambda)$ of S to be $S \backslash \{s_{\lambda_1}, s_{\lambda_1+\lambda_2}, \ldots, s_{\lambda_1+\ldots+\lambda_{r-1}}\}$. Moreover, for every subset J of S, there is a composition λ ($= \lambda(J)$) such that $J = J(\lambda)$. Thus, corresponding to each composition λ, there is a standard parabolic subgroup of W whose Coxeter generator set is $J(\lambda)$. The longest element $w_{J(\lambda)}$ of $W_{J(\lambda)}$, if λ is a composition with r parts, can be described in row-form by concatenating the sequences $(\widehat{\lambda}_{i+1}, \ldots, \widehat{\lambda}_i + 1)$ for $i = 0, \ldots, r-1$, where $\widehat{\lambda}_0 = 0$, $\widehat{\lambda}_r = n$, and $\widehat{\lambda}_{i+1} = \lambda_{i+1} + \widehat{\lambda}_i$.

A *partition* is a compositon whose terms are non-increasing. A composition or partition is *improper* if some of its parts are 0, and is otherwise *proper*. Unless otherwise stated explicitly, we will use the terms composition and partition to mean proper composition and proper partition, respectively. We use the notation $\lambda \vDash n$ (respectively, $\lambda \vdash n$) to say that λ is a composition (respectively, partition) of n.

Let r' be the maximum part of the composition λ. Recall that the *conjugate* composition $\lambda' = (\lambda'_1, \ldots, \lambda'_{r'})$ of λ is defined by $\lambda'_i = |\{j : 1 \le j \le r \text{ and } i \le \lambda_j\}|$ for $1 \le i \le r'$. It is immediate that λ' is a partition of n with r' parts.

If λ and μ are compositions of n, write $\lambda \trianglelefteq \mu$ if, for all k, $\sum_{1 \le i \le k} \lambda_i \le \sum_{1 \le i \le k} \mu_i$. In this case, we say that λ is dominated by μ, or μ dominates λ. This differs from the definition of domination in [8], though both definitions coincide if λ and μ are partitions. If $\lambda \trianglelefteq \mu$ and $\lambda \ne \mu$, we write $\lambda \triangleleft \mu$.

A *diagram* D is a finite subset $\mathbb{Z}^2$. A *row index* of D is an element $i \in \mathbb{Z}$, such that for some $j \in \mathbb{Z}$, $(i, j) \in D$. A *column index* of D is defined analogously. Let $\mathcal{R}(D)$ and $\mathcal{C}(D)$ denote the sets of row indices and column indices of D, respectively, and let $r_D = |\mathcal{R}(D)|$ and $c_D = |\mathcal{C}(D)|$. The *size* of the diagram D is $|D|$. The elements of D are the *nodes* of the diagram. The diagram D will be called *principal* if for any i with $\min \mathcal{R}(D) \le i \le \max \mathcal{R}(D)$ there is some j with $(i, j) \in D$ and for any j with $\min \mathcal{C}(D) \le j \le \max \mathcal{C}(D)$ there is some i with $(i, j) \in D$. A principal diagram may be loosely described as one without empty rows or columns.

We say that two diagrams D_1 and D_2 are equivalent if there are order-preserving bijections $\theta \colon \mathcal{R}(D_1) \to \mathcal{R}(D_2)$ and $\varphi \colon \mathcal{C}(D_1) \to \mathcal{C}(D_2)$ such that $(i, j) \in D_1$ if, and only if, $(i\theta, j\varphi) \in D_2$. Every equivalence class of diagrams has a principal diagram which is unique up to translations in $\mathbb{Z}^2$.

We will usually assume without comment that a principal diagram has row indices $\{1, \ldots, r_D\}$ and column indices $\{1, \ldots, c_D\}$.

Let $\mathcal{R}(D) = \{i_1, i_2, \ldots, i_{r_D}\}$ and $\mathcal{C}(D) = \{j_1, j_2, \ldots, j_{c_D}\}$, where $i_1 < i_2 < \cdots < i_{r_D}$ and $j_1 < j_2 < \cdots < j_{c_D}$. There are two compositions which are naturally associated with a diagram D, the *row-composition* λ_D and the *column-composition* μ_D defined by $\lambda_{D,k} = |\{(i,j) \in D \colon i = i_k\}|$ for $k = 1, \ldots, r_D$ and $\mu_{D,k} = |\{(i,j) \in D \colon j = j_k\}|$ for $k = 1, \ldots, c_D$. We write $J(D) = J(\lambda_D)$ and $J'(D) = J(\mu_D)$, thereby associating two standard parabolic subgroups of W with the diagram D.

Let λ and μ be compositions. A diagram D with $\lambda_D = \lambda$ and $\mu_D = \mu$ will be called a (λ, μ)-*diagram*. We will write $\mathcal{D}^{(\lambda,\mu)}$ for the set of principal (λ, μ)-diagrams. We denote by $\mathcal{D}^{(\lambda)}$ the set $\bigcup_{\mu \vDash n} \mathcal{D}^{(\lambda,\mu)}$ of principal diagrams with $\lambda_D = \lambda$.

It is easily seen $\mathcal{D}^{((2,1),(3))} = \emptyset$. We have, however, the following criterion for $\mathcal{D}^{(\lambda,\mu)} \neq \emptyset$.

Result 10 (Gale-Ryser)**.** (See [12, Theorem 1.4.17].) *Let $\lambda, \mu \vDash n$. There is a diagram D with $\lambda_D = \lambda$ and $\mu_D = \mu$ if, and only if, $\lambda'' \trianglelefteq \mu'$ (or, equivalently, $\mu'' \trianglelefteq \lambda'$).*

A *Young diagram* is a diagram D for which λ_D is a partition and $\mu_D = \lambda'_D$ and its *shape*, which is denoted by $\operatorname{sh}(D)$, is defined to be λ_D. A *special diagram* is a diagram obtained from a Young diagram by permuting the rows and columns. We characterize special diagrams in the following proposition.

Proposition 3.1. (Compare [7, Lemma 5.2]) *Let D be a diagram. The following statements are equivalent. (i) D is special; (ii) $\lambda''_D = \mu'_D$; (iii) for every pair of nodes $(i,j), (i',j')$ of D with $i \neq i'$ and $j \neq j'$, at least one of (i',j) and (i,j') is also a node of D.*

Proof. (i) $\Rightarrow$ (ii): Let E be the Young diagram corresponding to D. Then λ_E and μ_E are partitions, $\mu_E = \lambda'_E$, and λ_D and μ_D are compositions which are rearrangements of λ_E and μ_E, respectively. Hence, $\lambda''_D = \lambda_E$ and $\mu''_D = \mu_E$. So, $\lambda''_D = \mu'_E = \mu'''_D = \mu'_D$.

(ii) $\Rightarrow$ (i), (iii): Let E be obtained from D by rearranging the rows and columns so that $\lambda_E = \lambda''_D$ and $\mu_E = \mu''_D$. Since $\mu'_E = \mu'_D = \lambda''_D = \lambda_E$, E is a Young diagram. Hence, D is special.

Now suppose that (i,j) and (i',j') are nodes of D and in the rearrangement of D into E, the quadruple (i,j), (i',j'), (i,j'), (i',j) maps to the quadruple $(\bar{i},\bar{j})$, $(\bar{i}',\bar{j}')$, $(\bar{i},\bar{j}')$ and $(\bar{i}',\bar{j})$. Then $(\bar{i},\bar{j})$ and $(\bar{i}',\bar{j}')$ are nodes of E. If $\bar{i} < \bar{i}'$ then $(\bar{i},\bar{j}')$ is a node of E and if $\bar{i}' < \bar{i}$ then $(\bar{i}',\bar{j})$ is a node of E. Hence, at least one of (i',j) and (i,j') is a node of D.

(iii) $\Rightarrow$ (i): Construct diagram E from D as above. Then E clearly satisfies property (iii). Hence, the nodes on any row (respectively, column) of E are in the same columns (respectively, rows) as the nodes on the preceding row (respectively, column). So, E is a Young diagram and D is special. □

Since it is immediate that $\mathcal{D}^{(\lambda,\lambda')}$ consists of a single diagram if λ is a partition, it follows easily that $\mathcal{D}^{(\lambda,\mu)}$ consists of a single special diagram if λ and μ are compositions with $\lambda'' = \mu'$.

A *D-tableau* is a bijection $t \colon D \to \{1, \ldots, |D|\}$. We refer to $(i,j)t$, where $(i,j) \in D$, as the (i,j)-*entry* of t. In the case that the underlying diagram is a Young diagram, the tableau t is called a *Young tableau* and its *shape* $\operatorname{sh}(t)$ is the shape of the underlying diagram.

Now let D be a general diagram and let t be a D-*tableau*. The k-th *row* of t is the image of the k-th row of D and the ℓ-th *column* of t is the image of the ℓ-th column of D. We denote by t_k the set of elements on the k-th row of t. We say t is *row-standard* if it is increasing on rows, that is, if $(i, j'), (i, j'') \in D$ and $j' < j''$ then $(i, j')t < (i, j'')t$. Similarly, we say t is *column-standard* if it is increasing on columns, that is, if $(i', j), (i'', j) \in D$ and $i' < i''$ then $(i', j)t < (i'', j)t$. We say that t is *standard* if $(i', j')t < (i'', j'')t$ for any $(i', j'), (i'', j'') \in D$ with $i' \leq i''$ and $j' \leq j''$. Note that a standard D-tableau is row-standard and column-standard, but the converse is not true, in general.

We illustrate these concepts with an example. The diagram $\{(1, 2), (1, 4), (3, 2), (4, 1), (4, 4), (4, 5), (5, 0), (5, 1), (5, 4)\}$ is equivalent to the principal diagram $D = \{(1, 3), (1, 4), (2, 3), (3, 2), (3, 4), (3, 5), (4, 1), (4, 2), (4, 4)\}$. $\lambda_D = (2, 1, 3, 3)$, $\mu_D = (1, 2, 2, 3, 1)$. A sketch of diagram D and three D-tableaux t_1, t_2 and t_3 are given in Table 1. The sketch of D is a pattern of $\times$'s describing to the relative positions of its entries. The tableau t_1 is row-standard but not column-standard, the tableau t_2 is row-standard and column-standard but not standard, and the tableau t_3 is standard.

$$
\begin{array}{ccccc} & & \times & \times & \\ & & \times & & \\ & \times & & \times & \times \\ \times & \times & & \times & \end{array}
\qquad
\begin{array}{ccccc} & & 4 & 9 & \\ & & 7 & & \\ & 3 & & 5 & 6 \\ 1 & 2 & & 8 & \end{array}
\qquad
\begin{array}{ccccc} & & 3 & 5 & \\ & & 9 & & \\ & 1 & & 6 & 7 \\ 2 & 4 & & 8 & \end{array}
\qquad
\begin{array}{ccccc} & & 2 & 4 & \\ & & 6 & & \\ & 1 & & 7 & 8 \\ 3 & 5 & & 9 & \end{array}
$$

$$D \qquad\qquad\qquad t_1 \qquad\qquad\qquad t_2 \qquad\qquad\qquad t_3$$

TABLE 1. A diagram D with some D-tableaux

In the special case that $\lambda_D = \lambda$ and $\mu_D = \lambda'$, then D is a λ-diagram and any D-tableau is a λ-tableau in the sense of [19].

The group W acts on the set of D-tableaux in the obvious way—if $w \in W$, an entry i is replaced by iw and tw denotes the tableau resulting from the action of w on the tableau t. This action on D-tableaux corresponds to the action by letter permutations of Dipper and James [8, p.21]. There are two subgroups of W associated with a D-tableau t, the *row group* $\mathtt{R}_t$ consisting of those permutations which map each row of t into itself, and the *column group* $\mathtt{C}_t$ which behaves similarly on columns.

We construct two special D-tableaux t^D and t_D. Let t^D be obtained by filling D with $1, \ldots, |D|$ by rows, filling rows from top to bottom and filling each row from left to right, and let t_D be obtained by filling D-diagram with $1, \ldots, |D|$ by columns, filling columns from left to right and filling each column from top to bottom. Both t^D and t_D are standard D-tableaux. Moreover, $\mathtt{R}_{t^D} = W_{J(D)}$ and $\mathtt{C}_{t_D} = W_{J'(D)}$. For each D-tableau t, we define an element $w_t \in W$ by $t^D w_t = t$. The row-form of w_t is obtained by concatenating the rows of t beginning at the top. In particular, we write w_D for w_{t_D}, so that $t^D w_D = t_D$.

For diagram D in Table 1, we have $t^D = \begin{array}{ccccc} & & 1 & 2 & \\ & & 3 & & \\ & 4 & & 5 & 6 \\ 7 & 8 & & 9 & \end{array}$, $t_D = \begin{array}{ccccc} & & 4 & 6 & \\ & & 5 & & \\ & 2 & & 7 & 9 \\ 1 & 3 & & 8 & \end{array}$ and $w_D = (1, 4, 2, 6, 9, 8, 3, 5, 7)$ and w_D has row-form $[4, 6, 5, 2, 7, 9, 1, 3, 8]$.

Translating from [12] into the present context, we have the following more explicit characterization of all the elements of $\mathcal{X}_{J(D)}$.

Result 11 ([8, Lemma 1.1])**.** *Let D be a diagram. Then $\mathcal{X}_{J(D)} = \{w \in W : t^D w$ is row-standard$\}$.*

Lemma 3.2. *If $\lambda, \mu \vDash n$ and the diagrams $D_1, D_2 \in \mathcal{D}^{(\lambda,\mu)}$ are different then $w_{D_1} \neq w_{D_2}$.*

Proof. From the hypothesis, D_1 has a node (i,j) which is not a node of D_2. Suppose $t_{D_1}(i,j) = r$. Since $\mu_{D_1} = \mu_{D_2}$, $r = t_{D_2}(i',j)$ for some $i' \neq i$. It follows that $rw_{D_1}^{-1}$ and $rw_{D_2}^{-1}$ are on the i-th and i'-th rows of t^{D_1} and t^{D_2}, respectively. Since $\lambda_{D_1} = \lambda_{D_2}$, t^{D_1} and t^{D_2} contain exactly the same entries on corresponding rows. Hence, $rw_{D_1}^{-1} \neq rw_{D_2}^{-1}$. So, $w_{D_1} \neq w_{D_2}$. □

If P, Q are subgroups of a group G and $d \in G$, the double coset PdQ is said to have the *trivial intersection property* if $d^{-1}Pd \cap Q = \{1\}$ or, equivalently, every element of the double coset PdQ has a unique representation of the form udv with $u \in P$ and $v \in Q$.

Lemma 3.3. *Let D be any diagram.*

(i) The double coset $W_{J(\lambda_D)} w_D W_{J(\mu_D)}$ has the trivial intersection property;

(ii) $w_D \in \mathcal{X}_{J(\lambda_D),J(\mu_D)}$;

(iii) w_D is the unique element of minimum length in $W_{J(\lambda_D)} w_D W_{J(\mu_D)}$;

(iv) $l(uw_D v) = l(u) + l(w_D) + l(v)$ for all $u \in W_{J(\lambda_D)}$ and $v \in W_{J(\mu_D)}$;

Proof. Since $W_{J(\mu_D)}$ is the column-group of t_D and $w_D^{-1} W_{J(\lambda_D)} w_D$ is the row-group of t_D, (i) follows immediately. From Result 11, $w_D \in \mathcal{X}_{J(\lambda_D)}$. If D' denotes the diagram obtained from D by transposition, the equation $t^D w_D = t_D$ leads to $t_{D'} w_D = t^{D'}$. So, $w_D^{-1} = w_{D'} \in \mathcal{X}_{J(\mu)}$, completing (ii). Using Result 1, (iii) and (iv) follow. □

James and Kerber [12] associate with each double coset of a pair of Young subgroups a certain matrix over the integers which is a $(0,1)$-matrix if, and only if, the double coset has the trivial intersection property by establishing the following two results.

Result 12 ([12, Lemma 1.3.8])**.** *Let $J, K \subseteq S$, let $\lambda, \mu \vDash n$ satisfy $J(\lambda) = J$ and $J(\mu) = K$, let $E \in \mathcal{D}^{(\lambda)}$ and $F \in \mathcal{D}^{(\mu)}$. and let $g, h \in W$. Then $g \in W_J h W_K$ if, and only if, $|t_i^E g \cap t_k^F| = |t_i^E h \cap t_k^F|$ for all i and k, where t_i denotes the set of entries in the ith row of a tableau t.*

From Result 12, we see that the double coset $W_J h W_K$ is characterized by the matrix $Z^{J,K}(h) = [z_{i,k}]$, where $z_{i,k} = |t_i^E h \cap t_k^F|$ for all i and k. An immediate consequence is that each $g \in W_J h W_K$ has exactly $\prod_{i,k} z_{i,k}!$ expressions of the form $g = uhv$ with $u \in W_J$ and $v \in W_K$ and $|W_J h W_K| = (\prod_i \lambda_i! \prod_k \mu_k!) \,/\, \left(\prod_{i,k} z_{i,k}!\right)$. So, $W_J h W_K$ has the trivial intersection property if, and only if, $Z^{\lambda,\mu}(h)$ is a $(0,1)$-matrix, and such a $(0,1)$-matrix corresponds in an obvious way with a diagram $D \in \mathcal{D}^{(\lambda,\mu)}$ whose nodes correspond to the positions of the 1's in the matrix.

Result 13 ([12, Theorem 1.3.10 and Corollary 1.3.13])**.** *With the notation of the preceding paragraph and Result 12, the mapping $W_J h W_K \mapsto Z^{J,K}(h)$ establishes a bijection between the set of (W_J, W_K)*

double cosets in W and the set of $n \times n$ matrices with non-negative integer entries satisfying $\sum_j z_{j,k} = \mu_k$ and $\sum_j z_{i,j} = \lambda_i$ for all i and k.

Consequently, the number of (W_J, W_K) double cosets in W with the trivial intersection property is equal to the number of $(0,1)$ matrices with whose i-th row-sum is λ_i and whose k-th column-sum is μ_k for all i and k. Moreover, the double cosets $W_J w_D W_K$, $D \in \mathcal{D}^{(\lambda,\mu)}$, are precisely the distinct (W_J, W_K) double cosets with the trivial intersection property.

Let $\mathcal{Y}_{J(D)}$ be the set of prefixes of w_D. We show that this set is related to the set of standard D-tableaux in an analagous way to the corresponding result when D is the diagram of a partition which has been established in [8, Lemma 1.5]. We first need a technical lemma.

It will be convenient to say that a node (i', j') of a diagram is *north-east* of a node (i, j) if $i' < i$ and $j' > j$; we extend this notion to a tableau t by saying that an entry k' is *north-east* of an entry k if the node at which k' occurs in t is north-east of the node at which k occurs. It is easy to see that if $t = t^D w$ and $k+1$ is north-east of k in t then $l(ws_k) < l(w)$.

Lemma 3.4. *Let D be a diagram of size n, $1 \le k < n$, and $w \in W$ be such that $t^D w$ is a standard D-tableau. Then $l(ws_k) < l(w)$ if, and only if, $k+1$ is north-east of k in $t^D w$. In this case, $t^D w s_k$ is also a standard D-tableau.*

Proof. First, suppose that $l(ws_k) < l(w)$. So, $(k+1)w^{-1} < kw^{-1}$. Let k and $k+1$ occur in $t^D w$ at the nodes (a_k, b_k) and (a_{k+1}, b_{k+1}), respectively.

Since $t^D w$ is standard, $a_{k+1} < a_k$ or $b_{k+1} < b_k$. Since $(k+1)w^{-1}$ and kw^{-1} occur at the nodes (a_{k+1}, b_{k+1}) and (a_k, b_k), respectively, of t^D, either $a_{k+1} = a_k$ and $b_{k+1} < b_k$ or $a_{k+1} < a_k$. Hence, $a_{k+1} < a_k$ and $b_{k+1} > b_k$; that is, the node of $k+1$ in $t^D w$ is north-east of the node of k.

The converse follows from the remarks preceding the lemma. It is immediate that $t^D w s_k$ is also standard in this case. □

Proposition 3.5 (Compare [8, Lemma 1.5])**.** *Let D be a diagram. Then the mapping $u \mapsto t^D u$ is a bijection of the set $\mathcal{Y}_{J(D)}$ of prefixes of w_D to the set of standard D-tableaux.*

Proof. If u is a proper prefix of w_D, then us is a prefix of w_D for some $s \in S$ with $l(us) = l(u) + 1$. Note first that $t^D w_D$ is standard. By induction, $t^D us$ is standard. Hence, by Lemma 3.4, $t^D u$ is standard.

Now, let $u \in W$ be such that $t^D u$ is standard. Let (a_l, b_l) be the node containing l, for $l = 1, \dots, |D|$. Let $N_u = \{(l, m) : 1 \le l < m \le |D| \text{ and } b_l > b_m\}$ and $n_u = |N_u|$. We argue, by induction on n_u, that u is a prefix of w_D. If $n_u = 0$ then, since $t^D u$ is standard, it must be t_D. Hence, $u = w_D$. Now suppose $n_u > 0$. Then, for some k with $1 \le k < |D|$, $b_k > b_{k+1}$; for example, let k be the maximum first coordinate of an element of N_u. Since $t^D u$ is standard, $a_k < a_{k+1}$. Hence, $k+1$ is north-east of k in $t^D us_k$. Thus, $t^D us_k$ is standard and, by Lemma 3.4, $l(us_k) > l(u)$. So, u is a prefix of us_k. Also, $N_{us_k} \subseteq N_u$ and $(k, k+1) \in N_u \backslash N_{us_k}$. So, $n_{us_k} < n_u$. By induction, us_k is a prefix of w_D. Hence, u is a prefix of w_D. □

We now develop a simple algorithm for finding a reduced form for an element of $\mathcal{Y}_{J(D)}$. This algorithm is a special case of Algorithm A of [11], and it is a generalization of the algorithm given in [19, Proposition 2.10].

Algorithm 1. *Let w be a prefix of w_D, where D is a principal diagram, let $n = |D|$ and let $t = t^D w$. Then t is a standard D-tableau.*

1. *Let $t_0 = t^D$.*
2. *For i from 1 to n do the following:*
   - 2a. *Let $k(i)$ be the entry at the node of t_{i-1} which is occupied by i in t.*
   - 2b. *Let $g_i = s_{k(i)-1} \cdots s_i$.*
   - 2c. *Form t_i from t_{i-1} by replacing each j satisfying $i \le j \le k(i) - 1$ by $j + 1$ and replacing $k(i)$ by i.*
3. *Note that $t_n = t_{n-1} = t$ and $g_1 \cdots g_n$ gives a reduced form for w if non-trivial factors are replaced by the corresponding expressions in 2b and trivial factors are ignored.*

We find a reduced form for $w = [2, 4, 6, 1, 7, 8, 3, 5, 9]$ using Algorithm 1.

If $D = \begin{array}{ccccc} & & \times & \times & \\ & & \times & & \\ & \times & & \times & \times \\ \times & \times & & \times & \end{array}$, then $t = t^D w$ is standard.

$$\underset{t_0}{\begin{array}{ccccc} & & 1 & 2 & \\ & & 3 & & \\ & \mathbf{4} & & 5 & 6 \\ 7 & 8 & & 9 & \end{array}} \qquad \underset{t_1 = t_2}{\begin{array}{ccccc} & & 2 & 3 & \\ & & 4 & & \\ & 1 & & 5 & 6 \\ \mathbf{7} & 8 & & 9 & \end{array}} \qquad \underset{t_3 = t_4}{\begin{array}{ccccc} & & 2 & 4 & \\ & & 5 & & \\ & 1 & & 6 & 7 \\ 3 & \mathbf{8} & & 9 & \end{array}} \qquad \underset{t_5 = t_6 = t_7 = t_8 = t_9 = t}{\begin{array}{ccccc} & & 2 & 4 & \\ & & 6 & & \\ & 1 & & 7 & 8 \\ 3 & 5 & & 9 & \end{array}}$$

We get $g_1 = s_3 s_2 s_1$, $g_2 = 1$, $g_3 = s_6 s_5 s_4 s_3$, $g_4 = 1$, $g_5 = s_7 s_6 s_5$, $g_6 = 1$, $g_7 = 1$, $g_8 = 1$, $g_9 = 1$, and $w = s_3 s_2 s_1 s_6 s_5 s_4 s_3 s_7 s_6 s_5$. The bold entry in each tableau indicates the next position to be 'correctly' filled.

It is clear that other diagrams such as $\begin{array}{ccccc} & & \times & \times & \times \\ & \times & & \times & \times \\ \times & \times & & \times & \end{array}$ or $\begin{array}{cccc} & \times & \times & \times \\ \times & & \times & \times \\ \times & \times & \times & \end{array}$ could have been used in this case. For all such diagrams, Algorithm 1 will find the same reduced form for w.

Proposition 3.6. *Let D be a diagram of size n, let w be a prefix of w_D and let $t = t^D w$. Let $g_1, \dots, g_n$ be the elements defined by the preceding algorithm, where $g_i = s_{k(i)-1} \cdots s_i$. Then $l(g_i) = k(i) - i$ for $i = 1, \dots, n$, and $g_1, \dots, g_n$ gives a reduced form for w of length $\sum_{i=1}^{n}(k(i) - i)$ if factors which are trivial are ignored.*

Proof. By Proposition 3.5, t is a standard D-tableau. For $i = 0, \dots, n$, let t'_i be the D-tableau formed by placing $1, \dots, i$ at the same nodes as they occupy in t (their ' final' positions) and by filling the remaining nodes with $i + 1, \dots, n$, filling the currently unoccupied nodes on each row from left to right and filling the rows from top to bottom, and define $h_i \in W$ by $t^D h_i = t'_i$. Each t'_i is a standard D-tableau. We will show that $t'_i = t_i$ for all i. Clearly, $t'_0 = t^D = t_0$ and $t'_{n-1} = t'_n = t$.

Suppose that $i \ge 1$ and $t'_{i-1} = t_{i-1}$. Thus, $1, \dots, i-1$ are in their final positions in t_{i-1}. If i is not in its final position, then that position is occupied by some $k(i) > i$. Only numbers from $\{1, \dots, i-1\}$

occupy nodes west, north-west or north of that position. Hence, $i, i+1, \ldots, k(i)-1$ are north-east of $k(i)$ in t_{i-1}. Define the D-tableaux $t_{i-1}^{(j)}$ for $j = k(i), \ldots, i$ by $t_{i-1}^{(k(i))} = t_{i-1}$ and $t_{i-1}^{(j-1)} = t_{i-1}^{(j)} s_{j-1}$ for $j \geq i+1$, and define $h_{i-1}^{(j)} \in W$ by $t^D h_{i-1}^{(j)} = t_{i-1}^{(j)}$ for $i \leq j \leq k(i)$. In particular, $h_{i-1}^{(k(i))} = h_{i-1}$. As $t_{i-1}^{(j-1)}$ is obtained from $t_{i-1}^{(j)}$ by exchanging j with $j-1$, which is on a higher row of $t_{i-1}^{(j)}$, we get $l(h_{i-1}^{(j-1)}) = l(h_{i-1}^{(j)}) + 1$.

Moreover, $t_i = t_{i-1}^{(i)}$ and this is clearly t'_i. This completes the induction showing that $t_i = t'_i$. Thus, $h_i = h_{i-1}^{(i)} = h_{i-1}^{(k(i))} s_{k(i)-1} \cdots s_i = h_{i-1} g_i$ and $l(h_i) = l(h_{i-1}) + k(i) - i$. So, $w = h_n = g_1 \cdots g_n$ since $h_0 = 1$ and $l(w) = \sum_{i=1}^{n} k(i) - i$ and the expression for w given by $g_1 \cdots g_n$ with trivial g_i's ignored and non-trivial g_i's replaced by the corresponding words $s_{k(i)-1} \cdots s_i$ is necessarily reduced. □

We make some observations about the elements $w_D \in W$, where D is a diagram of size n. Our first lemma shows that every element in $\mathcal{X}_{J(\lambda)}$, where λ is a composition, has this form. Since $\mathcal{X}_{J((1^n))} = W$, every element of W has this form. In general, an element of W will have an expression of the form w_D for many different diagrams D of size n.

Recall that we denote by $\mathcal{D}^{(\lambda)}$ the set of principal diagrams with $\lambda_D = \lambda$.

Proposition 3.7. *Let $\lambda \vDash n$ and let $d \in \mathcal{X}_{J(\lambda)}$. Then $d = w_D$ for some diagram $D \in \mathcal{D}^{(\lambda)}$.*

Proof. We need to show that there is a diagram $D \in \mathcal{D}^{(\lambda)}$ such that $t^D d = t_D$. To construct such a D, start with any diagram $\tilde{D}$ such that $\lambda_{\tilde{D}} = \lambda$ and form the tableau $t^{\tilde{D}} d$. Suppose that symbol i appears in row r_i of $t^{\tilde{D}} d$. For each symbol i, $i = 1, \ldots, n$ we introduce in that order, a corresponding node of D at position (a_i, b_i), where $a_i = r_i$ (in particular $a_i = a_j$ if $r_i = r_j$) as follows: The node of D corresponding to symbol 1 is at position $(r_1, 1)$. Suppose that $1 \leq k < n$ and that we have already introduced nodes of D at positions (a_1, b_1),..., (a_k, b_k) corresponding to the symbols $1, \ldots, k$. If $r_{k+1} > r_k$ then introduce the node of D corresponding to symbol $k+1$ at position (r_{k+1}, b_k), otherwise introduce the node of D corresponding to symbol $k+1$ at position $(r_{k+1}, b_k + 1)$. The fact that $t^{\tilde{D}} d$ is row-standard, ensures that in the above construction, for each $j \geq 1$ the jth row of $t^{\tilde{D}} d$ not only contains exactly the same entries as the jth row of $t^D d$ but also that these entries appear in precisely the same order. Consequently, $t^D d = t_D$ and $d = w_D$. □

For $d \in \mathcal{X}_{J(\lambda)}$, denote by $D(d, \lambda)$ the diagram D satisfying $\lambda_D = \lambda$ and $d = w_D$ constructed in the proof of Proposition 3.7. Also, let $\mathcal{D}_d^{(\lambda)}$ be the set of principal diagrams $D \in \mathcal{D}^{(\lambda)}$ for which $w_D = d$. We will see in Proposition 3.8 that $D(d, \lambda)$ is an optimal diagram in this set.

Proposition 3.8. *Let $\lambda \vDash n$, let $d \in \mathcal{X}_{J(\lambda)}$, let $D = D(d, \lambda)$ and let $E \in \mathcal{D}_d^{(\lambda)}$. Then the set of columns of E may be partitioned into sets of consecutive columns so that, for $j \geq 1$,*

(i) for any two columns in the j-th set, the nodes in the column with lesser column index have row indices which are less than all the indices of the nodes in the column with greater column index;

(ii) the row indices of the nodes occurring in columns of the j-th set are precisely the row indices of the nodes in the j-th column of D.

In particular, D is the unique diagram in $\mathcal{D}_d^{(\lambda)}$ with the minimum number of columns.

Proof. Since $w_E = d = w_D$ and $D, E \in \mathcal{D}^{(\lambda)}$, the tableaux t_E and t_D have the same entries on each row. Clearly, 1 is the leading entry in the first column in both t_D and t_E. Let $j \geq 1$ and let k be the leading entry in the j-th column of t_D and let l be the final entry of that column. Let i and i' be the row indices of k and l, respectively, in t_D. If $k > 1$, then $k-1$ appears on the $(i-1)$-th column of t_D and its row index is at least i, by the construction of D. Hence, k is the leading entry in the column containing it in t_E. Let this column be the j'-th column. Again from the construction of D, if $l < n$ then $l+1$ is on the $(j+1)$-th column of t_D and its row index is at most i'. Hence, l is the final entry in the column containing it in t_E. Let this column be the j''-th column.

In the construction of w_E from E, the numbers $k,\ldots,\ l$ have increasing row indices and non-decreasing column indices. The association of the j-th column of D and the consecutive columns of E from the j'-th to the j''-th has the properties described in the statement of the lemma. □

4. Hecke algebra module homomorphisms

For the rest of the paper we take W to be the symmetric group S_n and $S = \{(i, i+1) : i = 1, \ldots, n-1\}$ and consider S_n to act on the right of the set $\{1, \ldots, n\}$.

If $\lambda \vdash n$, let $\mathbb{T}(\lambda)$ be the set of standard λ-tableaux. We recall the Robinson-Schensted bijection (see, for example Sagan [20]) which is a bijection from the set of pairs of standard tableaux of the same shape to the set of elements of the symmetric group. Following Geck [10], we write $\pi_\lambda(P, Q)$ for the element of W corresponding to the tableaux pair (P, Q) where $P, Q \in \mathbb{T}(\lambda)$. If $w = \pi_\lambda(P, Q)$, we say that λ is the *shape* of w and denote it by $\mathrm{sh}\,(w)$. As, unlike Geck, our action of W on $\{1, \ldots, n\}$ is on the right, his explicit description of left and right cells of W becomes our description of right and left cells, respectively. Geck in [10] gives an algebraic proof for the description of Kazhdan-Lusztig cells in Weyl groups of type A, which was sketched by Kazhdan and Lusztig in [13] and proved completely by Ariki in [1] using the methods of [13]. The description in [10, Corollary 5.6] is that a right cell of W is the set of elements $\pi_\lambda(P, Q)$ with Q fixed, a left cell of W is the set of elements $\pi_\lambda(P, Q)$ with P fixed, and a two-sided cell of W is the set of elements $\pi_\lambda(P, Q)$ with λ fixed—the latter is denoted by $\mathfrak{R}(\lambda')$.

In the course of his proof, Geck also gives an algebraic proof of the fact (see [10, Theorem 5.3]) that if two elements of the symmetric group are in the same two-sided cell and comparable in the right preorder $\leq_R$ then they are in the same right cell. This was first proved by Lusztig in [15] using the deep connection between cells and primitive ideals in universal enveloping algebras and later extended by him in [17] to finite and affine Weyl groups using a geometric interpretation of the Kazhdan-Lusztig basis. Moreover, in [13, §5] it is proved that, if $\mathfrak{C}$, $\mathfrak{C}_1$ are two right cells contained in the same $\mathfrak{R}(\lambda)$ then the $\mathcal{H}$-modules $S_{\mathfrak{C}}$ and $S_{\mathfrak{C}_1}$ are isomorphic (see also [10, Corollary 5.8]).

Now, recall the definitions of x_λ, y_λ in [8, p.29] where $\lambda \vDash n$ and $J(\lambda)$ is the subset of S corresponding to λ, and some easy consequences obtained using the multiplication rules (2.1).

$$x_\lambda = \sum_{w \in W_{J(\lambda)}} T_w; \qquad y_\lambda = \sum_{w \in W_{J(\lambda)}} (-q)^{-l(w)} T_w; \tag{4.1}$$

and

$$T_s x_\lambda = q x_\lambda = x_\lambda T_s; \qquad T_s y_\lambda = -y_\lambda = y_\lambda T_s; \qquad y_\lambda^2 = y_\lambda \sum_{w \in W_{J(\lambda)}} q^{-l(w)}. \tag{4.2}$$

if $\lambda \neq (1^n)$ and $s \in J(\lambda)$. If $\lambda = (1^n)$, then $J(\lambda) = \emptyset$, $W_{J(\lambda)} = 1$ and $x_\lambda = y_\lambda = T_1$. The $\mathcal{H}$-modules $x_\lambda \mathcal{H}$ and $y_\lambda \mathcal{H}$ are free right $\mathcal{H}$-modules.

In the notation of [18, Section 3],

$$x_\lambda = q^{(1/2)l(w_{J(\lambda)})} C'_{w_{J(\lambda)}} \text{ and } y_\lambda = \left(-q^{-1/2}\right)^{l(w_{J(\lambda)})} C_{w_{J(\lambda)}} \tag{4.3}$$

Theorem 4.1 (Compare [8, Theorem 3.3])**.** *Let $\lambda, \mu \vDash n$ and let $D \in \mathcal{D}^{(\lambda,\mu)}$. Then*

$$x_\lambda T_{w_D} y_\mu = \sum_{u \in W_{J(\lambda)}} \sum_{v \in W_{J(\mu)}} (-q)^{-l(v)} T_{u w_D v}. \tag{4.4}$$

Also, the set $\{x_\lambda T_{w_D} y_\mu \colon D \in \mathcal{D}^{(\lambda,\mu)}\}$ is F-linearly independent in $\mathcal{H}$ of size $|\mathcal{D}^{(\lambda,\mu)}|$.

Proof. For each $u \in W_{J(\lambda)}$ and $v \in W_{J(\mu)}$, $T_u T_{w_D} T_v = T_{u w_D v}$ since $l(u w_D v) = l(u) + l(w_D) + l(v)$ by Lemma 3.3 (iii). Equation (4.4) follows immediately.

If $D_1, D_2 \in \mathcal{D}^{(\lambda,\mu)}$ and $D_1 \neq D_2$ then $W_{J(\lambda)} w_{D_1} W_{J(\mu)} \neq W_{J(\lambda)} w_{D_2} W_{J(\mu)}$ by Lemmas 3.2 and 3.3 (ii). Hence, $W_{J(\lambda)} w_{D_1} W_{J(\mu)} \cap W_{J(\lambda)} w_{D_2} W_{J(\mu)} = \emptyset$. So, the sets of T-basis elements occurring in $x_\lambda T_{w_{D_1}} y_\mu$ and $x_\lambda T_{w_{D_2}} y_\mu$ with non-zero coefficients are disjoint. Hence, the set $\{x_\lambda T_{w_D} y_\mu \colon D \in \mathcal{D}^{(\lambda,\mu)}\}$ is F-linearly independent in $\mathcal{H}$ and has $|\mathcal{D}^{(\lambda,\mu)}|$ elements. □

In the context of Proposition 2.2, we may take $e = y_\mu$, $f = x_\lambda$ and $k = \sum_{w \in W_{J(\mu)}} q^{-l(w)}$. Then $k \neq 0$ and $\varphi_h \in \mathrm{Hom}_{\mathcal{H}_F}(y_\mu \mathcal{H}_F, x_\lambda \mathcal{H}_F)$ is now the homomorphism given by left multiplication by $x_\lambda h y_\mu$. In the particular case that $h = T_{w_D}$, we will write φ_D for φ_h.

Theorem 4.2 (Compare [8, Theorem 3.4])**.** *Let $\lambda, \mu \vDash n$. Then $\{\varphi_D \colon D \in \mathcal{D}^{(\lambda,\mu)}\}$ is an F-basis of $\mathrm{Hom}_{\mathcal{H}_F}(y_\mu \mathcal{H}_F, x_\lambda \mathcal{H}_F)$ and $\{x_\lambda T_{w_D} y_\mu \colon D \in \mathcal{D}^{(\lambda,\mu)}\}$ is an F-basis of $x_\lambda \mathcal{H}_F y_\mu$.*

Proof. By [8, Theorem 3.3 (ii)], $\dim_F \mathrm{Hom}_{\mathcal{H}_F}(y_\mu \mathcal{H}_F, x_\lambda \mathcal{H}_F)$ is the number of $(W_{J(\lambda)}, W_{J(\mu)})$ double cosets with the trivial intersection property (recall that our q here is an indeterminate), and this is $|\mathcal{D}^{(\lambda,\mu)}|$ by Result 13. It suffices to show that $\{\varphi_D \colon D \in \mathcal{D}^{(\lambda,\mu)}\}$ is linearly independent. Suppose now that $\mathcal{D}^{(\lambda,\mu)} = \{D_1, \ldots, D_r\}$ and that $\alpha_1 \varphi_{D_1} + \cdots + \alpha_r \varphi_{D_r} = 0$ ($\alpha_i \in F$). Then $0 = y_\mu(\alpha_1 \varphi_{D_1} + \cdots + \alpha_r \varphi_{D_r}) = \alpha_1 x_\lambda T_{w_{D_1}} y_\mu + \cdots + \alpha_r x_\lambda T_{w_{D_r}} y_\mu$. By Theorem 4.1, $\alpha_1 = \cdots = \alpha_r = 0$. This establishes the first basis.

By Proposition 2.2 and its proof, the mapping $\theta \colon x_\lambda \mathcal{H}_F y_\mu \to \mathrm{Hom}_{\mathcal{H}_F}(y_\mu \mathcal{H}_F, x_\lambda \mathcal{H}_F)$ given by $x_\lambda h y_\mu \mapsto \varphi_h$, $h \in \mathcal{H}_F$, is an isomorphism of F-spaces. Since $x_\lambda T_{w_D} y_\mu \theta = \varphi_D$ for $D \in \mathcal{D}^{(\lambda,\mu)}$, the second basis is established. □

Combining Result 10 (Gale-Ryser) with Theorem 4.2 we get the following corollary.

Corollary 4.3 (Compare [8, Lemma 4.1])**.** *Let $\lambda, \mu \vDash n$. Then* *(i)* $x_\lambda \mathcal{H}_F y_\mu \neq 0$ *if, and only if,* $\lambda'' \trianglelefteq \mu'$*;*

(ii) *If, in addition,* $\lambda'' = \mu'$ *and* $w \in W$ *satisfies* $x_\lambda T_w y_\mu \neq 0$*, then* $x_\lambda T_w y_\mu = \pm q^i x_\lambda T_{w_D} y_\mu \neq 0$ *for some non-negative integer* i *and some uniquely determined diagram* D *with* $\lambda_D = \lambda$ *and* $\mu_D = \mu$ *and* $w \in W_{J(\lambda)} w_D W_{J(\mu)}$*.*

Proof. (i) From Theorem 4.2, $x_\lambda \mathcal{H}_F y_\mu \neq 0$ if, and only if, the set $\mathcal{D}^{(\lambda,\mu)} \neq \emptyset$. Result 10 now gives the desired result.

(ii) Since $\mu'' = \lambda'$, there is a unique diagram D with $\lambda_D = \lambda$ and $\mu_D = \mu$. By Theorem 4.1, $x_\lambda T_{w_D} y_\mu \neq 0$. By Theorem 4.2, $\{x_\lambda T_{w_D} y_\mu\}$ is a basis of the 1-dimensional F-space $x_\lambda \mathcal{H}_F y_\mu$.

Using Result 1, there is a unique $d \in \mathcal{X}_{J(\lambda),J(\mu)}$ which is in $W_{J(\lambda)} w W_{J(\mu)}$. Hence $w = u'dv'$ with $u' \in W_{J(\lambda)}$ and $v' \in W_{J(\mu)}$. Now, $x_\lambda T_w y_\mu = \pm q^i x_\lambda T_d y_\mu$. Since $x_\lambda T_w y_\mu$ is a non-zero element of $x_\lambda \mathcal{H}_F y_\mu$, it is a non-zero multiple of $x_\lambda T_{w_D} y_\mu$. So $T_d = T_{u w_D v}$ for some $u \in W_{J(\lambda)}$ and $v \in W_{J(\mu)}$. Hence, $d = w_D$ and the result follows. □

Remark 4.4. *(i) From Result 13, we see that there is a natural bijection from* $\mathcal{D}^{(\lambda,\mu)}$ *to the set of* (W_J, W_K) *double cosets with trivial intersection property, where* J *and* K *are the subsets of* S *with* $\lambda(J) = \lambda$ *and* $\lambda(K) = \mu$*. So,* $\dim_F x_\lambda \mathcal{H}_F y_\mu$ *is the number of such double cosets. In particular, if* $\lambda, \mu \vDash n$ *with* $\lambda'' = \mu'$*, we see that there exists a unique* $(W_{J(\lambda)}, W_{J(\mu)})$ *double coset with the trivial intersection property.*

(ii) Let $\nu = (\nu_1, \ldots, \nu_r)$ *be an* r*-part composition of* n *and let* μ *be a composition of* n *which is a rearrangement of* ν*. Let* π *be a permutation of* $\{1, \ldots, r\}$ *such that* $\nu_i = \mu_{i\pi}$ *for* $i = 1, \ldots, r$*. Define* $g \in W$ *by* $g\colon \nu_1 + \nu_2 + \cdots + \nu_{i-1} + j \mapsto \mu_1 + \mu_2 + \cdots + \mu_{i\pi-1} + j$*, for* $1 \leq i \leq r$ *and* $1 \leq j \leq \nu_i$*. Then* $g^{-1} J(\nu) g = J(\mu)$*, in particular we see that the subsets* $J(\mu)$ *and* $J(\nu)$ *of* S *are in the same Coxeter class. It also follows easily that* $g^{-1} W_{J(\nu)} g = W_{J(\mu)}$*.*

Following [8], we write $M_F^\lambda = x_\lambda \mathcal{H}_F$ and $S_F^\lambda = x_\lambda T_{w_D} y_{\lambda'} \mathcal{H}_F$, where D is the unique diagram in $\mathcal{D}^{(\lambda,\lambda')}$, and recall that $M_{F,w_{J(\lambda)}} = C_{w_{J(\lambda)}} \mathcal{H}_F = y_\lambda \mathcal{H}_F$. The $\mathcal{H}_F$-module S_F^λ is a called a *Specht* module. In view of the preceding corollary, S_F^λ is a non-zero $\mathcal{H}_F$-module and $S_F^\lambda = x_\lambda \mathcal{H}_F y_{\lambda'} \mathcal{H}_F$ also. Suppose now that $\mu \vDash n$ and μ is a rearrangement of λ' and let E be the unique diagram in $\mathcal{D}^{(\lambda,\mu)}$. As in the proof of [8, Lemma 4.3], there is an element $d \in \mathcal{X}_{J(\mu),J(\lambda')}$ with $d^{-1} W_{J(\mu)} d = W_{J(\lambda')}$ and, consequently, $T_d^{-1} y_\mu T_d = y_{\lambda'}$. So, $x_\lambda T_{w_E} y_\mu \mathcal{H}_F = x_\lambda \mathcal{H}_F y_\mu \mathcal{H}_F = x_\lambda \mathcal{H}_F y_{\lambda'} \mathcal{H}_F = S_F^\lambda$. Since $\mathcal{H}_F$ is semisimple by [9, Theorem 4.3], it is then easy to see that S_F^λ is the unique common constituent of $x_\lambda \mathcal{H}_F$ and $y_\mu \mathcal{H}_F$.

The following lemma and theorem collect together certain useful relations between some of the $\mathcal{H}_F$-modules mentioned above. They essentially extend [18, Lemma 3.4 and Theorem 3.5] to compositions.

Lemma 4.5. *Let* $\lambda, \mu \vDash n$ *with* $\lambda'' = \mu'$*. Then* $S_F^\lambda \cong S^\bullet_{F,w_{J(\lambda)}} \cong S_{F,w_{J(\mu)}}$ *as* $\mathcal{H}_F$*-modules. In particular* $S_{F,w_{J(\mu)}}$ *is the unique common constituent of* $M_{F,w_{J(\mu)}}$ *and* M_F^λ*.*

Proof. Using Sagan [20, Theorem 3.2.3] and [10, Corollary 5.6], we see that $w_0 w_{J(\lambda)} \sim_{LR} w_{J(\lambda')}$. Moreover, in view of Remark 4.4(ii) and Proposition 2.1, we get $w_{J(\lambda)} \sim_{LR} w_{J(\mu)}$. By [10, Corollary 5.8], $S_{F,w_0 w_{J(\lambda)}} \cong S_{F,w_{J(\lambda')}} \cong S_{F,w_{J(\mu)}}$. Hence, $S^\bullet_{F,w_{J(\lambda)}} \cong S_{F,w_{J(\mu)}}$ since $S_{F,w_0 w_{J(\lambda)}} \cong S^\bullet_{F,w_{J(\lambda)}}$ by Result 9. It follows that $S_{F,w_{J(\mu)}}$ is a composition factor of both $y_\mu\mathcal{H}_F$ $(= M_{F,w_{J(\mu)}})$ and $x_\lambda\mathcal{H}_F$ $(= M_F^\lambda)$. But, as we have seen, S_F^λ is the unique common constituent of $x_\lambda\mathcal{H}_F$ and $y_\mu\mathcal{H}_F$. This completes the proof. □

Theorem 4.6. *Let $\lambda, \mu \vDash n$ and suppose that $\lambda'' = \mu'$. Let D be the unique diagram in $\mathcal{D}^{(\lambda,\mu)}$. Then φ_D is an $\mathcal{H}_F$-module homomorphism with $\ker\varphi_D = \hat{M}_{F,w_{J(\mu)}}$ and induces a natural isomorphism between $S_{F,w_{J(\mu)}}$ and S_F^λ. Moreover, the set $\{x_\lambda T_{w_D} C_u : u \in \mathfrak{C}(\mu)\}$ is an F-basis for S_F^λ.*

Proof. By Corollary 4.3, $x_\lambda\mathcal{H}_F y_\mu \neq 0$. By definition, the image of φ_D is S_F^λ. Moreover, $\hat{M}_{F,w_{J(\mu)}} \subseteq \ker\varphi_D$, since otherwise $\hat{M}_{F,w_{J(\mu)}}$ would have a composition factor isomorphic to S_F^λ contrary to Lemma 4.5. Hence, $\{x_\lambda T_{w_D} C_w : w \sim_R w_{J(\mu)}\}$ is an F-spanning set for S_F^λ. Since $\dim_F S_F^\lambda = \dim_F S_{F,w_{J(\mu)}} = |\mathfrak{C}(\mu)|$, this F-spanning set is an F-basis. Finally, considering dimensions, we obtain $\hat{M}_{F,w_{J(\mu)}} = \ker\varphi_D$ as required. □

We close this section by establishing the following result which will turn out to be useful in the next section.

Proposition 4.7. *Let $\lambda, \mu \vDash n$ with $\lambda'' = \mu'$ and let $d \in \mathcal{X}_{J(\lambda)}$. Then statements (i), (ii) and (iii) below are equivalent to one another and any one of them implies statement (iv).*

(i) $C'_{w_{J(\lambda)}d}\mathcal{H}_F y_\mu \neq 0$;

(ii) $(C'_{w_{J(\lambda)}d}\mathcal{H}_F y_\mu)\mathcal{H}_F = S_F^\lambda$;

(iii) S_F^λ *($\cong S^\bullet_{F,w_{J(\lambda)}} \cong S_{F,w_{J(\mu)}}$) is the unique common composition factor of* $C'_{w_{J(\lambda)}d}\mathcal{H}_F$ *and* $y_\mu\mathcal{H}_F$;

(iv) $w_{J(\lambda)}d \sim_R w_{J(\lambda)}$.

Proof. (i)⇒(ii): From Result 8, $C'_{w_{J(\lambda)}d} = C'_{w_{J(\lambda)}}h$ for some $h \in \mathcal{H}_F$. Hence, (i) implies that $(C'_{w_{J(\lambda)}d}\mathcal{H}_F y_\mu)\mathcal{H}_F$ is a non-zero $\mathcal{H}_F$-submodule of the irreducible $\mathcal{H}_F$-module $(C'_{w_{J(\lambda)}}\mathcal{H}_F y_\mu)\mathcal{H}_F = S_F^\lambda$. Hence, (ii) follows.

(ii)⇒(iii): (ii) implies that S_F^λ is an $\mathcal{H}_F$-submodule of $C'_{w_{J(\lambda)}d}\mathcal{H}_F \subseteq x_\lambda\mathcal{H}_F$ and a homomorphic image of $y_\mu\mathcal{H}_F$. Since $\mathrm{Hom}_{\mathcal{H}_F}(y_\mu\mathcal{H}_F, x_\lambda\mathcal{H}_F)$ is 1-dimensional as an F-space by Theorem 4.2 and $\mathcal{H}_F$ is semisimple by [9, Theorem 4.3], (iii) follows.

(iii)⇒(i): Semisimplicity of $\mathcal{H}_F$ together with (iii) imply that $\dim_F \mathrm{Hom}_{\mathcal{H}_F}(y_\mu\mathcal{H}_F, C'_{w_{J(\lambda)}d}\mathcal{H}_F) \neq 0$. Now, (i) follows from Proposition 2.2, taking $e = y_\mu$ and $f = C'_{w_{J(\lambda)}d}$.

(iii)⇒(iv): $d \in \mathcal{X}_{J(\lambda)}$ implies $w_{J(\lambda)}d \leq_R w_{J(\lambda)}$. Also, $C'_{w_{J(\lambda)}d}\mathcal{H}_F \subseteq M_{F,w_{J(\lambda)}}\jmath$. Since $S_F^\lambda \cong S^\bullet_{F,w_{J(\lambda)}} \cong M_{F,w_{J(\lambda)}}\jmath/\hat{M}_{F,w_{J(\lambda)}}\jmath$, (iii) implies that $C'_{w_{J(\lambda)}d} \notin \hat{M}_{F,w_{J(\lambda)}}\jmath$. Hence, $w_{J(\lambda)}d \not<_R w_{J(\lambda)}$. So, (iv) follows. □

5. **Double cosets and elements of parabolic cells**

In this section, we investigate how far the elements of minimum length in $(W_{J(\lambda)}, B)$ double cosets with the trivial intersection property (where B is a subgroup of W conjugate to $W_{J(\lambda')}$) can help us to determine reduced forms for the elements in the parabolic cell $\mathfrak{C}(\lambda)$, where $\lambda \vDash n$. Recall from

Result 4 that $\mathfrak{C}(\lambda) \subseteq w_{J(\lambda)}\mathcal{X}_{J(\lambda)}$. Also, if $d \in \mathcal{X}_{J(\lambda)}$ then $d = w_D$ for some diagram D with $\lambda_D = \lambda$, by Proposition 3.7, and Algorithm 1 gives a method of obtaining a reduced form for d.

As in [19], we say that $w \in W$ has a *decreasing cover of type* ν if $\nu \vdash n$ and the row-form of w has disjoint decreasing subsequences $C_1, C_2, \ldots$ such that $(|C_1|, |C_2|, \ldots) = \nu$. We define an *increasing cover of type* ν in a similar manner.

From [20, Theorem 3.7.3], we see that if $w \in W$ has an increasing cover of type ν and a decreasing cover of type ν' for some $\nu \vdash n$, then $\mathrm{sh}\,(w) = \nu$.

Now suppose that D is a special diagram with $\lambda_D = \lambda$ (and $\mu''_D = \lambda'$). Then it is easy to see $w_{J(\lambda)}w_D$ has a decreasing cover of type λ'' and an increasing cover of type λ'. So, $\mathrm{sh}\,(w_{J(\lambda)}w_D) = \lambda'$. Hence, $w_{J(\lambda)}w_D \in \mathfrak{R}(\lambda'')$. Since $w_{J(\lambda)} \in \mathfrak{R}(\lambda'')$ also, and $w_{J(\lambda)}w_D \leq_R w_{J(\lambda)}$ from Result 4, we get $w_{J(\lambda)}w_D \sim_R w_{J(\lambda)}$ (see the discussion at the beginning of Section 4). Note that, by Lemma 3.2, w_D is the unique element of minimum length the double coset $W_{J(\lambda)}w_D W_{J(\mu_D)}$, which has the trivial intersection property, and $W_{J(\mu_D)}$ is standard parabolic and of type λ', since $\mu''_D = \lambda'$. In particular, $W_{J(\mu_D)}$ is conjugate in W to $W_{J(\lambda')}$.

Now let $Z(\lambda) = \{d \in \mathcal{X}_{J(\lambda)} \colon w_{J(\lambda)}d \in \mathfrak{C}(\lambda)\}$ and let $\hat{Z}(\lambda) = \{d \in W \colon d$ is a prefix of w_D for some special diagram D with $\lambda_D = \lambda\}$. In the foregoing remarks, we have established the following proposition.

Proposition 5.1. *Let* $\lambda \vDash n$. *Then* $\hat{Z}(\lambda) \subseteq Z(\lambda)$.

In the case that λ is a partition, $\hat{Z}(\lambda) = Z(\lambda)$ by [18, Lemma 3.3(iv)]. We also get $\hat{Z}(\lambda) = Z(\lambda)$ for some other compositions in the next two propositions.

Proposition 5.2. *Let* $\lambda \vDash n$ *and suppose that* λ *is a rearrangement of a hook partition. Then* $\hat{Z}(\lambda) = Z(\lambda)$.

Proof. Assume the hypothesis. Suppose that λ has r parts and that all parts of λ, except possibly the kth part, are equal to 1. Let $d \in Z(\lambda)$. We will show that $d \in \hat{Z}(\lambda)$ by constructing a special tableau D' with $\lambda_{D'} = \lambda$ such that d is a prefix of $w_{D'}$. Set $D = D(d, \lambda)$.

Since $w_{J(\lambda)}d \sim_R w_{J(\lambda)}$, $\mathrm{sh}\,(w_{J(\lambda)}d) = \mathrm{sh}\,(w_{J(\lambda)}) = \lambda'$. By [20, Theorem 3.5.2], the row form of $w_{J(\lambda)}d$, obtained by concatenating the rows of $t^D w_{J(\lambda)}d$, has an increasing subsequence of length r. Since a row in $t^D w_{J(\lambda)}d$ is strictly decreasing, an increasing subsequence of length r must contain one entry from each row. Hence, there is a sequence $(1, l_1)$,..., (r, l_r) of nodes of D, all but one uniquely determined, so that the corresponding sequence of entries $(i, l_i)t_D$ $(i = 1, \ldots, r)$ of t_D is increasing. Since t_D is a standard tableau the sequence l_i, $(i = 1, \ldots, r)$, is weakly increasing.

Form a new tableau t' from t_D by first moving the entry at the node (i, l_i) to the node (i, l_k) for $1 \leq i \leq r$ and $i \neq k$, and then removing any empty columns. Let D' be the underlying diagram of t'. Then D' is a (principal) special diagram with $\lambda_{D'} = \lambda$ and t' is a standard D'-tableau. As $t' = t^{D'}d$, we conclude that d is a prefix of $w_{D'}$, as required. □

If D' is as in the proof of Proposition 5.2, let $l = l_k$; that is, the l-th column of D' has r nodes, all other columns have one node, the k-th row has $n - r + 1$ nodes and all other rows have one node. Any standard D' tableau necessarily has the entry $k+l-1$ at the (k,l) node, and the entries $1,\dots,k+l-2$ at the nodes in the first $k-1$ rows and those on the k-th row up to the $(l-1)$-th column.

Now let w be a prefix of $w_{D'}$ and let $t = t^{D'}w$. Then t is standard. Let $m_1,\dots,m_{l-1}$ denote the entries in t at nodes $(k,1),\dots,(k,l-1)$, respectively, and let $m'_1,\dots,m'_{r-k}$ denote the entries in t at nodes $(k+1,l),\dots,(r,l)$, respectively. Then $i \le m_i$ for $1 \le i \le l-1$ and $k+l+i-1 \le m'_i$ for $1 \le i \le k-r$. Note that

$$s_1 \cdots s_{m_1-1} s_2 \cdots s_{m_2-1} \cdots s_{l-1} \cdots s_{m_{l-1}-1} s_{l+k} \cdots s_{m'_1-1} s_{l+k+1} \cdots s_{m'_2-1} \cdots s_{l+r-1} \cdots s_{m'_{r-k}-1}$$

is a reduced form for w if trivial factors are removed (apply Algorithm 1).

Proposition 5.3. *If $\lambda = (r,t,s)$ with $r \ge s > t \ge 1$, then $\hat{Z}(\lambda) = Z(\lambda)$.*

Proof. Then $\lambda' = (3^t, 2^{s-t}, 1^{r-s})$. Let $d \in Z(\lambda)$. Then $\mathrm{sh}\,(w_{J(\lambda)}d) = \mathrm{sh}\,(w_{J(\lambda)}) = \lambda'$. By repeated application of [20, Theorem 3.7.3], we find that the row-form of $w_{J(\lambda)}d$ has an increasing cover $C_1,\dots,C_r$ be of type λ'. [First, it has an increasing subsequence of length 3; then it has a 2-increasing subsequence of length 6 or 5 in which both subsequences have length at most 3; etc.]

Let $D = D(d,\lambda)$. Since the rows of t_D correspond to decreasing subsequences in the row-form of $w_{J(\lambda)}d$, an increasing subsequence has elements from different rows and in columns which are non-strictly increasing. Clearly, the elements on the second row appear in the t increasing subsequences of length 3. Since the $s-t$ increasing subsequences consist of an element on the first row and an element on the third row, the $r-s$ subsequences of length 1 involve only elements on the first row. We may choose the cover described above so that $C_1,\dots,C_s$ are the subsequences of lengths at least 2 and their first row nodes $(1,j_1),\dots,(1,j_s)$ satisfy $j_1 < \dots < j_s$. Moreover, replacing the cover by another if necessary, we may assume that their third row nodes $(3,j'_1),\dots,(3,j'_s)$ satisfy $j'_1 < \cdots < j'_s$ and $j_i \le j'_i$ for $i = 1,\dots,s$. Let $C_{k_1},\dots,C_{k_t}$ be the subsequences of length 3. Again, replacing the cover by another if necessary, we may assume that their second row nodes $(3,j''_{k_1}),\dots,(3,j''_{k_t})$ satisfy $j'_{k_1} < \cdots < j'_{k_t}$ and $j_{k_i} \le j''_{k_i} \le j'_{k_i}$ for $i = 1,\dots,s$.

If D' denotes the diagram obtained from D by moving the nodes of C_k into the j_k-th column, then D' is a special diagram with $\lambda_{D'} = \lambda$ and $t^{D'}d$ is a standard D'-tableau. So, d is a prefix of $w_{D'}$. Thus, $Z(\lambda) \subseteq \hat{Z}(\lambda)$ and the opposite inclusion is true by Proposition 5.1. ☐

However, there are compositions λ for which $\hat{Z}(\lambda) \ne Z(\lambda)$; for example, if $\lambda = (2,1,1,2)$ then $Z(\lambda) = \{1,\ (2,3),\ (4,5),\ (2,4,3),\ (3,4,5),\ (2,3)(4,5),\ (2,3,4,5),\ (2,4,5,3),\ (2,5,4,3)\}$ and the two special diagrams have corresponding w_D which are $(2,5,4,3)$ and $(2,3,4,5)$ and whose non-trivial prefixes are $(2,3)$, $(2,4,3)$, $(4,5)$, and $(3,4,5)$. Thus, $Z(\lambda)\backslash\hat{Z}(\lambda) = \{(2,3)(4,5),\ (2,4,5,3)\}$. Note also that $d = (2,3)(4,5)$ is the unique element of shortest length in the double coset $W_{J(\lambda)}de^{-1}W_{J(\lambda')}e$, where $e = (3,5)(4,6)$, and this double coset has the trivial intersection property. But there is no corresponding result for $d = (2,4,5,3)$.

One of our main aims in this section is to prove the following theorem which provides a subset of $Z(\lambda)$ which is often larger than $\hat{Z}(\lambda)$ but may still be smaller than $Z(\lambda)$. Before proving the theorem, we need to establish a preliminary lemma.

Theorem 5.4. *Let $d \in \mathcal{X}_{J(\lambda)}$ where $\lambda \vDash n$. Suppose further that there exists $e \in W$ such that the double coset $W_{J(\lambda)}d(e^{-1}W_{J(\lambda')}e)$ has the trivial intersection property and d is an element of minimum length in this double coset. Then $d \in Z(\lambda)$.*

For the rest of the paper, let $h \mapsto \overline{h}$ denote the specialization of $\mathcal{H}_F$ determined by $q^{\frac{1}{2}} \mapsto 1$. Then $\overline{\mathcal{H}_F}$ is the group algebra FW and for $w \in W$ we may write $\overline{T_w}$ as w. So, $\overline{x_\lambda} = \sum_{w \in W_{J(\lambda)}} w$ and $\overline{y_{\lambda'}} = \sum_{w \in W_{J(\lambda')}} (-1)^{-l(w)} w$.

Lemma 5.5. *Let $\lambda, \mu \vDash n$ with $\lambda'' = \mu'$, and let $c, e \in W$. Then*

(i) There exists a unique $(W_{J(\lambda)}, e^{-1}W_{J(\mu)}e)$ double coset with the trivial intersection property.

(ii) The double coset $W_{J(\lambda)}c(e^{-1}W_{J(\mu)}e)$ has the trivial intersection property if, and only if, $\bar{x}_\lambda c(e^{-1}\bar{y}_\mu e) \neq 0$.

Proof. (i) Since by Remark 4.4(i) there is a unique $(W_{J(\lambda)}, W_{J(\mu)})$ double coset with the trivial intersection property and the map $w \mapsto we$ maps the set of $(W_{J(\lambda)}, W_{J(\mu)})$ double cosets bijectively to the set of $(W_{J(\lambda)}, e^{-1}W_{J(\mu)}e)$ double cosets, mapping each double coset to one of the same size, there is a unique $(W_{J(\lambda)}, e^{-1}W_{J(\mu)}e)$ double coset with the trivial intersection property.

(ii) From the proof of (i), the double coset $W_{J(\lambda)}c(e^{-1}W_{J(\mu)}e)$ has the trivial intersection property if, and only if, the double coset $W_{J(\lambda)}(ce^{-1})W_{J(\mu)}$ has the trivial intersection property. Let D be the unique diagram in $\mathcal{D}^{(\lambda,\mu)}$. Since $W_{J(\lambda)}w_D W_{J(\mu)}$ is the unique $(W_{J(\lambda)}, W_{J(\mu)})$ double coset with the trivial intersection property, we get that $W_{J(\lambda)}(ce^{-1})W_{J(\mu)}$ has the trivial intersection property if, and only if, $W_{J(\lambda)}(ce^{-1})W_{J(\mu)} = W_{J(\lambda)}w_D W_{J(\mu)}$. That is, if, and only if, $ce^{-1} = uw_D v$, for some $u \in W_{J(\lambda)}$ and $v \in W_{J(\mu)}$ (since clearly $ce^{-1} \in W_{J(\lambda)}(ce^{-1})W_{J(\mu)}$).

For any $w \in W$, $\bar{T}_w = w$. If $ce^{-1} = uw_D v$, as above, then $\bar{x}_\lambda(ce^{-1})\bar{y}_\mu = \pm\bar{x}_\lambda w_D \bar{y}_\mu \neq 0$ from equation (4.2). Hence, $\bar{x}_\lambda c(e^{-1}\bar{y}_\mu e) \neq 0$.

Conversely, suppose that $\bar{x}_\lambda c(e^{-1}\bar{y}_\mu e) \neq 0$. Then, $\bar{x}_\lambda(ce^{-1})\bar{y}_\mu \neq 0$. So, $x_\lambda T_{ce^{-1}} y_\mu \neq 0$. Hence, $ce^{-1} \in W_{J(\lambda)}w_D W_{J(\mu)}$ from Corollary 4.3. This completes the proof of the lemma. □

Proof of Theorem 5.4. In view of Proposition 4.7 it suffices to show that $C'_{w_{J(\lambda)}d}T_{e^{-1}}C_{w_{J(\lambda')}} \neq 0$. From Result 8, for any $J \subseteq S$, $C'_{w_J d} = C'_{w_J}\sum_{w \in \mathcal{X}_J,\ w \leq d}(-1)^{l(w)}a_w T_w$, where $a_w \in A$ for $w \leq d$, and a_d is a power of $q^{\frac{1}{2}}$. Let $J = J(\lambda)$ and $y = T_{e^{-1}}C_{w_{J(\lambda')}}T_e$. Then

$$C'_{w_{J(\lambda)}d}T_{e^{-1}}C_{w_{J(\lambda')}}T_e = (-1)^{l(d)}a_d C'_{w_{J(\lambda)}}T_d y + \sum_{w \in \mathcal{X}_{J(\lambda)},\ w<d}(-1)^{l(w)}a_w C'_{w_{J(\lambda)}}T_w y \tag{5.1}$$

Since $\overline{x_\lambda}$ and $\overline{y_{\lambda'}}$ are non-zero multiples of $\overline{C'_{w_{J(\lambda)}}}$ and $\overline{C_{w_{J(\lambda')}}}$, respectively, by (4.3), the righthand side of equation (5.1) becomes a non-zero multiple of $(-1)^{l(d)}\overline{x_\lambda}d\overline{y} + \sum_{w \in \mathcal{X}_J,\ w<d}(-1)^{l(w)}\overline{a_w}\,\overline{x_\lambda}w\overline{y}$, as $\overline{a_d} = 1$, and $\overline{y}$ is a non-zero multiple of $e^{-1}\overline{y_{\lambda'}}e$.

Hence, in order to show that the right hand side of equation (5.1) is non-zero, it is enough to show that $\bar{x}_\lambda d\bar{y} \neq 0$ and $\bar{x}_\lambda w\bar{y} = 0$ whenever $w \in \mathcal{X}_{J(\lambda)}$ and $w < d$.

Set $V = e^{-1}W_{J(\lambda')}e$. By hypothesis, $W_{J(\lambda)}dV$ has the trivial intersection property. Hence, $\overline{x_\lambda}d\overline{y} \neq 0$ by Lemma 5.5(ii). However, for any $w < d$, since $l(w) < l(d)$ and d has minimum length in $W_{J(\lambda)}dV$, we have $W_{J(\lambda)}wV \neq W_{J(\lambda)}dV$. By Lemma 5.5(i), $W_{J(\lambda)}wV$ does not have the trivial intersection property. Hence, $\overline{x_\lambda} w \overline{y} = 0$ by item (ii) of the same lemma. So, from equation (5.1), $\overline{C'_{w_{J(\lambda)}d} T_{e^{-1}} C_{w_{J(\lambda')}} T_e} \neq 0$. Hence, $C'_{w_{J(\lambda)}d} T_{e^{-1}} C_{w_{J(\lambda')}} T_e \neq 0$. So, $C'_{w_{J(\lambda)}d} T_{e^{-1}} C_{w_{J(\lambda')}} \neq 0$. $\square$

We illustrate Theorem 5.4 with the example $\lambda = (2, 1^r, 2)$, where $r \geq 2$. First note that, by [10, Corollary 5.6], $|\mathfrak{C}(\lambda)| = (r+1)(r+4)/2$. With $d' = w_{D'}$ and $d'' = w_{D''}$, where $t_{D'}$ and $t_{D''}$ are the tableaux described in Table 2, Proposition 5.1 shows that d', d'' and all their prefixes are in $Z(\lambda)$. Since each of d' and d'' have $r+1$ proper prefixes and they have one prefix in common, these elements account for $2r+3$ elements of $Z(\lambda)$.

$$
\begin{array}{cc} 1 & r+3 \\ 2 & \\ \vdots & \\ r+1 & \\ r+2 & r+4 \end{array}
\qquad
\begin{array}{cc} 1 & 3 \\ & 4 \\ & \vdots \\ & r+3 \\ 2 & r+4 \end{array}
\qquad
\begin{array}{cc} 1 & i+2 \\ \vdots & \\ i & \\ & i+3 \\ & \vdots \\ i+1 & r+4 \end{array}
\qquad
\begin{array}{ccc} 1 & i+1 & \\ \vdots & & \\ i & & \\ & i+2 & \\ & \vdots & \\ & j-2 & \\ & & j \\ & & \vdots \\ & j-1 & r+4 \end{array}
$$

$$
t_{D'} \qquad\qquad t_{D''} \qquad\qquad t_{D_i} \qquad\qquad t_{D_{i,j}}
$$

TABLE 2. Tableaux relating to $\mathfrak{C}((2, 1^r, 2))$

Now let $d_i = w_{D_i}$, $2 \leq i \leq r$, and $d_{i,j} = w_{D_{i,j}}$, $2 \leq i \leq j-3 \leq r$, where t_{D_i} and $t_{D_{i,j}}$ are the tableaux described in Table 2.

If $e \in W$ satisfies $\{1, \ldots, r+2\}e = \{1, \ldots, r+4\} \setminus \{i+1, j-1\}$ then it is easy to see, by considering the action on $t_{D_{i,j}}$, that $d_{i,j}$ is the unique element of minimum length in the double coset $W_{J(\lambda)} d_{i,j} (e^{-1} W_{J(\lambda')} e)$ and that $d_{i,j}$ has a unique expression of the form $u d_{i,j} v$ where $u \in W_{J(\lambda)}$ and $v \in e^{-1} W_{J(\lambda')} e$. Hence, this double coset has the trivial intersection property. By Theorem 5.4, $d \in Z(\lambda)$. This contributes a further $r(r-1)/2$ elements to $Z(\lambda)$. Nothing further is contributed by their prefixes since each prefix of $d_{i,j}$ is either $d_{i',j'}$ for some i' and j' or is a prefix of d' or d''. Indeed, an easy calculation shows that $d_{i,j+1}$ and $d_{i-1,j}$ are prefixes of $d_{i,j}$.

The remaining elements of $Z(\lambda)$ in this case are the elements d_i, $2 \leq i \leq r$. None of these are of minimum length in the double cosets of the form $W_{J(\lambda)} d_i (e^{-1} W_{J(\lambda')} e)$, $e \in W$, which have the trivial intersection property. So, Theorem 5.4 does not apply to them.

Now let $e \in W$ and let $d \in \mathcal{X}_{J(\lambda)}$ where $\lambda \vDash n$. Also set $T(d, e) = \{c \in \mathcal{X}_{J(\lambda)} \colon c \leq d$ and $W_{J(\lambda)} c (e^{-1} W_{J(\lambda')} e)$ has the trivial intersection property$\}$. It is easy to see using Proposition 4.7 that $\overline{C'_{w_{J(\lambda)}d}}(e^{-1} \bar{y}_{\lambda'} e) \neq 0$ implies that $d \in Z(\lambda)$. This is because $\overline{C'_{w_{J(\lambda)}d}}(e^{-1} \bar{y}_{\lambda'} e) \neq 0$ implies $C'_{w_{J(\lambda)}d} T_{e^{-1}} y_{\lambda'} T_e \neq 0$ which, in turn, implies that $C'_{w_{J(\lambda)}d} T_{e^{-1}} y_{\lambda'} \neq 0$. Working as in the proof of

Theorem 5.4, compare in particular with the discussion before equation (5.1), we can use Result 8 in order to express $\overline{C'_{w_{J(\lambda)}d}}(e^{-1}\bar{y}_{\lambda'}e)$ as a sum $\sum_{c\in\mathcal{X}_{J(\lambda)},\, c\leq d}\bar{\alpha}_c\bar{x}_\lambda c(e^{-1}\bar{y}_{\lambda'}e)$ where $\alpha_c\in A$ (note that $\bar{\alpha}_c=(-1)^{l(c)}\bar{a}_c$ in the notation of the above proof). Invoking Lemma 5.5(ii), we see that this last sum in fact equals $\sum_{c\in T(d,e)}\bar{\alpha}_c\bar{x}_\lambda c(e^{-1}\bar{y}_{\lambda'}e)$.

Observe that in the special case the double coset $W_{J(\lambda)}d(e^{-1}W_{J(\lambda')}e)$ has the trivial intersection property and d is an element of minimum length in this double coset (that is, when the hypothesis of the Theorem 5.4 is satisfied) we have $T(d,e)=\{d\}$. This is because for all $c\in\mathcal{X}_{J(\lambda)}$ with $c<d$ we have $W_{J(\lambda)}c(e^{-1}W_{J(\lambda')}e)\neq W_{J(\lambda)}d(e^{-1}W_{J(\lambda')}e)$. Recalling that $\bar{\alpha}_d\neq 0$, we see that there is precisely one non-zero term in the last sum mentioned above.

Next we consider more closely the situation when $|T(d,e)|=1$, so $T(d,e)=\{x\}$ for some $x\in W$ but now we allow for the possibility $x\neq d$. Clearly the condition $\bar{\alpha}_x\neq 0$ implies that again there is precisely one non-zero term in the sum $\sum_{c\in T(d,e)}\bar{\alpha}_c\bar{x}_\lambda c(e^{-1}\bar{y}_{\lambda'}e)$ and this is sufficient for us in order to conclude that $d\in Z(\lambda)$.

In [21, Lemma 1.4.5 (ii), (iii)] it is shown that $P_{w,d}=1$ whenever $w<d$ and $l(d)-l(w)\leq 2$. Comparing also with Results 7 and 8 (see in particular the comment at the end of Result 7) we see that $\bar{\alpha}_c\neq 0$ whenever $c\in\mathcal{X}_{J(\lambda)}$, $c\leq d$ and $l(d)-l(c)\leq 2$.

We thus get the following generalization of Theorem 5.4.

Theorem 5.6. *Let $d\in\mathcal{X}_{J(\lambda)}$, where $\lambda\vDash n$, and suppose that $|T(d,e)|=1$ for some $e\in W$. Suppose further that the unique element $x\in T(d,e)$ satisfies $l(d)-l(x)\leq 2$. Then $d\in Z(\lambda)$.*

Consider again the example $\lambda=(2,1^r,2)$ above with $r\geq 2$ and $2\leq i\leq r$. Then $d_i=[1,i+2,2,\ldots,i,i+3,\ldots,r+3,i+1,r+4]=s_{r+2}s_{r+1}\cdots s_{i+2}s_2s_3\cdots s_{i+1}$. Let $c_i=[1,i,2,\ldots,i-1,i+2,\ldots,r+3,i+1,r+4]=s_{r+2}\cdots s_{i+2}s_2\cdots s_{i-1}s_{i+1}$, and $e_i=[1,\ldots,i-1,i+2,\ldots,r+4,i,i+1]=s_{r+2}\cdots s_is_{r+3}\cdots s_{i+1}$. The forms above involving the Coxeter generators are reduced. Hence, $c_i\leq d_i$, $l(d_i)=r+1$ and $l(c_i)=r$.

$W_{J(\lambda)}=\mathrm{Sym}(\{1,2\})\times\mathrm{Sym}(\{r+3,r+4\})$ and $e_i^{-1}W_{J(\lambda')}e_i=\mathrm{Sym}(\{1,\ldots,i-1,i+2,\ldots r+4\})\times\mathrm{Sym}(\{i,i+1\})$, where $\mathrm{Sym}(X)$ denotes the symmetric group on the set X. The elements of the $(W_{J(\lambda)},e_i^{-1}W_{J(\lambda')}e_i)$ double coset containing c_i are obtained from the row form of c_i by permuting the entries in the first two positions arbitrarily, and by permuting the entries in the last two positions arbitrarily, and then by permuting the entries $1,\ldots,i-1,i+2,\ldots r+4$ arbitrarily and finally by permuting the entries i and $i+1$ arbitrarily. Since the sets $\{1,\ldots,i-1,i+2,\ldots r+4\}$ and $\{i,i+1\}$ appear in their natural order in c_i and this remains so if the first two positions or last positions are interchanged, c_i is the unique element of shortest length in the double coset.

Observe that each element of the double coset has one of the entries i or $i+1$ in one of the first two positions. Hence, d_i is not in the double coset and, by considering lengths, if an element c of the double coset satisfies $c\leq d_i$ then $c=c_i$.

The double coset has the trivial intersection property since an easy calculation shows $c_ie_i^{-1}=s_{r+3}w_D$, where $\{D\}=\mathcal{D}^{(\lambda,\lambda')}$. Hence, $T(d_i,e_i)=\{c_i\}$. It now follows from Theorem 5.6 that $d_i\in Z(\lambda)$.

Remark 5.7. *In* [13] *it is conjectured that all Kazhdan–Lusztig polynomials have non-negative coefficients. Braden and MacPherson* [3] *have shown that for finite and affine Weyl groups a monotonicity result concerning the coefficients of Kazhdan–Lusztig polynomials holds, which implies the non-negativity conjecture (see* [2, pages 171–172] *for a discussion on the coefficients of Kazhdan–Lusztig polynomials and relevant references).*
Since the constant term of the Kazhdan–Lusztig polynomials is 1 *(see for example* [21, Lemma 1.4.5(i)]*), we get that all coefficients $\bar{\alpha}_c$ in the sum $\sum_{c\in T(d,e)} \bar{\alpha}_c\bar{x}_\lambda c(e^{-1}\bar{y}_{\lambda'}e)$ considered above are non-zero in view of the Braden–MacPherson result. From this we see that the additional hypothesis that $l(x) \geq l(d)-2$ for $x \in T(d,e)$ in Theorem 5.6 is not actually needed. Thus, a sufficient condition for the element $d \in \mathcal{X}_{J(\lambda)}$ to belong to $Z(\lambda)$ is the existence of $e \in W$ with $|T(d,e)| = 1$.*

Finally, let us consider the situation when the composition λ has at least 2 parts and the element $d \in \mathcal{X}_{J(\lambda)}$ satisfies $T(d,e) \neq \emptyset$ for some $e \in W$. Fix $b \in T(d,e)$. Since there exists a unique $(W_{J(\lambda)}, e^{-1}W_{J(\lambda')}e)$ double coset with the trivial intersection property, we have $T(d,e) \subseteq W_{J(\lambda)}b(e^{-1}W_{J(\lambda')}e)$. Now let $c \in T(d,e)$. It follows that c has a unique representation of the form $c = u(c,e,b)bv(c,e,b)$ where $u(c,e,b) \in W_{J(\lambda)}$ and $v(c,e,b) \in e^{-1}W_{J(\lambda')}e$. Set $\alpha_{(e,b)}(c) = (-1)^{l(v(c,e,b))}$ $(=$ sgn$v(c,e,b))$. The assumption that λ has at least 2 parts ensures that the even permutations inside $e^{-1}W_{J(\lambda')}e$ form a subgroup of this group of index 2. Moreover, when we express $e^{-1}\bar{y}_{\lambda'}e$ as a linear combination of the usual basis consisting of the elements of W, the even permutations (resp. odd permutations) in $e^{-1}W_{J(\lambda')}e$ occur with coefficient $+1$ (resp. -1). It follows that $v(c,e,b)\, e^{-1}\bar{y}_{\lambda'}e = \alpha_{(e,b)}(c)\, e^{-1}\bar{y}_{\lambda'}e$ and so $\bar{x}_\lambda c(e^{-1}\bar{y}_{\lambda'}e) = \alpha_{(e,b)}(c)\, \bar{x}_\lambda b(e^{-1}\bar{y}_{\lambda'}e)$ as elements of the group algebra.

The next result gives another sufficient condition for the element $d \in \mathcal{X}_{J(\lambda)}$ to belong to $Z(\lambda)$.

Theorem 5.8. *Let $d \in \mathcal{X}_{J(\lambda)}$, where the composition λ has at least two parts, and suppose that there exists $e \in W$ with $T(d,e) \neq \emptyset$ and $l(c) \geq l(d)-2$ for all $c \in T(d,e)$. Suppose further that there exists $b \in T(d,e)$ such that $\sum_{c\in T(d,e)}(-1)^{l(c)-l(b)}\alpha_{(e,b)}(c) \neq 0$. Then $d \in Z(\lambda)$.*

Proof. As before, we express $\overline{C'_{w_{J(\lambda)}d}}(e^{-1}\bar{y}_{\lambda'}e)$ as a sum $\sum_{c\in T(d,e)} \bar{\alpha}_c\bar{x}_\lambda c(e^{-1}\bar{y}_{\lambda'}e)$, where $\alpha_c \in A$ and it is enough to show that this sum is non-zero. The assumption that $l(c) \geq l(d)-2$ for all $c \in T(d,e)$ ensures that $\bar{\alpha}_c = (-1)^{l(c)-l(b)}\bar{\alpha}_b$ whenever $c \in T(d,e)$ (see the comment below Result 7 and [21, Lemma 1.4.5 (ii), (iii)]). Moreover, as we have already observed in the discussion immediately before the statement of this theorem, for each $c \in T(d,e)$ we have $\bar{x}_\lambda c(e^{-1}\bar{y}_{\lambda'}e) = \alpha_{(e,b)}(c)\,\bar{x}_\lambda b(e^{-1}\bar{y}_{\lambda'}e)$. We conclude that $\sum_{c\in T(d,e)} \bar{\alpha}_c\bar{x}_\lambda c(e^{-1}\bar{y}_{\lambda'}e) = \bar{x}_\lambda b(e^{-1}\bar{y}_{\lambda'}e)\sum_{c\in T(d,e)}(-1)^{l(c)-l(b)}\alpha_{(e,b)}(c)$, and the result now follows easily. □

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