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ON BIPARTITE DIVISOR GRAPH FOR CHARACTER DEGREES

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Abstract. The concept of the bipartite divisor graph for integer subsets has been considered in [M. A. Iranmanesh and C. E. Praeger, Bipartite divisor graphs for integer subsets, *Graphs Combin.*, **26** (2010) 95–105.]. In this paper, we will consider this graph for the set of character degrees of a finite group G and obtain some properties of this graph. We show that if G is a solvable group, then the number of connected components of this graph is at most 2 and if G is a non-solvable group, then it has at most 3 connected components. We also show that the diameter of a connected bipartite divisor graph is bounded by 7 and obtain some properties of groups whose graphs are complete bipartite graphs.

1. Introduction

Let G be a finite group and $\mathrm{Irr}(G)$ be the set of all irreducible complex characters of G. We write $\mathrm{cd}(G) = \{\chi(1) \mid \chi \in \mathrm{Irr}(G)\}$ to denote the set of all character degrees of G, and we use $\mathrm{cd}^*(G)$ for the set $\mathrm{cd}(G) \setminus \{1\}$. The connection between the structure of G and the set $\mathrm{cd}(G)$ has been studied extensively by many authors. A favorite field in this subject is the graphs associated to $\mathrm{cd}(G)$. [7] is a good overview of graphs associated to the set $\mathrm{cd}(G)$. The prime graph $\Delta(\mathrm{cd}(G))$ and the common-divisor graph $\Gamma(\mathrm{cd}(G))$ are two important graphs associated to $\mathrm{cd}(G)$. Let n be a positive integer and $\pi(n)$ be the set of prime divisors of n. Also suppose that X is a set of positive integers, $X^* = X \setminus \{1\}$ and define $\rho(X)$ to be the set of prime divisors of elements of X, i.e.

$$\rho(X) = \bigcup_{n \in X} \pi(n)$$

The prime graph $\Delta(X)$ is the graph with vertex set $\rho(X)$ and two vertices p and q are connected if there exist some $n \in X$ which is divisible by pq. The common-divisor graph $\Gamma(X)$ is the graph with vertex

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set X^* and two vertices m and n are connected if $\gcd(m,n) > 1$. Motivated by [3], in this paper, we introduce the bipartite divisor graph for character degrees of a finite group and obtain some properties of this graph. The concept of the bipartite divisor graph for integer subsets has been considered in [4] and the bipartite divisor graph for group conjugacy class sizes has been studied in [3] and [5]. The bipartite divisor graph $B(X)$ is the graph with vertex set the disjoint union[1] $\rho(X) \cup X^*$ and edge set

$$E = \{\{p, n\} \mid p \in \rho(X), n \in X^* \text{ and } p|n\}.$$

Here we introduce some notations, which will be used throughout the paper.

Notations. Let $\mathcal{G}$ be a graph. We use $n(\mathcal{G})$ to denote the number of connected components of $\mathcal{G}$. The distance between two vertices x and y is denoted by $d_{\mathcal{G}}(x,y)$ and $\operatorname{diam}(\mathcal{G})$ is used for the diameter of the graph $\mathcal{G}$. (Remind that we don't define the distance between vertices that belong to different connected components of $\mathcal{G}$, and the diameter of $\mathcal{G}$ is the maximum distance between vertices in the same connected component, i.e. the largest diameter among the connected components). We also use $\rho(G)$, $\Gamma(G)$, $\Delta(G)$ and $B(G)$ to represent the sets $\rho(\operatorname{cd}(G))$, $\Gamma(\operatorname{cd}(G))$, $\Delta(\operatorname{cd}(G))$ and $B(\operatorname{cd}(G))$ respectively.

Some graph-theoretic properties of $B(X)$ are closely related to properties of $\Delta(X)$ and $\Gamma(X)$. The following lemma [4, Lemma 1((c),(d))] describes the relation between the number of connected components and diameter of $\Gamma(X)$, $\Delta(X)$ and $B(X)$.

Lemma 1.1. *Let X be a set of positive integers and let $B = B(X)$, $\Gamma = \Gamma(X)$ and $\Delta = \Delta(X)$. Then*

(i) $n(B) = n(\Gamma) = n(\Delta)$

(ii) $|\operatorname{diam}(\Gamma) - \operatorname{diam}(\Delta)| \leq 1$ *and one of the following occurs:*

  (a) $\operatorname{diam}(B) = 2\max\{\operatorname{diam}(\Gamma), \operatorname{diam}(\Delta)\}$ *or*

  (b) $\operatorname{diam}(B) = 2\operatorname{diam}(\Gamma) + 1 = 2\operatorname{diam}(\Delta) + 1.$

In this paper we consider the graphs $B(X)$, $\Delta(X)$ and $\Gamma(X)$ for the set $X = \operatorname{cd}(G)$. Figure 1 shows the graphs associated to the set of character degrees for group $\mathrm{SL}_3(3)$.

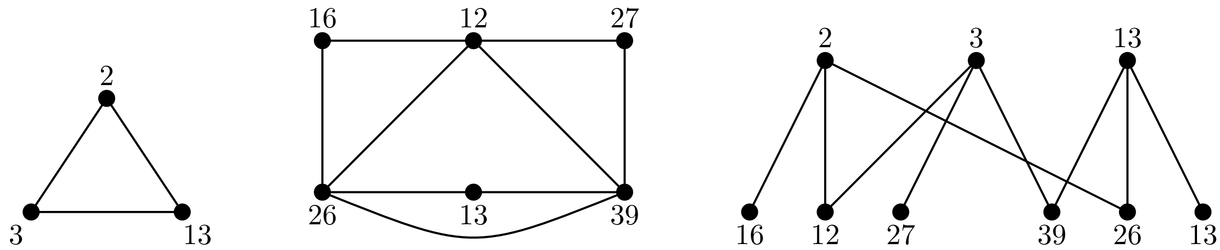
There is useful information about the number of connected components and diameter of $\Gamma(G)$ and $\Delta(G)$ and by lemma 1.1, we can use them to find the number of connected components and diameter of $B(G)$.

2. Number of connected components of $B(G)$

In this section we discuss about the number of connected components of $B(G)$. The following theorem describes the number of connected components of $B(G)$ for solvable groups.

Theorem 2.1. *Let G be a solvable group. Then $n(B(G)) \leq 2$. Furthermore, $n(B(G)) = 2$ if and only if G belongs to one of the six families stated in section 2 of [8].*

[1] We mention that $\rho(X)$ and X^* are not necessarily distinct and disjoint union just asserts that $\rho(X)$ is one part of vertices of $B(X)$ and X^* is the other part of vertices of $B(X)$.

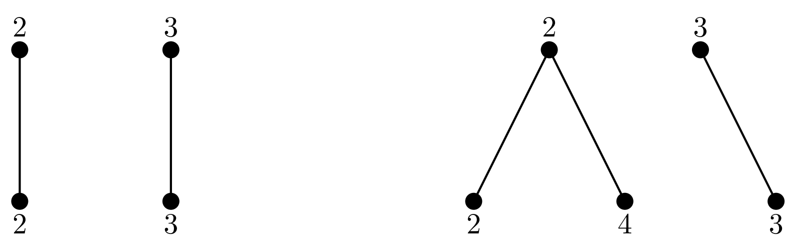


(a) $\Delta(\mathrm{SL}_3(3))$ (b) $\Gamma(\mathrm{SL}_3(3))$ (c) $B(\mathrm{SL}_3(3))$

FIGURE 1. Graphs associated to $\mathrm{cd}(\mathrm{SL}_3(3))$

Proof. By lemma 1.1, we know that the number of connected components of $B(G)$ and $\Delta(G)$ are equal. By of [7, corollary 4.2] if G is a solvable group, then $n(\Delta(G)) \leq 2$. Therefore, we have $n(B(G)) \leq 2$. Again by lemma 1.1, we have $n(B(G)) = 2$ if and only if $n(\Delta(G)) = 2$. By [8, main theorem], $n(\Delta(G)) = 2$ if and only if G belongs to one of the six families that are presented in section 2 of [8] and accordingly, $n(B(G)) = 2$ if and only if G belongs to a family of groups stated in section 2 of [8]. □

Remark 2.2. *By theorem 2.1 and the fact that solvable groups whose bipartite divisor graph has two connected components exist we can say that the upper bound* 2*, is the best upper bound for the number of connected componnets of* $B(G)$ *when* G *is a solvable group. Figure 2 shows the bipartite divisor graphs of* S_4 *and* $\mathrm{GL}_2(3)$ *which have exactly two connected components.*



(a) $B(S_4)$ (b) $B(\mathrm{GL}_2(3))$

FIGURE 2. Bipartite divisor graphs of S_4 and $\mathrm{GL}_2(3)$ with two connected components.

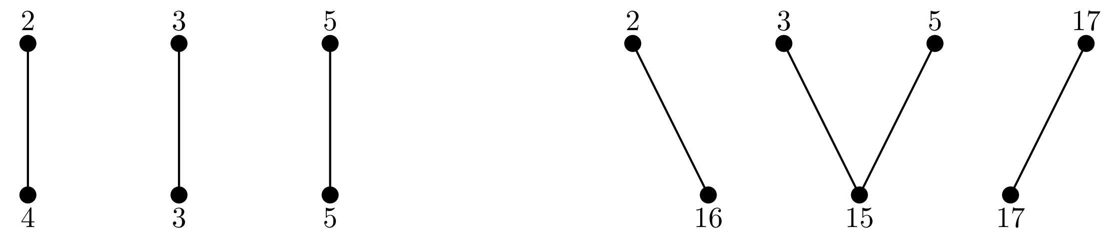
Theorem 2.3. *Let* G *be a non-solvable group. Then the following are true:*

(a) $n(B(G)) \leq 3$.
(b) $n(B(G)) = 3$ *if and only if* $G = S \times A$ *where* $S \cong \mathrm{PSL}(2, 2^n)$ *for some integer* n *and* A *is an abelian group.*
(c) *if* $n(B(G)) = 2$ *then* G *has normal subgroups* $N \subseteq K$ *so that* $K/N \cong \mathrm{PSL}(2, p^n)$ *where* p *is a prime and* n *is an integer so that* $p^n \geq 4$*. Furthermore,* G/K *is abelian and* N *is abelian or metabelian.*

Proof. By [7, theorem 6.4] we know that if G is a non-solvable group, then $n(\Delta(G)) \leq 3$. Also by lemma 1.1 we know that $n(B(G)) = n(\Delta(G))$ and therefore, we have that $n(B(G)) \leq 3$ for non-solvable groups.

If $n(B(G)) = 3$ then by lemma 1.1 we have $n(\Delta(G)) = 3$ and by [7, part 2 of theorem 6.4] case (b) occurs. If $n(B(G)) = 2$ then $n(\Delta(G)) = 3$ and by [7, part 3 of theorem 6.4] case (c) occurs. □

Remark 2.4. *Theorem 2.3 describes non-solvable groups whose bipartite divisor graph has three connected components. Therefore, the upper bound 3 is the best upper bound for the number of connected components of a non-solvable group. In figure 3 we see bipartite divisor graph of* $\mathrm{PSL}_2(4)$ *and* $\mathrm{PSL}_2(16)$ *with three connected components.*



(a) $B(\mathrm{PSL}_2(4))$ (b) $B(\mathrm{PSL}_2(16))$

FIGURE 3. Bipartite divisor graphs of $\mathrm{PSL}_2(4)$ and $\mathrm{PSL}_2(16)$ with three connected components.

3. Diameter of $B(G)$

In this section we focus on upper bound for the diameter of $B(G)$. We note that both solvable and non-solvable groups have the same upper bound for the diameter of $\Delta(G)$ and $\Gamma(G)$, therefore, we express the theorem for both of solvable and non-solvable groups.

Theorem 3.1. *Let G be a finite group. Then* $\mathrm{diam}(B(G)) \leq 7$.

Proof. By [7, corollary 4.2 and theorems 6.5 and 7.2] we know that the upper bound for the diameters of graphs $\Delta(G)$ and $\Gamma(G)$ is 3. Now by using part (ii) of lemma 1.1 we have two cases. If the case (a) occurs, then since $\mathrm{diam}(\Delta(G)) \leq 3$ and $\mathrm{diam}(\Gamma(G)) \leq 3$ we have $\mathrm{diam}(B(G)) \leq 6$. If the case (b) occurs, then since $\mathrm{diam}(\Delta(G)) \leq 3$ and $\mathrm{diam}(\Gamma(G)) \leq 3$ we would have $\mathrm{diam}(B(G)) \leq 7$. Therefore, in any cases we have $\mathrm{diam}(B(G)) \leq 7$. □

Remark 3.2. *The bound* 7 *is the best upper bound for the diameter of a bipartite divisor graph, because there are groups whose bipartite divisor graph has diameter 7. For example, Lewis in [9] found a solvable group whose diameter of its prime graph is 3. He showed that character degrees of this group are as follows:*

$$\mathrm{cd}(G) = \{1, a_1 = 3, a_2 = 5, a_3 = 3 \cdot 5, a_4 = 7 \cdot 31 \cdot 151, a_5 = 2^7 \cdot 7 \cdot 31 \cdot 151, a_6 = 2^{12} \cdot 31 \cdot 151,$$
$$a_7 = 2^{12} \cdot 3 \cdot 31 \cdot 151, a_8 = 2^{12} \cdot 7 \cdot 31 \cdot 151, a_9 = 2^{13} \cdot 7 \cdot 31 \cdot 151, a_{10} = 2^{15} \cdot 3 \cdot 31 \cdot 151\}.$$

Therefore, the bipartite divisor graph of this group is the graph in figure 4. We have $d_{B(G)}(a_2, 7) = 7$ *and the following is a path between these vertices:*

$$a_2 - 5 - a_3 - 3 - a_7 - 31 - a_8 - 7$$

Hence the upper bound 7 occurs for this graphs and therefore is the best upper bound.

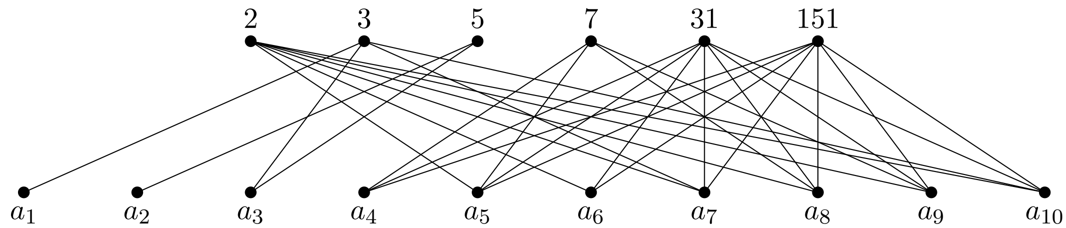


FIGURE 4. Bipartite divisor graph of a solvable group with diameter 7

4. Groups with complete bipartite divisor graph

Classification of groups whose bipartite divisor graph has a particular property could be a research problem in this field. In this section, we find properties of groups whose bipartite divisor graph is the complete bipartite graph. For an example of groups whose bipartite graph is complete let G be a p-group then it is obvious that $B(G)$ is the complete bipartite graph $K_{1,n}$ where the first index is used for the number of elements of $\rho(G)$ and the second index denotes the number of elements of $\mathrm{cd}^*(G)$. If G is a group such that $B(G) = K_{r,1}$ then we have $\mathrm{cd}(G) = \{1, m\}$ and groups with this property have been studied by many authors, for example, chapter 12 of [6], [1] and [2]. Now we return to the general case.

Lemma 4.1. *Let G be a group such that $d_{B(G)}(x, y) = 2$ for all distinct $x, y \in \mathrm{cd}^*(G)$. Then G is solvable.*

Proof. Suppose that $d_{B(G)}(x, y) = 2$ for all distinct $x, y \in \mathrm{cd}^*(G)$, then it is obvious that $d_{\Gamma(G)}(x, y) = 1$ for all distinct $x, y \in \mathrm{cd}^*(G)$ and therefore $\Gamma(G)$ is a complete graph. Hence by [7, theorem 7.3] G is a solvable group. □

Corollary 4.2. *Let G be a group whose bipartite divisor graph is a complete bipartite graph. Then G is solvable.*

Proof. Suppose that $B(G)$ is a complete bipartite graph. Then $d_{B(G)}(x, y) = 2$ for all distinct $x, y \in \mathrm{cd}^*(G)$ and by lemma 4.1, G is solvable. □

Lemma 4.3. *Let G be a non-abelian group such that $B(G)$ is a complete bipartite graph. If G is nilpotent, then $G = P \times A$ where P is a non-abelian Sylow p-subgroup of G and A is an abelian p-complement.*

Proof. Suppose that $B(G)$ is a complete bipartite graph. If G is nilpotent, then every Sylow p-subgroup P of G is isomorphic to a quotient of G and therefore,

$$\mathrm{cd}(P) \subseteq \mathrm{cd}(G). \tag{4.1}$$

If G has non-abelian Sylow subgroups for two distinct primes, then by equation (4.1), $\mathrm{cd}(G)$ contains two coprime elements, and this is a contradiction, since $B(G)$ is a complete graph. Hence G contains a non-abelian Sylow p-subgroup P for exactly one prime p and therefore, $G = P \times A$, where P is a non-abelian Sylow p-subgroup and A is an abelian p-complement. □

Theorem 4.4. *Let G be a group such that every prime divisor of $|G|$ divides some $a \in \mathrm{cd}(G)$, i.e. $\pi(|G|) = \rho(G)$. If $B(G)$ is a complete bipartite graph then G is a non-abelian p-group for some prime p.*

Proof. Suppose that $p \in \pi(|G|)$ then by hypothesis we have $p \in \rho(G)$ and therefore, p is one of the vertices of $B(G)$. Since $B(G)$ is a complete bipartite graph therefore, p is connected to every element of $\mathrm{cd}^*(G)$ which means that degree of every non-linear irreducible character of G is divisible by p. Therefore, by [6, corollary 12.2] G has a normal p-complement. Since p was an arbitrary element of $\pi(|G|)$ therefore, G has a normal p-complement for every prime p i.e. G is p-nilpotent for every prime p and therefore G is nilpotent. Hence by lemma 4.3, $G = P \times A$ which P is a non-abelian p-group and A is abelian. In particular, every element of $\mathrm{cd}(G)$ is a p-power. If A is non-trivial, then there exists a prime number $q \neq p$ such that $q||G|$ and by hypothesis q must divide some elements of $\mathrm{cd}(G)$ which is a contradiction. Thus A is trivial and therefore, $G = P$ is a non-abelian p-group. □

Theorem 4.5. *Let G be a group whose $B(G)$ is a complete bipartite graph. Then one of the following cases occurs:*

(a) *$G = A \rtimes H$, where A is an abelian normal Hall subgroup of G and H is abelian, i.e. G is metabelian.*
(b) *$G = A \rtimes H$, where A is an abelian normal Hall subgroup of G and H is a non-abelian p-group for some prime p. In particular, $\rho(G) = \{p\}$.*

Proof. Suppose that $B(G)$ is a complete bipartite graph, and recall that G is a solvable group by corollary 4.2. Let $\pi = \pi(|G|) \setminus \rho(G)$. For each $p \in \pi$, the prime p is, by definition, coprime to every $b \in \mathrm{cd}^*(G)$. Thus, by [6, corollary 12.34], G has a normal, abelian Sylow p-subgroup. Let A be the direct product of these normal, abelian Sylow subgroups, the product taken over $p \in \pi$. Hence, A is a normal, abelian, Hall π-subgroup. Let H be a π-complement, which makes H a Hall $\rho(G)$-subgroup. If H is abelian, we have the conclusion (a).
Therefore, assume that H is non-abelian. Observe that $H \cong G/A$, and it follows that

$$\mathrm{cd}(H) = \mathrm{cd}(G/A) \subseteq \mathrm{cd}(G).$$

Recalling that $\pi(|H|) = \rho(G)$, and that $B(G)$ is complete, for each $p \in \pi(|H|)$, p divides all degrees in $\mathrm{cd}^*(G)$, and therefore all degrees in $\mathrm{cd}^*(H)$. Accordingly, $B(H)$ is a complete bipartite graph. Theorem 4.4 applies, and H is a non-abelian p-group. Also, since H is a π-complement therefore $\rho(G) = \{p\}$ and the conclusion (b) follows. □

Corollary 4.6. *Let G be a group such that $B(G)$ is a complete bipartite graph. If $|\rho(G)| > 1$, then G' is abelian.*

Proof. In conclusion (b) of Theorem 4.5, $\rho(G) = \{p\}$. Since $|\rho(G)| > 1$, we are in conclusion (a), and G is metabelian, as needed. $\square$

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