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ON CENTRAL ENDOMORPHISMS OF A GROUP

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Abstract. Let Γ be a normal subgroup of the full automorphism group $Aut(G)$ of a group G, and assume that $Inn(G) \leq \Gamma$. An endomorphism σ of G is said to be Γ-*central* if σ induces the the identity on the factor group $G/C_G(\Gamma)$. Clearly, if $\Gamma = Inn(G)$, then a Γ-central endomorphism is a *central* endomorphism. In this article the conditions under which a Γ-central endomorphism of a group is an automorphism are investigated.

1. Introduction

Let G be a group. An endomorphism σ of G is said to be *central* if σ acts trivially on $G/Z(G)$. Clearly, a central endomorphism of G is also *normal*, i.e., it commutes with every inner automorphism of G. Conversely, it is easy to show that every normal endomorphism which is surjective is central. Therefore, an automorphism is central if and only if it is normal. It follows that the set $Aut_c(G)$ of all central automorphisms of G is a normal subgroup of the full automorphism group $Aut(G)$ of G.

Let $Inn(G)$ be the group of all inner automorphisms of G. Then the centre $Z(G)$ of G coincides with the centralizer $C_G(Inn(G))$ (i.e., the set of all points of G fixed under the action of $Inn(G)$). This remark suggests to study central endomorphisms in a more general context. For this purpose, let G be a group and consider a normal subgroup Γ of $Aut(G)$ containing $Inn(G)$. We define the Γ-*centre* of G as the (characteristic) subgroup $C_G(\Gamma)$ of G consisting of all elements g in G such that $g^\gamma = g$ for every $\gamma \in \Gamma$. Clearly,

$$C_G(Aut(G)) \leq C_G(\Gamma) \leq Z(G).$$

The subgroup $C_G(Aut(G))$ is called the *absolute centre* (or *autocentre*) of G and firstly it was studied by Hegarty [7] in 1994 (see also [2], [5], [8]).

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An endomorphism σ of G is said to be Γ-*central* if it induces the identity on the factor group $G/C_G(\Gamma)$, i.e., if $g^{-1}g^\sigma \in C_G(\Gamma)$ for all $g \in G$. Clearly, the $Inn(G)$-central endomorphisms of G are exactly the central endomorphisms of G, while the $Aut(G)$-central endomorphisms of G are the so-called *autocentral* endomorphisms of G (see [7]).

The aim of this short article is to investigate the conditions under which a Γ-central endomorphism of a group is an automorphism. Recall that Adney and Yen [1] proved that a necessary and sufficient condition for a finite group G to having every central endomorphism an automorphism is that G is *purely non-abelian*, i.e., G has no abelian non-trivial direct factors. This result was extended by Franciosi, de Giovanni and Newell [4] to a wider class containing all periodic groups with no large abelian sections. Most of our notation is standard and can be found in [9].

2. **Results and proofs**

Let G be a group. In the following we denote by Γ a normal subgroup of $Aut(G)$ such that $Inn(G) \leq \Gamma$. Moreover, let $E\Gamma_c(G)$ be the monoid consisting of all Γ-central endomorphisms of G. Clearly, if σ is an element of $E\Gamma_c(G)$, then $f_\sigma : x \in G \mapsto x^{-1}x^\sigma \in C_G(\Gamma)$ defines a homomorphism from G to $C_G(\Gamma)$. Therefore we may consider the map

$$f : \sigma \in E\Gamma_c(G) \mapsto f_\sigma \in Hom(G, C_G(\Gamma)).$$

Lemma 2.1. *The map f is bijective and $E\Gamma_c(G) = \{\alpha : x \in G \mapsto xx^h \in G \mid h \in Hom(G, C_G(\Gamma))\}$.*

Proof. Let $\alpha, \beta \in E\Gamma_c(G)$ such that $f_\alpha = f_\beta$. Then $x^{-1}x^\alpha = x^{-1}x^\beta$, for every $x \in G$. Therefore $\alpha = \beta$. Now let h be an element of $Hom(G, C_G(\Gamma))$. Clearly, the rule $x^{\alpha_h} = xx^h$, for each $x \in G$, defines a Γ-central endomorphism α_h of G such that $f(\alpha_h) = h$. □

Denote by $A\Gamma_c(G)$ the group of all Γ-central automorphisms of G. It is easy to show that $A\Gamma_c(G)$ is a normal subgroup of $Aut(G)$. In particular, if $\Gamma = Aut(G)$, we obtain the group $Var(G)$ of *autocentral* automorphisms of G introduced by Hegarty [7]. As an immediate consequence of Lemma 2.1 we have that if h is an element of $Hom(G, C_G(\Gamma))$ such that $x^h = x^{-1}$ for some $x \in G \setminus \{1\}$, then G has a Γ-central endomorphism which is not an automorphism.

Now suppose that the group G has a direct decomposition $G = H \times K$, where H is a non-trivial Γ-central direct factor. Then the rule $x^h = x^{-1}$ if $x \in H$, and $x^h = 1$ if $x \in K$, defines a homomorphism $h : G \longrightarrow C_G(\Gamma)$. Therefore $A\Gamma_c(G) \neq E\Gamma_c(G)$.

Our next result contains the quoted theorem of Adney and Yen, but is proved independently of it.

Theorem 2.2. *Let G be a group satisfying the maximal and minimal conditions on normal subgroups, and let σ be a Γ-central endomorphism of G. If G has no non-trivial Γ-central direct factors, then σ is an automorphism.*

Proof. By Lemma 2.1 there exists a homomorphism h from G to $C_G(\Gamma)$ such that $x^\sigma = xx^h$ for all x in G. Therefore $Ker\sigma^n$ is contained in $C_G(\Gamma)$ for every positive integer n. Moreover, since σ is a normal endomorphism of G, Fitting's lemma (see, for instance, [9, 3.3.4]) yields that $G = (Im\sigma^m) \times (Ker\sigma^m)$

for some positive integer m. It follows by hypothesis that $Ker\sigma^m = \{1\}$, and hence $G = Im\sigma$. Thus σ is an automorphism of G. $\square$

Note that there exist infinite groups with no non-trivial Γ-central direct factors containing Γ-central endomorphisms which are not automorphisms.

Remark 2.3. *There exists a non-periodic purely non-abelian group G such that $|Aut_c(G)| = 2$ and $Hom(G, Z(G))$ is infinite cyclic.*

Proof. Consider the central extensions of an infinite cyclic group C by S_3, the symmetric group of degree 3. As by Universal Coefficients Theorem (see, for instance, [9, 11.4.18])

$$H^2(S_3, C) \cong Ext(\mathbb{Z}_2, \mathbb{Z}) \oplus Hom(M(S_3), \mathbb{Z}) \cong Ext(\mathbb{Z}_2, \mathbb{Z}) \cong \mathbb{Z}_2,$$

then there exists a unique central non-split extension $\Sigma : C \longrightarrow G \longrightarrow S_3$ of C by S_3. Suppose that $G = H \times K$, where H is a non-trivial abelian subgroup of G. Clearly, $C = H \times (K \cap C)$ and so $K \cap C \neq \{1\}$ since Σ is not split. It follows that H is finite. This contradiction shows that G is purely non-abelian. Moreover, as $Hom(G/C, C) = \{1\}$, then $|Aut_c(G)| = 2$. Finally, $Hom(G, Z(G))$ is infinite cyclic. For, firstly we note that G' is finite by Schur's theorem since G/C is finite. Then the factor group $G'C/G'$ is infinite and hence $G' \cap C = \{1\}$. Thus $G/G' \cong \mathbb{Z} \times F$, for some finite group F. Therefore $Hom(G, C) \cong Hom(G/G', C) \cong Hom(\mathbb{Z}, C) \oplus Hom(F, C) \cong Hom(\mathbb{Z}, C) \cong \mathbb{Z}$. $\square$

Recall that in 1994 Franciosi, de Giovanni and Newell [4] proved that *if G is a purely non-abelian group such that $Z(G)$ or G/G' is periodic with finite abelian section rank, then every central endomorphism of G is an automorphism* (here a group is said to have *finite abelian section rank* if it has no infinite sections of prime exponent). In order to find an analogue version of this result for Γ-central endomorphisms, we define the subgroups

$$E_\Gamma(G) = [G, C_\Gamma(E\Gamma_c(G))]$$

and

$$A_\Gamma(G) = [G, C_\Gamma(A\Gamma_c(G))]$$

of G. Clearly, as

$$Inn(G) \leq C_\Gamma(E\Gamma_c(G)) \leq C_\Gamma(A\Gamma_c(G)),$$

it follows that

$$G' \leq E_\Gamma(G) \leq A_\Gamma(G).$$

It is well known that a central endomorphism of a group G fixes each element of the commutator subgroup G' of G. The next lemma shows that a similar result holds for Γ-central endomorphisms.

Lemma 2.4. *Let α be a Γ-central endomorphism of a group G. Then α acts trivially on the subgroup $E_\Gamma(G)$.*

Proof. Let x be an element in G. As α is Γ-central, then $x^\alpha = xa$ for some $a \in C_G(\Gamma)$. If β belongs to $C_\Gamma(E\Gamma_c(G))$, then $[x, \beta]^\alpha = (x^{-1}x^\beta)^\alpha = (x^\alpha)^{-1}x^{\beta\alpha} = (xa)^{-1}(xa)^\beta = a^{-1}x^{-1}x^\beta a^\beta = x^{-1}x^\beta = [x, \beta]$. It follows that α fixes every element in $E_\Gamma(G)$. $\square$

Recall that a group G is said to be *co-hopfian* if every injective endomorphism of G is an automorphism. This means that G is not isomorphic to a proper subgroup. It is easy to show that every group satisfying Min, the minimal condition on subgroups, is co-hopfian.

Lemma 2.5. *Let G be a periodic abelian group with finite abelian section rank. Then G is co-hopfian.*

Proof. Let α be an injective endomorphism of G. By hypothesis every p-primary component G_p of G satisfies Min. It follows that $G_p^\alpha = G_p$, and hence $G^\alpha = G$. □

In the following we need a technical result due to Endimioni [3].

Lemma 2.6. *Let G be a group and let α be an endomorphism of G. Consider a group class X closed under taking subgroups and union of chain of X-subgroups, and such that for every normal subgroup H of G containing $Ker\alpha$, if $H/ker\alpha \in X$, then $H \in X$. Under these conditions, for any normal X-subgroup K_0 of G such that $K_0^\alpha \leq K_0$, there exists a normal X-subgroup K_1 of G containing K_0 such that $K_1^\alpha \leq K_1$ and the endomorphism induced by α on G/K_1 is injective.*

Theorem 2.7. *Let G be a group such that $C_G(\Gamma)$ or $G/E_\Gamma(G)$ is periodic with finite abelian section rank. If σ is a Γ-central endomorphism of G such that $Ker\sigma$ is finite, then the factor group $G/Im\sigma$ is finite. In particular, σ is an automorphism if σ is injective.*

Proof. By Lemma 2.1 there exists a homomorphisms h from G to $C_G(\Gamma)$ such that $x^\sigma = xx^h$ for all $x \in G$. Moreover, σ fixes every element of $E_\Gamma(G)$ by Lemma 2.4. It follows that $x^h = 1$ for every $x \in E_\Gamma(G)$, and hence $Imh \cong G/Kerh$ is periodic with finite abelian section rank. Let y be an element in Imh. Then $y = x^h$, for some $x \in G$. Therefore $y^\sigma = yy^h = x^h x^{h^2} = (xx^h)^h \in Imh$. Thus Imh is σ-invariant, and hence σ induces the identity on G/Imh and an endomorphism σ_1 in Imh. Clearly, $Ker\sigma_1 = Ker\sigma$ and in order to prove that $G/Im\sigma$ is finite, it is enough to show the finiteness of $Imh/Im\sigma_1$. So we may replace G with Imh and hence σ with σ_1.

Let Π be the set of primes which divide the order of $Ker\sigma$, and let X be the class of Π-groups. By Lemma 2.6 (with $K_0 = Ker\sigma$) there exists a (normal) σ-invariant subgroup K_1 of G containing $Ker\sigma$ and such that the induced endomorphism $\bar{\sigma}$ on G/K_1 is injective. By Lemma 2.5 $\bar{\sigma}$ is an automorphism. Clearly K_1 satisfies Min, so that if we denote by σ_0 the restriction of σ to K_1, it follows that $K_1 = (Im\sigma_0)(Ker\sigma^n)$, for some positive integer n. But $Ker\sigma^n$ is finite, and hence $Im\sigma_0$ has finite index in K_1. Therefore $G/Im\sigma$ is finite, as required.

Finally, suppose that σ is injective. By Lemma 2.5 the restriction of σ to Imh is an automorphism, so that $Imh \leq Im\sigma$. Let y be an element in G. Then $y^\sigma = yy^h$ and hence $y = y^\sigma(y^h)^{-1}$. It follows that $Im\sigma = G$. The statement is proved. □

Corollary 2.8. *Let G be a group having no Γ-central non-trivial direct factors and such that $C_G(\Gamma)$ or $G/E_\Gamma(G)$ is periodic with finite abelian section rank. Then every Γ-central endomorphism of G is an automorphism. In particular, $E_\Gamma(G) = A_\Gamma(G)$.*

Proof. Let σ be a Γ-central endomorphism of G, and let h be a homomorphism $G \longrightarrow C_G(\Gamma)$ such that $x^\sigma = xx^h$, for every $x \in G$ (see Lemma 2.1). Then the same argument used in the proof of Lemma 2.14

in [4] (replacing the centre $Z(G)$ of G with $C_G(\Gamma)$), yields that $x^h \neq x^{-1}$ for all $x \in G \setminus \{1\}$. It follows that σ is injective, and hence it is an automorphism by Theorem 2.7. □

The consideration of the group defined in the Remark 2.3 shows that the Corollary 2.8 does not hold if $G/C_G(\Gamma)$ or $E_\Gamma(G)$ is periodic with finite abelian section rank. However G is *isoclinic* to a group H in which every central endomorphism is an automorphism. Following P. Hall [6], two groups G_1 and G_2 are said to be *isoclinic* if there exist isomorphisms $\alpha : G_1/Z(G_1) \longrightarrow G_2/Z(G_2)$ and $\beta : G_1' \longrightarrow G_2'$ such that, if $\alpha(x_1Z(G_1)) = x_2Z(G_2)$ and $\alpha(y_1Z(G_1)) = y_2Z(G_2)$, then $\beta([x_1, y_1]) = [x_2, y_2]$.

Proposition 2.9. *Let G be a group such that $G/C_G(\Gamma)$ or $E_\Gamma(G)$ is periodic with finite abelian section rank. Then G is isoclinic to a purely non-abelian group H in which every central endomorphism is an automorphism.*

Proof. By a classical P. Hall's theorem (see [6]) there exists a group H isoclinic to G such that $Z(H) \leq H'$ (i.e., H is a *stem* group). It follows that $H/Z(H)$ or H' is periodic with finite abelian section rank. Let σ be a central endomorphism of H. Clearly, σ is injective since it fixes every element in $Z(H)$. Therefore by Theorem 2.7 σ is an automorphisms of H. Finally, H is purely non-abelian since every central endomorphism of H is an automorphism. □

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