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ON THE SEMI COVER-AVOIDING PROPERTY AND $\mathscr{F}$-SUPPLEMENTATION

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Abstract. In this paper, we investigate the influence of some subgroups of Sylow subgroups with semi cover-avoiding property and $\mathscr{F}$-supplementation on the structure of finite groups and generalize a series of known results.

1. Introduction

Throughout the paper, all groups are finite. We use conventional notions and notation, as in Huppert [17]. G always denotes a group, $|G|$ is the order of G, $O_p(G)$ is the maximal normal p-subgroup of G and $\Phi(G)$ is the Frattini subgroup of G.

Let L/K be a normal factor of a group G. A subgroup H of G is said to cover L/K if $HL = HK$, and H is said to avoid L/K if $H \cap L = H \cap K$. If H covers or avoids every chief factor of G, then H is said to have the cover-avoiding property in G. This conception was first studied by Gaschütz (see [5]) to study the solvable groups, later by Gillam (see [6]) and Ezquerro (see [3]), et al. More recently, in Fan et al. (see [4]) introduced the semi cover-avoiding property, which is the generalization not only of the cover-avoiding property but also of c-normality (see [22]). A subgroup H of a group G is said to have the semi cover-avoiding property in G, if there exists a chief series of G such that H either covers or avoids every G-chief factor of this series. The results in Guo and Shum (see [10]) and Wang (see [22]) were extended with the requirement that the certain subgroups of G have the semi cover-avoiding property (see [9] and [16]). More recently, many authors presented some conditions for a group to be p-nilpotent and supersolvable under the condition that some subgroups of Sylow subgroup have the semi cover-avoiding property (see [15], [19] and [27]).

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A subgroup H of a group G is said to be complemented in G if G has a subgroup K such that $G = HK$ and $H \cap K = 1$. A subgroup H of a group G is said to be supplemented in G if there exists a subgroup K of G such that $G = HK$. Obviously, a complemented subgroup is a special supplemented subgroup. Recently, by considering some other special supplemented subgroups, many authors obtained a series of new characterization theorems for soluble groups and supersolvable groups. For example, Wang introduced the concept of c-supplemented subgroup [21] (a subgroup H of a group G is said to be a c-supplemented in G if there exists a subgroup K of G such that $G = HK$ and $H \cap K \leq H_G$, where H_G is the maximal normal subgroup of G contained in H). In 2007, A. Y. Alsheik Ahmad, et al., introduced the concept of $\mathscr{U}_c$-normal subgroup [1] (A subgroup H of a group G is called $\mathscr{U}_c$-normal in G if there exists a subnormal subgroup T of G such that $G = HT$ and $(H \cap T)H_G/H_G$ is contained in the $\mathscr{U}$-hypercenter $Z_\infty^{\mathscr{U}}(G/H_G)$, where $\mathscr{U}$ is the class of the finite supersoluble groups). As a promotion of above a series of subgroups, W. Guo introduced the concept of $\mathscr{F}$-supplemented subgroup [8] (A subgroup H of a group G is $\mathscr{F}$-supplemented in G if there exists a subgroup T of G such that $G = HT$ and $(H \cap T)H_G/H_G$ is contained in the $\mathscr{F}$-hypercenter $Z_\infty^{\mathscr{F}}(G/H_G)$, where $\mathscr{F}$ is a formation of finite groups). In [8], by using some $\mathscr{F}$-supplemented subgroups, W. Guo has given some conditions under which a finite group belongs to some formations.

A subgroup that satisfies the cover-avoiding property does not necessary need to be $\mathscr{F}$-supplemented and vice-versa. In this paper, we will try an attempt to unify the two concepts and establish the structure of groups under the assumption that all maximal subgroups of a Sylow subgroup either have the semi cover-avoiding property or are $\mathscr{F}$-supplemented subgroups. Some new results are obtained and a series of previously known results are generalized, such as in [9], [11], [12], [13], [14], [18], [19], [21], [23], [24] and [25].

2. **Preliminaries**

In this section, we list some lemmas which will be useful for the proofs of our main results.

Lemma 2.1. [9, Lemmas 2.5 and 2.6] *Suppose that H has the semi cover-avoiding property in G.*

(1) *If $H \leq L \leq G$, then H has the semi cover-avoiding property in L.*

(2) *If $N \trianglelefteq G$ and $N \leq H \leq G$, then H/N has the semi cover-avoiding property in G/N.*

(3) *If H is a π-subgroup and N is a normal π'-subgroup of G, then HN/N has the semi cover-avoiding property in G/N.*

Lemma 2.2. [8, Lemma 2.2] *Let G be a group and $H \leq K \leq G$. Then*

(1) *If H is $\mathscr{F}$-supplemented in G and $\mathscr{F}$ is s-closed, then H is $\mathscr{F}$-supplemented in K.*

(2) *Suppose that H is normal in G. Then K/H is $\mathscr{F}$-supplemented in G/H if and only if K is $\mathscr{F}$-supplemented in G.*

(3) *Suppose that H is normal in G. Then, for every $\mathscr{F}$-supplemented subgroup E in G satisfying $(|H|, |E|) = 1$, HE/H is $\mathscr{F}$-supplemented in G/H.*

(4) *H is $\mathscr{F}$-supplemented in G if and only if there exists a subgroup T of G such that $G = HT$, $H_G \leq T$ and $(H/H_G) \cap (T/H_G) \leq Z_\infty^{\mathscr{F}}(G/H_G)$.*

Lemma 2.3. [9, Lemma 3.1] *Let p be a prime dividing the order of the group G with $(|G|, p-1) = 1$ and let P be a p-Sylow subgroup of G. If there is a maximal subgroup P_1 of P such that P_1 has the semi cover-avoiding property in G, then G is p-solvable.*

Lemma 2.4. [24, Lemma 2.8] *Let M be a maximal subgroup of G and P a normal p-subgroup of G such that $G = PM$, where p is a prime. Then $P \cap M$ is a normal subgroup of G.*

Lemma 2.5. [26, Lemma 2.7] *Let G be a group and p a prime dividing $|G|$ with $(|G|, p-1) = 1$.*

(1) *If N is normal in G of order p, then $N \leq Z(G)$.*

(2) *If G has cyclic Sylow p-subgroup, then G is p-nilpotent.*

(3) *If $M \leq G$ and $|G : M| = p$, then $M \trianglelefteq G$.*

Lemma 2.6. [7, Main Theorem] *Suppose that G has a Hall π-subgroup where π is a set of odd primes. Then all Hall π-subgroups of G are conjugate.*

Lemma 2.7. [18, Lemma 2.6] *Let $H \neq 1$ be a solvable normal subgroup of a group G. If every minimal normal subgroup of G which is contained in H is not contained in $\Phi(G)$, then the Fitting subgroup $F(H)$ of H is the direct product of minimal normal subgroups of G which are contained in H.*

Lemma 2.8. [20, Lemma 2.16] *Let $\mathscr{F}$ be a saturated formation containing $\mathscr{U}$. Suppose that G is a group with a normal subgroup N such that $G/N \in \mathscr{F}$. If N is cyclic, then $G \in \mathscr{F}$.*

3. **Main results**

Theorem 3.1. *Let p be a prime dividing the order of a group G with $(|G|, p-1) = 1$ and H a normal subgroup of G such that G/H is p-nilpotent. Suppose that there exists a Sylow p-subgroup P of H such that every maximal subgroup of P either has the semi cover-avoiding property or is $\mathscr{N}_p$-supplemented in G, where $\mathscr{N}_p$ is the class of all p-nilpotent groups. Then G is p-nilpotent.*

Proof. We distinguish two cases:

Case I. $H = G$.

Suppose that the theorem is false and let G be a counterexample of minimal order. We will derive a contradiction in several steps.

(1) $O_{p'}(G) = 1$.

Assume that $O_{p'}(G) \neq 1$. Then $PO_{p'}(G)/O_{p'}(G)$ is a Sylow p-subgroup of $G/O_{p'}(G)$. Suppose that $M/O_{p'}(G)$ is a maximal subgroup of $PO_{p'}(G)/O_{p'}(G)$. Then there exists a maximal subgroup P_1 of P such that $M = P_1O_{p'}(G)$. By the hypothesis of the theorem, P_1 either has the semi cover-avoiding property or is $\mathscr{N}_p$-supplemented in G. Then $M/O_{p'}(G) = P_1O_{p'}(G)/O_{p'}(G)$ either has the semi cover-avoiding property or is $\mathscr{N}_p$-supplemented in $G/O_{p'}(G)$ by Lemmas 2.1 and 2.2. It is clear that $(|G/O_{p'}(G)|, p-1) = 1$. The minimal choice of G implies that $G/O_{p'}(G)$ is p-nilpotent, and so G is p-nilpotent, a contradiction. Therefore, we have $O_{p'}(G) = 1$.

(2) $O_p(G) \neq 1$.

If not, suppose that $O_p(G) = 1$. If there is a maximal subgroup of P which has the semi cover-avoiding property in G, then G is p-solvable by Lemma 2.3. Since $O_{p'}(G) = 1$ by Step (1), we have $O_p(G) \neq 1$, a contradiction. Thus we may assume that all maximal subgroups of P are $\mathscr{N}_p$-supplemented in G. Let L be an arbitrary maximal subgroup of P. Then G has a subgroup T of G such that $G = LT$ and $(L \cap T)L_G/L_G$ is contained in the $\mathscr{N}_p$-hypercenter $Z_\infty^{\mathscr{N}_p}(G/L_G)$. Since $O_p(G) = 1$, obviously $L_G = 1$. It follows that $L \cap T \leq Z_\infty^{\mathscr{N}_p}(G)$. If $Z_\infty^{\mathscr{N}_p}(G) \neq 1$, we can take a minimal normal N of G which contained in $Z_\infty^{\mathscr{N}_p}(G)$. By Step (1), N is not a p'-group. Consequently, N is a p-group and so $O_p(G) \neq 1$, a contradiction. Therefore we have $Z_\infty^{\mathscr{N}_p}(G) = 1$ and so every maximal subgroup of P is complemented in G. If $p \neq 2$, then G is odd from the assumption that $(|G|, p-1) = 1$. By the Feit-Thompson Theorem, G is solvable. It follows that $O_p(G) \neq 1$ by Step (1), a contradiction. If $p = 2$, we get also G is solvable by [2, Lemma 3], the same contradiction.

(3) If $N \leq O_p(G)$, then G/N is p-nilpotent. Consequently, G is solvable.

Suppose that M/N is a maximal subgroup of P/N. Then M is a maximal subgroup of P. By the hypothesis of the theorem, M either has the semi cover-avoiding property or is $\mathscr{N}_p$-supplemented in G. Then M/N either has the semi cover-avoiding property or is $\mathscr{N}_p$-supplemented in G/N by Lemmas 2.1 and 2.2. Therefore G/N satisfies the hypothesis of the theorem. The minimal choice of G implies that G/N is p-nilpotent. If p is odd, then $(|G|, p-1) = 1$ implies that G is odd order, hence G is solvable. If $p = 2$, then G/N is solvable, and so G is solvable.

(4) $O_p(G)$ is the unique minimal normal subgroup of G.

Let N be a minimal normal subgroup of G. Since G is solvable by Step (3), N is an elementary abelian subgroup. Note that $O_{p'}(G) = 1$, then we have N is a p-subgroup and so $N \leq O_p(G)$. Step (3) implies that $G/O_p(G)$ is p-nilpotent. Since the class of all p-nilpotent groups is a saturated formation, N is a unique minimal normal subgroup of G and $N \nleq \Phi(G)$. Choose M to be a maximal subgroup of G such that $G = NM$. Obviously, $G = O_p(G)M$ and so $O_p(G) \cap M$ is normal in G by Lemma 2.4. The uniqueness of N yields $N = O_p(G)$.

(5) The final contradiction.

By the proof in Step (4), G has a maximal subgroup M such that $G = MO_p(G)$ and $G/O_p(G) \cong M$ is p-nilpotent. Clearly, $P = O_p(G)(P \cap M)$. Furthermore, $P \cap M < P$. Thus, there exists a maximal subgroup V of P such that $P \cap M \leq V$. Hence, $P = O_p(G)V$. By the hypothesis, V either has the semi cover-avoiding property or is $\mathscr{N}_p$-supplemented in G.

First, we assume that V has the semi cover-avoiding property in G. Since $O_p(G)$ is the unique minimal normal subgroup of G, V covers or avoids $O_p(G)/1$. If V covers $O_p(G)/1$, then $VO_p(G) = V$, i.e., $O_p(G) \leq V$. It follows that $P = O_p(G)V = V$, a contradiction. If V avoids $O_p(G)/1$, then $V \cap O_p(G) = 1$. Since $V \cap O_p(G)$ is a maximal subgroup of $O_p(G)$, we have that $O_p(G)$ is of order p and so $O_p(G)$ lies in $Z(G)$ by Lemma 2.5. By Step (3), we have $G/O_p(G)$ is p-nilpotent. Then $G/Z(G)$ is p-nilpotent, and so G is p-nilpotent, a contradiction.

Now, we may assume that V is $\mathscr{N}_p$-supplemented in G. Then there is a subgroup T of G such that $G = VT$ and $(V \cap T)V_G/V_G$ is contained in the $\mathscr{N}_p$-hypercenter $Z_\infty^{\mathscr{N}_p}(G/V_G)$. If $V_G \neq 1$, then $O_p(G) = V_G \leq V$. It follows that $P = O_p(G)V = V$, a contradiction. Thus we may assume $V_G = 1$. Consequently, we have $V \cap T \leq Z_\infty^{\mathscr{N}_p}(G)$. If $Z_\infty^{\mathscr{N}_p}(G) \neq 1$, then $O_p(G) \leq Z_\infty^{\mathscr{N}_p}(G)$ and $|O_p(G)| = p$. It follows that G is p-nilpotent as above, a contradiction. Now assume that $Z_\infty^{\mathscr{N}_p}(G) = 1$. Then $V \cap T = 1$, and so $|T|_p = p$. By Lemma 2.5, T is p-nilpotent. Let $T_{p'}$ be the normal p-complement of T. Since M is p-nilpotent, we may suppose M has a normal Hall p'-subgroup $M_{p'}$ and $M \leq N_G(M_{p'}) \leq G$. The maximality of M implies that $M = N_G(M_{p'})$ or $N_G(M_{p'}) = G$. If the latter holds, then $M_{p'} \trianglelefteq G$ and $M_{p'}$ is actually the normal p-complement of G, which is contrary to the choice of G. Hence, we may assume $M = N_G(M_{p'})$. By applying Lemma 2.6 and the Feit-Thompson Theorem, there exists $g \in G$ such that $T_{p'}^g = M_{p'}$. Hence, $T^g \leq N_G(T_{p'}^g) = N_G(M_{p'}) = M$. However, $T_{p'}$ is normalized by T, so g can be considered as an element of V. Thus, $G = VT^g = VM$ and $P = V(P \cap M) = V$, a contradiction.

Case II. $H < G$.

By Lemmas 2.1 and 2.2, every maximal subgroup of P has the semi cover-avoiding property or is $\mathscr{N}_p$-supplemented in H. By Case I, H is p-nilpotent. Now, let $H_{p'}$ be the normal p-complement of H. Then $H_{p'} \trianglelefteq G$. Assume $H_{p'} \neq 1$ and consider $G/H_{p'}$. Applying Lemmas 2.1 and 2.2, it is easy to see that $G/H_{p'}$ satisfies the hypotheses for the normal subgroup $H/H_{p'}$. Therefore, by induction $G/H_{p'}$ is p-nilpotent and so G is p-nilpotent. Hence, we may assume $H_{p'} = 1$ and so $H = P$ is a p-group. Since G/H is p-nilpotent, we can let K/H be the normal p-complement of G/H. By The Schur-Zassenhaus Theorem, there exists a Hall p'-subgroup $K_{p'}$ of K such that $K = HK_{p'}$. A new application of Case I yields K is p-nilpotent and so $K = H \times K_{p'}$. Hence, $K_{p'}$ is a normal p-complement of G and G is p-nilpotent. □

Corollary 3.2. *Let P be a Sylow p-subgroup of a group G, where p is the smallest prime divisor of $|G|$. If every maximal subgroup of P either has the semi cover-avoiding property or is $\mathscr{N}_p$-supplemented in G, then G is p-nilpotent.*

Proof. It is clear that $(|G|, p-1) = 1$ if p is the smallest prime dividing the order of G and so the corollary follows immediately from Theorem 3.1. □

Corollary 3.3. *Suppose that every maximal subgroup of any Sylow subgroup of a group G either has the semi cover-avoiding property or is $\mathscr{U}$-supplemented in G, where $\mathscr{U}$ is the class of all supersolvable groups. Then G is a Sylow tower group of supersolvable type.*

Proof. Let p be the smallest prime dividing $|G|$ and P a Sylow p-subgroup of G. By Corollary 3.2, G is p-nilpotent. Let T be the normal p-complement of G. By Lemmas 2.1 and 2.2, every maximal subgroup of any Sylow subgroup of T has the semi cover-avoiding property or is $\mathscr{U}$-supplemented in T. Thus T satisfies the hypothesis of the Corollary. It follows by induction that T, and hence G is a Sylow tower group of supersolvable type. □

Corollary 3.4. [19, Theorem 3.3] *Let G be a group, p a prime dividing the order of G, and P a Sylow p-subgroup of G. If $(|G|, p-1) = 1$ and every maximal subgroup of P has the semi cover-avoiding property in G, then G is p-nilpotent.*

Corollary 3.5. [9, Theorem 3.2] *Let P be a Sylow p-subgroup of a group G, where p is the smallest prime divisor of $|G|$. If P is cyclic or every maximal subgroup of P has the semi cover-avoiding property in G, then G is p-nilpotent.*

Proof. If P is cyclic, by Lemma 2.5, we have G is p-nilpotent. Thus we may assume that every maximal subgroup of P has the semi cover-avoiding property in G. By Corollary 3.2, G is p-nilpotent. □

Corollary 3.6. [11, Theorem 3.4] *Let G be a group and P a Sylow p-subgroup of G, where p is the smallest prime dividing $|G|$. If all maximal subgroups of P are c-normal in G, then G is p-nilpotent.*

Corollary 3.7. [12, Theorem 3.2] *Let G be a group and P a Sylow p-subgroup of G, where p is the smallest prime dividing $|G|$. If all maximal subgroups of P are c-supplemented in G, then G is p-nilpotent.*

Corollary 3.8. [13, Theorem 3.1] *Let P be a Sylow p-subgroup of a group G, where p is a prime divisor of $|G|$ with $(|G|, p-1) = 1$. If every maximal subgroup of P is c-supplemented in G, then G is p-nilpotent.*

Corollary 3.9. [21, Theorem 3.1] *Let p be a prime dividing the order of a group G with $(|G|, p-1) = 1$. Suppose that every maximal subgroup of P is c-supplemented in G and $G \in C_{p'}$, then $G/O_p(G)$ is p-nilpotent and $G \in D_{p'}$.*

Theorem 3.10. *Let $\mathscr{F}$ be a saturated formation containing $\mathscr{U}$. A group $G \in \mathscr{F}$ if and only if there is a normal subgroup H of G such that $G/H \in \mathscr{F}$ and every maximal subgroup of any noncyclic Sylow subgroup of H either has the semi cover-avoiding property or is $\mathscr{U}$-supplemented in G.*

Proof. The necessity is obvious. We only need to prove the sufficiency. Suppose that the assertion is false and let G be a counterexample of minimal order.

(1) G has a minimal normal subgroup $N \leq H$ and N is an elementary abelian p-group, where p is the largest prime in $\pi(H)$.

By the hypothesis of the theorem, every maximal subgroup of any noncyclic Sylow subgroup of H either has the semi cover-avoiding property or is $\mathscr{U}$-supplemented in G. Consequently, by Lemmas 2.1 and 2.2 every one also either has the semi cover-avoiding property or is $\mathscr{U}$-supplemented in H. Applying Corollary 3.3, H is a Sylow tower group of supersolvable type. Let p be the largest prime divisor of $|H|$ and P a Sylow p-subgroup of H. Then P is normal in H. Obviously, P is normal in G. Therefore, G has a minimal normal subgroup $N \leq H$ and N is an elementary abelian p-group.

(2) $G/N \in \mathscr{F}$ and $N = P$ is the Sylow p-subgroup of H.

First, we want to prove that G/N satisfies the hypothesis of the theorem. In fact, $(G/N)/(H/N) \cong G/H \in \mathscr{F}$. Let P_1/N be a maximal subgroup of the Sylow p-subgroup P/N of H/N. Then P_1 is a maximal subgroup of the Sylow p-subgroup P of H. If P/N is noncyclic, then P is also noncyclic. By the hypothesis of the theorem, P_1 either has the semi cover-avoiding property or is $\mathscr{U}$-supplemented in G. By Lemmas 2.1 and 2.2, P_1/N either has the semi cover-avoiding property or is $\mathscr{U}$-supplemented in G/N. Let M_1/N be a maximal subgroup of the noncyclic Sylow q-subgroup QN/N of H/N, where $q \neq p$ and Q is a noncyclic Sylow q-subgroup of H. It is clear that $M_1 = Q_1N$, where Q_1 is a maximal subgroup of Q. By the hypothesis of the theorem, Q_1 either has the semi cover-avoiding property or is $\mathscr{U}$-supplemented in G. Hence M_1/N either has the semi cover-avoiding property or is $\mathscr{U}$-supplemented in G/N by Lemmas 2.1 and 2.2. We now have proved that G/N satisfies the hypothesis of the theorem. By the minimal choice of G, we have $G/N \in \mathscr{F}$. Since $\mathscr{F}$ is a saturated formation, N is the unique minimal normal subgroup of G contained in P and $N \nleq \Phi(G)$. By Lemma 2.7, it follows that $P = F(P) = N$.

(3) The final contradiction.

Let M be a maximal subgroup of N. By the hypothesis, M either has the semi cover-avoiding property or is $\mathscr{U}$-supplemented in G. First we assume that M is $\mathscr{U}$-supplemented in G. Then G has a subgroup T of G such that $G = MT$ and $T \cap M \leq Z_\infty^{\mathscr{U}}(G)$. Thus $G = NT$ and $N = N \cap MT = M(N \cap T)$. This implies that $N \cap T \neq 1$. Since $N \cap T$ is normal in G and N is a minimal normal subgroup of G, $N \cap T = N$. It follows that $T = G$, and so $M \leq N \cap Z_\infty^{\mathscr{U}}(G)$. By the minimality of N, $Z_\infty^{\mathscr{U}}(G) \cap N = 1$ or $N \leq Z_\infty^{\mathscr{U}}(G)$. If the latter holds, then $|N| = p$. By Step (2), $G/N \in \mathscr{F}$. Applying Lemma 2.8, $G \in \mathscr{F}$, a contradiction. Therefore $Z_\infty^{\mathscr{U}}(G) \cap N = 1$. It follows that $M = 1$ and $|N| = p$, the same contradiction as above.

Now we assume that M has the semi cover-avoiding property in G. Then there exists a chief series of G

$$1 = G_0 < G_1 < \cdots G_{n-1} < G_n = G$$

such that M covers or avoids every factor G_j/G_{j-1}. Since N is minimal normal in G, there exists j such that $G_j \cap N = N$ and $G_{j-1} \cap N = 1$. If M covers G_j/G_{j-1}, then $MG_j = MG_{j-1}$ and so $MG_j \cap N = MG_{j-1} \cap N$. Hence $M(G_j \cap N) = M(G_{j-1} \cap N)$, i.e., $MN = M$, a contradiction. If M avoids G_j/G_{j-1}, then $M \cap G_j = M \cap G_{j-1}$ and so $M \cap G_j \cap N = M \cap G_{j-1} \cap N$, i.e., $M = 1$. It follows that the same contradiction as above. □

Corollary 3.11. [19, Theorem 3.6] *Let $\mathscr{F}$ be a saturated formation containing $\mathscr{U}$. If there is a normal Hall subgroup H of G such that $G/H \in \mathscr{F}$ and every maximal subgroup of any Sylow subgroup of H has the semi cover-avoiding property in G, then $G \in \mathscr{F}$.*

Corollary 3.12. [18, Theorem 3.3] *Let H be a normal subgroup of a group G such that G/H is supersolvable. If every maximal subgroup of any Sylow subgroup of H is c-normal in G, then G is supersolvable.*

Corollary 3.13. [12, Theorem 4.2] *Let $\mathscr{F}$ be a saturated formation containing $\mathscr{U}$. If there is a normal subgroup H of G such that $G/H \in \mathscr{F}$ and every maximal subgroup of any Sylow subgroup of H is c-supplemented in G, then $G \in \mathscr{F}$.*

Corollary 3.14. [23, Theorem 4.1] *Let $\mathscr{F}$ be a saturated formation containing $\mathscr{U}$. If there is a normal subgroup H of G such that $G/H \in \mathscr{F}$ and every maximal subgroup of any noncyclic Sylow subgroup of H is c-supplemented in G, then $G \in \mathscr{F}$.*

Theorem 3.15. *Let $\mathscr{F}$ be a saturated formation containing $\mathscr{U}$. Suppose that G is a group with a solvable normal subgroup N such that $G/N \in \mathscr{F}$. If every maximal subgroup of each non-cyclic Sylow subgroup of $F(N)$ either has the semi cover-avoiding property or is $\mathscr{U}$-supplemented in G, then $G \in \mathscr{F}$.*

Proof. Suppose that the theorem is false and let G be a counterexample of minimal order. We distinguish two cases.

(1) $\Phi(G) \cap N \neq 1$.

Since $\Phi(G) \cap N \neq 1$, then there exists a prime p dividing the order of $\Phi(G) \cap N$. Let P_0 be the Sylow p-subgroup of $\Phi(G) \cap N$. Then $P_o \trianglelefteq G$. Since $(G/P_0)/(N/P_0) \cong G/N$, it follows that $(G/P_0)/(N/P_0) \in \mathscr{F}$. By [1 , p.270 Satz 3.5], $F(N/P_0) = F(N)/P_0$. Let P_1/P_0 be a maximal subgroup of the Sylow p-subgroup P/P_0 of $F(N)/P_0$. Then P_1 is a maximal subgroup of the Sylow p-subgroup P of $F(N)$. If P/P_0 is non-cyclic, then P is non-cyclic. By the hypothesis, P_1 either has the semi cover-avoiding property or is $\mathscr{U}$-supplemented in G. Hence P_1/P_0 either has the semi cover-avoiding property or is $\mathscr{U}$-supplemented in G/P_0 by Lemmas 2.1 and 2.2. Set Q_*/P_0 be a maximal subgroup of the non-cyclic Sylow q-subgroup of $F(N)/P_0$, where $p \neq q$. It is clear that $Q_* = Q_1^*P_0$, where Q_1^* is a maximal subgroup of the non-cyclic Sylow q-subgroup of $F(N)$. Then Q_1^* either has the semi cover-avoiding property or is $\mathscr{U}$-supplemented in G. Hence $Q_1^*P_0/P_o$ either has the semi cover-avoiding property or is $\mathscr{U}$-supplemented in G/P_o by Lemmas 2.1 and 2.2. Now we have proved that G/P_0 satisfies the hypotheses of the theorem. Therefore $G/P_0 \in \mathscr{F}$ by minimal choice of G. Since $P_o \leq \Phi(G)$ and $\mathscr{F}$ is a saturated formation, we have that $G \in \mathscr{F}$, a contradiction.

(2) $\Phi(G) \cap N = 1$.

If $N = 1$, nothing need to be proved. So assume $N \neq 1$. Then $F(N) \neq 1$ by the solvability of N. By Lemma 2.7, $F(N)$ is the direct product of some minimal normal subgroups of G. Let P be the Sylow p-subgroup of $F(N)$. We can denote $P = R_1 \times R_2 \times \cdots \times R_m$, where every R_i is a minimal normal subgroup of G. We will show that $|R_i| = p$ $(i = 1, 2, \cdots, m)$. If not, then there exists an index i such that $|R_i| > p$. Without loss of generality, suppose that $i = 1$. Since $R_1 \nleq \Phi(G)$, there exist a maximal subgroup M of G such that $G = R_1M$ and $R_1 \cap M = 1$. Then $G_p = R_1M_p$. Pick a maximal subgroup G_p^* of G_p containing M_p. Then $|R_1 : G_p^* \cap R_1| = |R_1G_p^*/G_p^*| = |G_p : G_p^*| = p$. Hence $R_1^* = G_p^* \cap R_1$ is a maximal subgroup of R_1. This implies that $P^* = R_1^*R_2 \cdots R_m$ is a maximal subgroup of P. Obviously, P is not cyclic. By the hypothesis, P^* either has the semi cover-avoiding property or is $\mathscr{U}$-supplemented in G. Let $K = R_2 \times \cdots \times R_m$.

First, we assume that P^* has the semi cover-avoiding property in G. By Lemma 2.1, P^*/K has the semi cover-avoiding property in G/K. Suppose that P^*/K cover-avoids a chief series $1 = \overline{K} \lhd G_1/K = \overline{G_1} \lhd \cdots \lhd G/K = \overline{G_n}$ of G/K. Let i be the smallest number in $\{1, 2, \cdots, n-1\}$ such that $\overline{G_{i+1}}/\overline{G_i}$ was covered by P^*/K in above chief series. Then we have $G_i \cap P^* = K$ and $G_{i+1} \leq G_iP^* = G_iR_1^*$. Hence

$G_{i+1} = G_i(R_1^* \cap G_{i+1})$ and $R_1^* \cap G_{i+1} > 1$. Since R_1 is a minimal normal subgroup of G, we have $R_1 \le G_{i+1}$ and $R_1 \cap G_i = 1$. Hence $|R_1| = |G_{i+1}/G_i| = |R_1^* \cap G_{i+1}| < |R_1|$, a contraction. Therefore, P^*/K does not cover any chief factor in above chief series. It follows that $P^*/K = 1$ and $|R_1| = p$, a contraction.

We now assume that P^* is $\mathscr{U}$-supplemented in G. By Lemma 2.2(4), there exists a subgroup T of G such that $G = P^*T$ and $P^*/P_G^* \cap T/P_G^* \le Z_\infty^{\mathscr{F}}(G/P_G^*)$. Obviously, $P_G^* = K$ and so $P^*/K \cap T/K \le Z_\infty^{\mathscr{F}}(G/K)$. If $P^*/K \cap T/K = 1$, then $P^* \cap T = K$ and so $G = P^*T = R_1^*KT = R_1^*T = R_1T$. It is easy to see that $R_1 \cap T \lhd G$. Since R_1 is a minimal normal in G, we have $R_1 \cap T = 1$ or $R_1 \cap T = R_1$. If the latter holds, then $T = G$ and $K = P^*$. In this case, $R_1^* = 1$ and R_1 is of order p, a contraction. Therefore we have $R_1 \cap T = 1$. It follows that $R_1^* \cap T = 1$. Then $|G| = |R_1^*||T| = |R_1||T|$ and $|R_1^*| = |R_1|$. This contraction shows that $P^*/K \cap T/K \neq 1$. Let $Z_\infty^{\mathscr{F}}(G/K) = V/K$. Then $P/K \cap V/K \lhd G/K$. Since $P \cap V \ge P^* \cap T \cap V = P^* \cap T > K$, we have $P/K \cap V/K \neq 1$. By the G-isomorphism $P/K \cong R_1$, we see that P/K is a chief factor of G contained in V/K. Therefore, P/K is of order p and so $|R_1| = p$, a contraction.

From above discussion, we can let $F(N) = L_1 \times L_2 \times \cdots \times L_n$, where every L_i is a normal subgroup of prime order. Obviously $G/C_G(L_i)$ is abelian. Since $C_G(F(N)) = \bigcap_{i=1}^n C_G(L_i)$, $G/C_G(F(N))$ is abelian. Hence $G/C_G(F(N)) \in \mathscr{U} \subseteq \mathscr{F}$. By the assumption, $G/N \in \mathscr{F}$, it implies $G/N \cap C_G(F(N)) = G/C_N(F(N)) \in \mathscr{F}$ by the properties of formations. Since N is solvable, $C_N(F(N)) \le F(N)$. Again, $F(N)$ is abelian, so $F(N) \le C_N(F(N))$. Thus $F(N) = C_N(F(N))$ and $G/F(N) \in \mathscr{F}$. By Theorem 3.10, $G \in \mathscr{F}$, a contradiction. □

Corollary 3.16. [25, Theorem 1] *Let $\mathscr{F}$ be a saturated formation containing $\mathscr{U}$, the class of all supersolvable groups. Suppose that G is a group with a solvable normal subgroup H such that $G/H \in \mathscr{F}$. If all maximal subgroups of all Sylow subgroups of $F(H)$ are c-normal in G, then $G \in \mathscr{F}$.*

Corollary 3.17. [24, Theorem 4.5] *Let $\mathscr{F}$ be a saturated formation containing $\mathscr{U}$, the class of all supersolvable groups. Suppose that G is a group with a solvable normal subgroup H such that $G/H \in \mathscr{F}$. If all maximal subgroups of all Sylow subgroups of $F(H)$ are c-supplemented in G, then $G \in \mathscr{F}$.*

Corollary 3.18. [14, Theorem 1.6] *Let $\mathscr{F}$ be a saturated formation containing $\mathscr{U}$, the class of all supersolvable groups. Suppose that G is a group with a solvable normal subgroup H such that $G/H \in \mathscr{F}$. If all maximal subgroups of all Sylow subgroups of $F(H)$ are complemented in G, then $G \in \mathscr{F}$.*

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