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A NOTE ON TRANSFER THEOREMS

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Abstract. In this paper, we generalize some transfer theorems. In particular, we derive one of the main results of Gagola (Contemp Math., **524** (2010) 49–60) from our results.

1. **Introduction**

Throughout the paper, we suppose G is a finite group, π is a set of prime numbers, p is a prime number and m is an integer. Let $\pi(m)$ be the set of all the prime divisors of m and denote $\pi(|G|)$ by $\pi(G)$. For any $m \in N_+$, $\exists$ a π-number m_π and a π'-number $m_{\pi'}$ such that $m = m_\pi m_{\pi'}$. To state our results, we need to recall some notations, most of which are standard. Define $O^\pi(G) = \bigcap\{N \mid N \trianglelefteq G$ and G/N is a π-group$\}$, $A^\pi(G) = \bigcap\{N \mid N \trianglelefteq G$ and G/N is an abelian π-group$\}$ and $E^m(G) = \langle x^m \mid x \in G\rangle G'$. Note that for any $H \leq G$, $A^\pi(H) = E^{|G|_\pi}(H)$. For any $H \leq G$, let $v_{G\to H/H'} : G \to H/H'$ be the transfer homomorphism, we denote the image of $v_{G\to H/H'}$ by $T_G(H)/H'$. The definition of $T_G(H)$ appears in [1] and [2]. Let $\pi \subseteq \pi(G)$ and $H \leq G$ be such that $[G : H]$ is a π'-number. In [5], we define $T_G^H(\pi) = \prod_{p\in\pi} (T_G(P)H')$, of which $P \in Syl_p(H)$, for any $p \in \pi$.

Transfer theory has been investigated by many authors, see [1],[2],[3] and [5].

In this paper, we prove the following theorems.

Theorem 1.1. *Let $H \leq G$ be such that $[G : H]$ is a π'-number. Then*
$T_G(H) = T_G^H(\pi)T_{O^\pi(G)}(H \cap O^\pi(G))$.

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Theorem 1.2. *Let $H \le G$ be such that $([G:H], m) = 1$ and $\pi = \pi(m)$. Then*
$H/E^m(H) = H \cap E^m(G)/E^m(H) \times T_G^H(\pi)E^m(H)/E^m(H)$ *and* $T_G(H)E^m(H) = T_G^H(\pi)E^m(H)$.
In particular, let $H \le G$ be such that $[G:H]$ is a π'-number. Then $H/A^\pi(H) = H \cap A^\pi(G)/A^\pi(H) \times T_G^H(\pi)A^\pi(H)/A^\pi(H)$.

Note that our Theorem 1.2 is a generalization of [2, Theorem B].

In [5], we have proved the following result See [5, Theorem A].

Theorem 1.3. *Let $H \le G$ be such that $[G:H]$ is a π'-number. Then*
$H' \cap O^\pi(G) = T_G^H(\pi) \cap O^\pi(G) = [H \cap O^\pi(G), H]$.

Along the way, we make a different approach to prove Theorem 1.3 and derive of [1, Theorem 2.7] from Theorem 1.1 and Theorem 1.3.

Theorem 1.4. *(Gagola)Let $H \le G$ and $N \trianglelefteq G$ be such that $([G:H],[G:N]) = 1$. Then $T_G(H) \cap N = [H \cap N, H]T_N(H \cap N)$.*

2. Preliminaries

Lemma 2.1. *Let $H \le G$ and $N \trianglelefteq G$ be such that $G = HN$. Then $E^m(G) = E^m(H)E^m(N)[N,G]$.*

The proof can be found on page 134 of [3].

The following lemma generalizes of [5, Theorem 5.20].

Lemma 2.2. *Let $H \le G$ be such that $([G:H], m) = 1$ and $E^m(H) \le K \le H \cap E^m(G)$. Then $K \trianglelefteq H$. Let $v_{G\to H/K} : G \to H/K$ be the transfer homomorphism. Then $Ker\ v_{G\to H/K} = E^m(G)$, $T_G(H) \cap E^m(G) = T_G(H) \cap E^m(H)$ and $G = T_G(H)E^m(G)$.*

Proof. It is not very hard to see $E^m(G) \le Ker\ v_{G\to H/K}$. Fix a right transversal T for H in G, for any $g \in Ker\ v_{G\to H/K}$, $\exists T_0 \subseteq T$ and $\{n_t \mid n_t \in N_+, t \in T_0\}$ such that $\sum_{t\in T_0} n_t = [G:H]$ and $v_{G\to H/K}(g) = \prod_{t\in T_0} tg^{n_t}t^{-1}K$. Since $g \in Ker\ v_{G\to H/K}$, $\prod_{t\in T_0} tg^{n_t}t^{-1} \in K \le E^m(G)$. In $G/E^m(G)$, $\prod_{t\in T_0} \bar{t}\bar{g}^{n_t}\bar{t}^{-1} = \bar{1}$. Note that $G/E^m(G)$ is abelian, we have $\bar{g}^{[G:H]} = \prod_{t\in T_0} \bar{t}\bar{g}^{n_t}\bar{t}^{-1} = \bar{1}$. Also $([G:H], |G/E^m(G)|) = 1$, so $\bar{g} = \bar{1}$, i.e. $g \in E^m(G)$. Hence $Ker\ v_{G\to H/K} \le E^m(G)$.

Observe that $G/E^m(G) \cong T_G(H)K/K \cong T_G(H)/(T_G(H) \cap K)$. In particular, let $K = E^m(H)$ or $K = H \cap E^m(G)$, we have $T_G(H)/(T_G(H) \cap E^m(H)) \cong T_G(H)/(T_G(H) \cap E^m(G))$. Since $E^m(H) \le H \cap E^m(G)$, we have $T_G(H) \cap E^m(G) = T_G(H) \cap E^m(H)$. Note that $|T_G(H)E^m(G)| = \frac{|T_G(H)||E^m(G)|}{|T_G(H)\cap E^m(G)|} = \frac{|G|}{|E^m(G)|}|E^m(G)| = |G|$, we have $G = T_G(H)E^m(G)$. □

Remark 2.1. *Let $H \le G$ be such that $[G:H]$ is a π'-number and let $m = |G|_\pi$. Then $Ker\ v_{G\to H/K} = A^\pi(G)$, $G = T_G(H)A^\pi(G)$ and $T_G(H) \cap A^\pi(G) = T_G(H) \cap A^\pi(H)$. In particular, let $\pi = \{p\} \subseteq \pi(G)$, $H = P \in Syl_p(G)$. Then $G = T_G(P)A^p(G)$ and $T_G(P) \cap A^p(G) = P'$, which is the statement of* [2, Lemma 2.4].

Lemma 2.3. *Let $H \le G$ be such that $[G:H]$ is a π'-number. Then $G = T_G(H)O^\pi(G)$.*

Proof. Since $([G:H],[G:O^\pi(G)]) = 1$, we have $G = HO^\pi(G)$. Then $A^\pi(G)/O^\pi(G) = (G/O^\pi(G))' = (HO^\pi(G)/O^\pi(G))' = H'O^\pi(G)/O^\pi(G)$. Hence $A^\pi(G) = H'O^\pi(G)$. By Remark 2.1, we have $G = T_G(H)A^\pi(G) = T_G(H)(H'O^\pi(G)) = T_G(H)O^\pi(G)$. □

Lemma 2.4. *Let $H \le G$ be such that $[G:H]$ is a π'-number. Then $T_G^H(\pi) \cap A^\pi(G) = H'$.*

See [5, Lemma 3.2].

Lemma 2.5. *Let $H \le G$ be such that $([G:H],m) = 1$ and $Z \le Z(G)$. Then $Z \cap E^m(H) = Z \cap H \cap E^m(G)$.*

Proof. It is only necessary to prove $Z \cap H \cap E^m(G) \le Z \cap E^m(H)$. Fix a right transversal T for H in G, for any $g \in Z \cap H \cap E^m(G)$, $g \in E^m(G) = Ker\ v_{G\to H/E^m(H)}$, $\exists T_0 \subseteq T$ and $\{n_t \mid n_t \in N_+, t \in T_0\}$ such that $\sum_{t\in T_0} n_t = [G:H]$ and $v_{G\to H/E^m(H)}(g) = \prod_{t\in T_0} tg^{n_t}t^{-1}E^m(H)$. Hence $\prod_{t\in T_0} tg^{n_t}t^{-1} \in E^m(H)$. Since $g \in Z(G)$, also $g^{[G:H]} = \prod_{t\in T_0} tg^{n_t}t^{-1} \in E^m(H)$. Note that $g \in H$ and $([G:H], |H/E^m(H)|) = 1$, we have $g \in E^m(H)$. Hence $Z \cap H \cap E^m(G) \le Z \cap E^m(H)$. □

Lemma 2.6. *$G' \cap O^\pi(G) = [O^\pi(G), G]$.*

Proof. We may assume $\pi \subseteq \pi(G)$, fix $p \in \pi$ and let $P \in Syl_p(G)$ and $T = PO^\pi(G)$. In $G/[O^\pi(G), G]$, $\overline{O^\pi(G)} \le Z(\overline{G})$ and $[\overline{G}:\overline{T}]$ is a π-number, by Lemma 2.5, $\overline{O^\pi(G)} \cap A^{\pi'}(\overline{T}) = \overline{O^\pi(G)} \cap A^{\pi'}(\overline{G})$. In $\overline{T}$, since $O^\pi(O^\pi(G)) = O^\pi(G)$ and $\overline{O^\pi(G)}$ is abelian, we have $\overline{O^\pi(G)}$ is an abelian π'-subgroup. Note that $\overline{P}$ is π-group, we have $\overline{P} \cap \overline{O^\pi(G)} = \overline{1}$. Also, $\overline{O^\pi(G)} \le Z(\overline{G})$, so $\overline{T} = \overline{P} \times \overline{O^\pi(G)}$ and $\overline{P} = A^{\pi'}(\overline{T}) = O^{\pi'}(\overline{T})$. Hence $\overline{G'} \cap \overline{O^\pi(G)} = \overline{G}' \cap \overline{O^\pi(G)} \le A^{\pi'}(\overline{G}) \cap \overline{O^\pi(G)} = A^{\pi'}(\overline{T}) \cap \overline{O^\pi(G)} = \overline{1}$, i.e. $\overline{G'} \cap \overline{O^\pi(G)} = \overline{1}$. Hence $G' \cap O^\pi(G) = [O^\pi(G), G]$. □

Lemma 2.7. *(Gagola-Isaacs)Let $P \in Syl_p(G)$. Then $P' \cap O^p(G) = T_G(P) \cap O^p(G) = [P \cap O^p(G), P]$.*

See [2, Theorem 3.1].

Lemma 2.8. *Let $l \in N_+$, $p \in \pi(G)$ and $H \le G$ be such that $[G:H]$ is a p'-number. Then $E^{p^l}(H) \cap O^p(G) = [H \cap O^p(G), H]E^{p^l}(H \cap O^p(G))$.*

Proof. Let $P \in Syl_p(H)$, then $P \in Syl_p(G)$. By Lemma 2.3, we have $H = T_G(P)(H \cap O^p(G))$. By Lemma 2.1, $E^{p^l}(H) = E^{p^l}(T_G(P))E^{p^l}(H \cap O^p(G))[H \cap O^p(G), H]$. By Lemma 2.7, we have

$$\begin{aligned} E^{p^l}(H) \cap O^p(G) &= E^{p^l}(H \cap O^p(G))[H \cap O^p(G), H](E^{p^l}(T_G(P)) \cap O^p(G)) \\ &\le E^{p^l}(H \cap O^p(G))[H \cap O^p(G), H](T_G(P) \cap O^p(G)) \\ &= E^{p^l}(H \cap O^p(G))[H \cap O^p(G), H]. \end{aligned}$$

Since the opposite is obvious, we have proved Lemma 2.8. □

Lemma 2.9. *Let $H \le G$ be such that $([G:H],m) = 1$ and let $N \trianglelefteq G$ be such that $G = HN$. Let $\pi = \pi(m)$, then $E^m(H) \cap O^\pi(N) = [H \cap O^\pi(N), H]E^m(H \cap O^\pi(N))$.*

Proof. We may assume $\pi \subseteq \pi(G)$. For any $p \in \pi$, let $P \in Syl_p(H)$, then $P \in Syl_p(G)$. Let $T = PO^\pi(N)$, then $H\cap T = P(H\cap O^\pi(N))$. Note that $[T : H\cap T]$ is a p'-number and $O^p(T) = O^\pi(N)$. By Lemma 2.8, we have $E^{m_p}(H\cap T)\cap O^\pi(N) = [H\cap O^\pi(N), H\cap T]E^{m_p}(H\cap O^\pi(N))$, of which $m_p \in N_+$ is a power of p and $m_{p'} \in N_+$ such that $(m_{p'}, p) = 1$ and $m = m_p m_{p'}$.

In $H/[H \cap O^\pi(N), H]$, we have $\overline{H \cap O^\pi(N)} \leq Z(\overline{H})$ and $[\overline{H} : \overline{H \cap T}]$ is a p'-number. By Lemma 2.5, we have $E^{m_p}(\overline{H}) \cap \overline{H \cap O^\pi(N)} = E^{m_p}(\overline{H \cap T}) \cap \overline{H \cap O^\pi(N)}$. Hence

$$\begin{aligned} E^{m_p}(H) \cap O^\pi(N) &= E^{m_p}(H \cap T)[H \cap O^\pi(N), H] \cap O^\pi(N) \\ &= [H \cap O^\pi(N), H](E^{m_p}(H \cap T) \cap O^\pi(N)) \\ &= [H \cap O^\pi(N), H]E^{m_p}(H \cap O^\pi(N)). \end{aligned}$$

Hence $E^{m_p}(\overline{H}) \cap \overline{H \cap O^\pi(N)} = E^{m_p}(\overline{H \cap O^\pi(N)})$.

Now we have $E^m(\overline{H}) \cap \overline{H \cap O^\pi(N)} = \bigcap_{p\in\pi}(E^{m_p}(\overline{H}) \cap \overline{H \cap O^\pi(N)}) = \bigcap_{p\in\pi} E^{m_p}(\overline{H \cap O^\pi(N)}) = E^m(\overline{H \cap O^\pi(N)})$. Hence $E^m(H) \cap O^\pi(N) = [H \cap O^\pi(N), H]E^m(H \cap O^\pi(N))$. □

Remark 2.2. *Let $H \leq G$ be such that $[G : H]$ is a π'-number and $N \trianglelefteq G$ be such that $G = HN$. In the statement of Lemma 2.9, let $m = |G|_\pi$, then $A^\pi(H) \cap O^\pi(N) = [H \cap O^\pi(N), H]A^\pi(H \cap O^\pi(N))$.*

Lemma 2.10. *Let $K \leq H \leq G$. Then $T_G(K) \leq T_G(H)$.*

Proof. Fix a right transversal T for K in H, by [5, Theorem 10.8], we have $T_G(K)/K' = v_{H\to K/K'}(T_G(H))$. Hence for any $k \in T_G(K)$, $\exists h \in T_G(H)$ such that $kK' = v_{H\to K/K'}(h)$. Note that $v_{H\to K/K'}(h) = \prod_{t\in T} th(t \cdot h)^{-1}K'$. Hence $kH' = \prod_{t\in T} th(t \cdot h)^{-1}H' = h^{[H:K]}H' = (hH')^{[H:K]} \in T_G(H)/H'$. Hence $T_G(K) \leq T_G(H)$. □

3. Main Results

The proof of Theorem 1.1. By Lemma 2.10, we have $T_G^H(\pi) \leq T_G(H)$. By [2, Lemma 2.1], we have $T_{O^\pi(G)}(H \cap O^\pi(G)) \leq T_G(H)$. For any $g \in G$, $\exists\{g_p|p \in \pi\} \cup \{g_{\pi'}\}$ such that $g = (\prod_{p\in\pi} g_p)g_{\pi'}$, of which g_p is a p-element, $g_{\pi'}$ is a π'-element. Then $v_{G\to H/H'}(g) = (\prod_{p\in\pi} v_{G\to H/H'}(g_p))v_{G\to H/H'}(g_{\pi'})$. Let $l_pH' = v_{G\to H/H'}(g_p)$, $\forall p \in \pi$, $l_{\pi'}H' = v_{G\to H/H'}(g_{\pi'})$. Note that $g_{\pi'} \in O^\pi(G)$, by [2, Lemma 2.1], we have $l_{\pi'}H' \in T_{O^\pi(G)}(H \cap O^\pi(G))H'/H'$. For any $p \in \pi$, let $P \in Syl_p(H)$, then $P \in Syl_p(G)$. Fix a right transversal T_p for P in H, let $S_pP' = v_{G\to P/P'}(g_p)$, then $S_pP' = v_{H\to P/P'}(l_p) = \prod_{t\in T_p} tl_p(t\cdot l_p)^{-1}P'$. Hence $S_pH' = (l_pH')^{[H:P]}$. Since g_p is a p-element, $l_pH' = v_{G\to H/H'}(g_p)$ is also a p-element. Hence $\exists v \in N_+$ such that $[H : P]v \equiv 1(mod\ \circ(l_pH'))$. Hence $l_pH' = (S_pH')^v = S_p^vH' \in T_G(P)H'/H'$. We have $T_G(H) \leq T_G^H(\pi)T_{O^\pi(G)}(H \cap O^\pi(G))$. This completes the proof. □

The proof of Theorem 1.2. By Lemma 2.2 and Dedekind's Lemma, we have $H = (H\cap E^m(G))T_G(H)$. Also by Lemma 2.2 and Dedekind's Lemma, $T_G(H)E^m(H) \cap (H \cap E^m(G)) = E^m(H)(T_G(H) \cap E^m(G)) = E^m(H)(T_G(H) \cap E^m(H)) = E^m(H)$.

Hence $H/E^m(H) = H \cap E^m(G)/E^m(H) \times T_G(H)E^m(H)/E^m(H)$.

By Theorem 1.1 and Lemma 2.2,

$$\begin{aligned}T_G(H) &= T_G^H(\pi)T_{O^\pi(G)}(H\cap O^\pi(G))\\ &\le T_G^H(\pi)(T_G(H)\cap O^\pi(G))\\ &\le T_G^H(\pi)(T_G(H)\cap E^m(G))\\ &= T_G^H(\pi)(T_G(H)\cap E^m(H)) \le T_G^H(\pi)E^m(H).\end{aligned}$$

Hence $T_G(H)E^m(H) = T_G^H(\pi)E^m(H)$.

We have $H/E^m(H) = H\cap E^m(G)/E^m(H)\times T_G^H(\pi)E^m(H)/E^m(H)$.

In particular, let $H\le G$ be such that $[G:H]$ is a π'-number, let $m=|G|_\pi$, then
$H/A^\pi(H) = H\cap A^\pi(G)/A^\pi(H)\times T_G^H(\pi)A^\pi(H)/A^\pi(H)$. □

The proof of Theorem 1.3. In the statement of Remark 2.2, let $N=G$. Then

$$\begin{aligned}A^\pi(H)\cap O^\pi(G) &= [H\cap O^\pi(G),H]A^\pi(H\cap O^\pi(G))\\ &= [H\cap O^\pi(G),H](H\cap O^\pi(G))'O^\pi(H)\\ &= [H\cap O^\pi(G),H]O^\pi(H).\end{aligned}$$

by Lemma 2.6, we have $H'\cap O^\pi(H) = [O^\pi(H),H]$. Hence

$$\begin{aligned}H'\cap O^\pi(G) &= H'\cap A^\pi(H)\cap O^\pi(G)\\ &= H'\cap[H\cap O^\pi(G),H]O^\pi(H) = [H\cap O^\pi(G),H](H'\cap O^\pi(H))\\ &= [H\cap O^\pi(G),H][O^\pi(H),H] = [H\cap O^\pi(G),H].\end{aligned}$$

By Lemma 2.4, we have $T_G^H(\pi)\cap O^\pi(G) = T_G^H(\pi)\cap A^\pi(G)\cap O^\pi(G) = H'\cap O^\pi(G)$. □

The proof of Theorem 1.4. Let $\pi=\pi(G/N)$, we have $[G:H]$ is a π'-number and $O^\pi(G)\le N$. By Theorem 1.1 and Theorem 1.3, we have

$$\begin{aligned}T_G(H)\cap O^\pi(G) &= T_G^H(\pi)T_{O^\pi(G)}(H\cap O^\pi(G))\cap O^\pi(G)\\ &= (T_G^H(\pi)\cap O^\pi(G))T_{O^\pi(G)}(H\cap O^\pi(G))\\ &= [H\cap O^\pi(G),H]T_{O^\pi(G)}(H\cap O^\pi(G)).\end{aligned}$$

By Remark 2.1, we have $N = T_N(H\cap N)O^\pi(G)$. It is not very hard to see $T_{O^\pi(G)}(H\cap O^\pi(G))\le T_N(H\cap N)$. Hence

$$\begin{aligned}T_G(H)\cap N &= T_G(H)\cap T_N(H\cap N)O^\pi(G)\\ &= T_N(H\cap N)(T_G(H)\cap O^\pi(G))\\ &= T_N(H\cap N)T_{O^\pi(G)}(H\cap O^\pi(G))[H\cap O^\pi(G),H]\\ &= T_N(H\cap N)[H\cap O^\pi(G),H]\le T_N(H\cap N)[H\cap N,H]\le T_G(H)\cap N.\end{aligned}$$

Hence $T_G(H)\cap N = T_N(H\cap N)[H\cap N,H]$. □

Acknowledgments

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