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TORSION UNITS FOR SOME PROJECTED SPECIAL LINEAR GROUPS

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Abstract. In this paper, we investigate the Zassenhaus conjecture for $PSL(4,3)$ and $PSL(5,2)$. Consequently, we prove that the Prime graph question is true for both groups.

1. Introduction

Let $\mathbb{Z}G$ be the integral group ring of a finite group G and $U(\mathbb{Z}G)$ be the unit group $\mathbb{Z}G$. It is well known that

$$U(\mathbb{Z}G) = \{\pm 1\} \times V(\mathbb{Z}G),$$

where $V(\mathbb{Z}G)$ is the group of units of augmentation one. Throughout this article, G is always a finite group and torsion units will always represent torsion units in $V(\mathbb{Z}G) \setminus \{1\}$. A very important conjecture in the theory of integral group rings is:

Conjecture 1. *If G is a finite group, then for each torsion unit $u \in V(\mathbb{Z}G)$ there exists $g \in G$, such that $|u| = |g|$ where $|u|$ and $|g|$ is the order of u and g respectively.*

Hans Zassenhaus formulated a stronger version of this conjecture in [37], which states that:

Conjecture 2. *A torsion unit in $V(\mathbb{Z}G)$ is rationally conjugate to a group element if it is conjugate to an element of G by a unit of the rational group algebra $\mathbb{Q}G$.*

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This conjecture was confirmed for some classes nilpotent groups in [36, 32] and cyclic-by-abelian groups in [18]. The Luthar-Passi Method (which was introduced in [27]) is the main investigative tool for simple groups G in relation to the Zassenhaus conjecture for $\mathbb{Z}G$. It was confirmed true for all groups up to order 71, A_5, S_5, central extensions of S_5 and other simple finite groups in [23, 27, 28, 4, 5]. Partial results were given for A_6 in [33] and the remaining cases were dealt with in [21]. It was also proved for $PSL(2,p)$ when $p = \{7, 11, 13\}$ in [22] and $PSL(2,p)$ where $p = \{8, 17\}$ in [19] and $PSL(2,p)$ when $p = \{19, 23\}$ in [2].

Let H be a group with a torsion part $t(H)$ (i.e. the set of elements of H of finite order) of finite exponent and $\#H$ be the set of primes dividing the order of elements from the set $t(H)$. The prime graph of H (denoted by $\pi(H)$) is a graph with vertices labeled by primes from $\#H$, such that vertices p and q are adjacent if and only if there is an element of order pq in the group H. The following, was composed as a problem in [30] (Problem 37):

Question 1. *(Prime Graph Question) If G is a finite group, then $\pi(G) = \pi(V(\mathbb{Z}G))$.*

This question was upheld for Frobenius and Solvable groups in [25] and was also confirmed for some Sporadic Simple groups in [16, 7, 6, 13, 15, 10, 9, 3, 8, 12, 14, 11] and some alternating groups in [34, 35]. Additionally, it was confirmed for the sympletic simple group $S_4(4)$ in [31]. We use the Luthar-Passi Method to obtain our results. Our results are the following:

Theorem 1.1. *Let $G = PSL(4,3)$ and u be a torsion unit of $V(\mathbb{Z}G)$. Subsequently, the following conditions hold:*

- (i) *If $|u| \in \{5\}$, then u is rationally conjugate to some $g \in G$.*
- (ii) *There are no elements of order 15, 26, 39 or 65 in $V(\mathbb{Z}G)$.*
- (iii) *If $|u| = 2$, then $\nu_{rx} = 0 \,\forall\, rx \notin \{\nu_{2a}, \nu_{2b}\}$ and*

$$(\nu_{2a}, \nu_{2b}) \in \{(2,-1), (1,0), (0,1), (-1,2)\}.$$

- (iv) *If $|u| = 3$, then $\nu_{rx} = 0 \,\forall\, rx \notin \{\nu_{3a}, \nu_{3b}, \nu_{3c}, \nu_{3d}\}$ and the possible values for $(\nu_{3a}, \nu_{3b}, \nu_{3c}, \nu_{3d})$ are listed in Table 1.*
- (v) *If $|u| = 13$, then $\nu_{rx} = 0 \,\forall\, rx \notin \{\nu_{3a}, \nu_{3b}\}$ and*

$$(\nu_{13a}, \nu_{13b}, \nu_{13c}, \nu_{13d}) \in \{(1,-1,0,1), (1,-1,1,0), (1,0,-1,1), (1,0,0,0), (0,0,0,1), (1,0,1,-1), (0,0,1,0), (0,1,-1,1), (0,1,0,0), (-1,1,0,1), (0,1,1,-1), (-1,1,1,0)\}.$$

Theorem 1.2. *Let $G = PSL(5,2)$ and u be a torsion unit of $V(\mathbb{Z}G)$. Subsequently, the following conditions hold:*

- (i) *If $|u| \in \{5, 7, 31\}$, then u is rationally conjugate to some $g \in G$.*
- (ii) *There are no elements of order 10, 35, 62, 93 155 or 217 in $V(\mathbb{Z}G)$.*
- (iii) *If $|u| = 2$, then $\nu_{rx} = 0 \,\forall\, rx \notin \{\nu_{2a}, \nu_{2b}\}$ and*

$$(\nu_{2a}, \nu_{2b}) \in \{(3,-2), (2,-1), (1,0), (0,1), (-1,2), (-2,3), (-3,4), (-4,5)\}.$$

(*iv*) *If* $|u| = 3$*, then* $\nu_{rx} = 0 \, \forall \, rx \notin \{\nu_{3a}, \nu_{3b}\}$ *and*

$$(\nu_{3a}, \nu_{3b}) \in \{(2, -1), (1, 0), (0, 1)\}.$$

Consequently, we obtain the following result:

Corollary 1.3. *The Prime Graph question is true for the integral group ring of the groups* $PSL(4, 3)$ *and* $PSL(5, 2)$*.*

Let $u = \sum a_g g$ be a torsion unit of $V(\mathbb{Z}G)$. Then, the sum $\sum_{g \in X^G} a_g \in \mathbb{Z}$ which is the partial augmentation (denoted by $\varepsilon_C(u)$) of u with respect to its conjugacy classes X^G in G. Let $\nu_i = \varepsilon_{C_i}(u)$ be the i-th partial augmentation of u. It was proved that $\nu_1 = 0$ and $\nu_j = 0$ if the conjugacy class C_j consists of a central element by G. Higman and S. D. Berman [1]. Therefore $\nu_2 + \nu_3 + \cdots + \nu_l = 1$ where l denotes the number of non-central conjugacy classes of G.

Proposition 1.4. *([17]) Let* u *be a torsion unit of* $V(\mathbb{Z}G)$*. The order of* u *divides the exponent of* G*.*

The following propositions provide relationships between the partial augmentations and the order of a torsion unit.

Proposition 1.5. [20, Proposition 3.1] *Let* u *be a torsion unit of* $V(\mathbb{Z}G)$*. Let* C *be a conjugacy class of* G*. If* p *is a prime dividing the order of a representative of* C *but not the order of* u *then the partial augmentation* $\varepsilon_C(u) = 0$*.*

Proposition 1.6. [22, Proposition 2.2] *Let* G *be a finite group and let* u *be a torsion unit in* $V(\mathbb{Z}G)$*.*

(*i*) *If* u *has order* p^n*, then* $\varepsilon_x(u) = 0$ *for every* x *of* G *whose* p*-part is of order strictly greater than* p^n*.*

(*ii*) *If* x *is an element of* G *whose* p*-part, for some prime, has order strictly greater than the order of the* p*-part of* u*, then* $\varepsilon_x(u) = 0$*.*

Proposition 1.7. [27] *and* [29, Theorem 2.5] *Let* u *be a torsion unit of* $V(\mathbb{Z}G)$ *of order* k*. Then* u *is conjugate in* $\mathbb{Q}G$ *to an element* $g \in G$ *iff for each* d *dividing* k *there is precisely one conjugacy class* C_{i_d} *with partial augmentation* $\varepsilon_{C_{i_d}}(u^d) \neq 0$*.*

For any character χ of G and any torsion unit u of $V(\mathbb{Z}G)$, clearly $\chi(u) = \sum_{i=2}^{l} \nu_i \chi(h_i)$ where h_i is a representative of a non-central conjugacy class C_i.

Proposition 1.8. [27, Theorem 1] *and* [22] *Let* p *be equal to zero or a prime divisor of* $|G|$*. Suppose that* u *is an element of* $V(\mathbb{Z}G)$ *of order* k*. Let* z *be a primitive* k*-th root of unity. Then for every integer* l *and any character* χ *of* G*, the number*

$$\mu_l(u, \chi, p) = \frac{1}{k} \sum_{d|k} Tr_{\mathbb{Q}(z^d)/\mathbb{Q}} \{\chi(u^d) z^{-dl}\}$$

is a non-negative integer.

We will use the notation $\mu_l(u,\chi,*)$ when $p=0$. The LAGUNA package [26] for the GAP system [24] is a very useful tool when calculating $\mu_l(u,\chi,p)$.

2. Proof of Theorem 1.1

Let $G = PSL(4,3)$. Clearly $|G| = 6065280 = 2^7\cdot 3^6\cdot 5\cdot 13$ and $exp(G) = 4680 = 2^3\cdot 3^2\cdot 5\cdot 13$. Initially, for any torsion unit of $V(\mathbb{Z}G)$ of order k:

$$\nu_{2a}+\nu_{2b}+\nu_{3a}+\nu_{3b}+\nu_{3c}+\nu_{3d}+\nu_{4a}+\nu_{4b}+\nu_{4c}+\nu_{5a}+\nu_{6a}+\nu_{6b}+\nu_{6c}+\nu_{6d}+\nu_{6e}+$$
$$\nu_{8a}+\nu_{9a}+\nu_{9b}+\nu_{10a}+\nu_{12a}+\nu_{12b}+\nu_{12c}+\nu_{13a}+\nu_{13b}+\nu_{13c}+\nu_{13d}+\nu_{20a}+\nu_{20b}=1.$$

In order to prove that the Zassenhaus Conjecture holds, we need to consider torsion units of $V(\mathbb{Z}G)$ of order 2, 3, 4, 5, 6, 8, 9, 10, 12, 13, 20, 15, 18, 24, 26, 39, 40 and 65 (by Proposition 1.4). For the purpose of this paper and due to the complexity of certain orders, we shall consider elements of order 2, 3, 5, 13, 15, 26, 39 and 65. We shall now consider each case separately.

Case (i). Let $u\in V(\mathbb{Z}G)$ where $|u|=2$. Case (i). Let $u\in V(\mathbb{Z}G)$ where $|u|=2$. Using Propositions 1.5 & 1.6,

$$\nu_{2a}+\nu_{2b}=1.$$

Applying Proposition 1.8, we obtain the following system of inequalities:

$$\mu_0(u,\chi_2,*)=\tfrac{1}{2}(2\gamma_1+26)\geq 0;\quad \mu_1(u,\chi_2,*)=\tfrac{1}{2}(-2\gamma_1+26)\geq 0;$$
$$\mu_0(u,\chi_2,3)=\tfrac{1}{2}(2\gamma_2+6)\geq 0;\quad \mu_1(u,\chi_2,3)=\tfrac{1}{2}(-2\gamma_2+6)\geq 0$$

where $\gamma_1=3\nu_{2a}+\nu_{2b}$ and $\gamma_2=\nu_{2a}-\nu_{2b}$. It follows the only possible integer solutions for (ν_{2a},ν_{2b}) are listed in Theorem 1.1.

Case (ii). Let $u\in V(\mathbb{Z}G)$ where $|u|=3$. Using Propositions 1.5 & 1.6,

$$\nu_{3a}+\nu_{3b}+\nu_{3c}+\nu_{3d}=1.$$

Applying Proposition 1.8, we obtain the following system of inequalities:

$$\mu_0(u,\chi_2,*)=\tfrac{1}{3}(-2\gamma_1+26)\geq 0;\quad \mu_1(u,\chi_2,*)=\tfrac{1}{3}(\gamma_1+26)\geq 0;$$
$$\mu_0(u,\chi_3,*)=\tfrac{1}{3}(-2\gamma_2+26)\geq 0;\quad \mu_1(u,\chi_3,*)=\tfrac{1}{3}(\gamma_2+26)\geq 0;$$
$$\mu_0(u,\chi_4,*)=\tfrac{1}{3}(6\gamma_3+39)\geq 0;\quad \mu_1(u,\chi_4,*)=\tfrac{1}{3}(-3\gamma_3+39)\geq 0$$

where $\gamma_1=\nu_{3a}+\nu_{3b}-8\nu_{3c}+\nu_{3d}$, $\gamma_2=\nu_{3a}-8\nu_{3b}+\nu_{3c}+\nu_{3d}$ and $\gamma_3=4\nu_{3a}+\nu_{3b}+\nu_{3c}+\nu_{3d}$. Clearly $\gamma_1\in\{1+3k\mid -9\leq k\leq 4\}$, $\gamma_2\in\{1+3k\mid -9\leq k\leq 4\}$ and $\gamma_3\in\{k\mid -6\leq k\leq 13\}$. It follows the only possible integer solutions for $(\nu_{3a},\nu_{3b},\nu_{3c},\nu_{3d})$ are listed in Table 1.

Case (iii). Let $u \in V(\mathbb{Z}G)$ where $|u| = 5$. By Proposition 1.5, $\nu_{kx} = 0$ for all

$$kx \in \{\nu_{2a}, \nu_{2b}, \nu_{3a}, \nu_{3b}, \nu_{3c}, \nu_{3d}, \nu_{4a}, \nu_{4b}, \nu_{4c}, \nu_{6a}, \nu_{6b}, \nu_{6c}, \nu_{6d}, \nu_{6e}, \nu_{8a}, \nu_{9a}, \nu_{9b}, \nu_{10a}, \nu_{12a}, \nu_{12b}, \nu_{12c}, \nu_{13a}, \nu_{13b}, \nu_{13c}, \nu_{13d}, \nu_{20a}, \nu_{20b}\}.$$

Therefore, u is rationally conjugated to some element $g \in G$ by Proposition 1.7.

Case (iv). Let $u \in V(\mathbb{Z}G)$ where $|u| = 13$. Using Propositions 1.5 & 1.6,

$$\nu_{13a} + \nu_{13b} + \nu_{13c} + \nu_{13d} = 1.$$

Applying Proposition 1.8, we obtain the following system of inequalities:

$$\begin{aligned}
\mu_1(u, \chi_{23}, *) &= \tfrac{1}{13}(\gamma_1 + 640) \geq 0; \quad \mu_7(u, \chi_3, 3) = \tfrac{1}{13}(-\gamma_1 + 10) \geq 0; \\
\mu_1(u, \chi_3, 3) &= \tfrac{1}{13}(\gamma_2 + 10) \geq 0; \quad \mu_2(u, \chi_{23}, *) = \tfrac{1}{13}(-\gamma_2 + 640) \geq 0; \\
\mu_2(u, \chi_3, 3) &= \tfrac{1}{13}(\gamma_3 + 10) \geq 0; \quad \mu_4(u, \chi_{23}, *) = \tfrac{1}{13}(-\gamma_3 + 640) \geq 0; \\
\mu_1(u, \chi_2, 3) &= \tfrac{1}{13}(\gamma_4 + 6) \geq 0; \quad \mu_2(u, \chi_2, 3) = \tfrac{1}{13}(\gamma_5 + 6) \geq 0
\end{aligned}$$

where $\gamma_1 = 10\nu_{13a} - 3\nu_{13b} - 3\nu_{13c} - 3\nu_{13d}$, $\gamma_2 = 3\nu_{13a} + 3\nu_{13b} - 10\nu_{13c} + 3\nu_{13d}$ and $\gamma_3 = 3\nu_{13a} - 10\nu_{13b} + 3\nu_{13c} + 3\nu_{13d}$, $\gamma_4 = 7\nu_{13a} + 7\nu_{13b} - 6\nu_{13c} - 6\nu_{13d}$ and $\gamma_5 = -6\nu_{13a} - 6\nu_{13b} + 7\nu_{13c} + 7\nu_{13d}$. Clearly $\gamma_1 \in \{10 + 13k \mid -50 \leq k \leq 0\}$, $\gamma_2 \in \{3 + 13k \mid -1 \leq k \leq 49\}$ and $\gamma_3 \in \{3 + 13k \mid -1 \leq k \leq 49\}$. It follows the only possible integer solutions for $(\nu_{3a}, \nu_{3b}, \nu_{3c}, \nu_{3d})$ are listed in Theorem 1.2.

Case (v). Let $u \in V(\mathbb{Z}G)$ where $|u| = 15$. There is only one partial augmentation for units of order 5 and 175 partial augmentations for units of order 3. Therefore we need to consider $175 \cdot 1 = 175$ cases for units of order 15. With the aid of LAGUNA ([26]) for the GAP system ([24]), we solved each case and it transpired that there are no solutions in each case.

Case (vi). Let $u \in V(\mathbb{Z}G)$ where $|u| = 26$. Using Propositions 1.5 & 1.6,

$$\nu_{2a} + \nu_{2b} + \nu_{13a} + \nu_{13b} + \nu_{13c} + \nu_{13d} = 1.$$

Let $\gamma_1 = 4\nu_{2a} - 4\nu_{2b} - \nu_{13a} - \nu_{13b} - \nu_{13c} - \nu_{13d}$, $\gamma_2 = 3\nu_{2a} + \nu_{2b}$ and $\gamma_3 = 10\nu_{13a} - 3\nu_{13b} - 3\nu_{13c} - 3\nu_{13d}$. We shall now separately consider the following cases involving $\chi(u^n)$ for $n \in \{2, 5\}$:

- $\chi(u^{13}) = k_1\chi(2a) + k_2\chi(2b)$ and $\chi(u^2) = m_1\chi(13a) + m_2\chi(13b) + m_3\chi(13c) + m_4\chi(13d)$ where $(k_1, k_2) \in \{(1, 0), (0, 1)\}$ and $(m_1, m_2, m_3, m_4) \in \{(1, -1, 0, 1), (1, -1, 1, 0), (1, 0, -1, 1), (1, 0, 0, 0), (0, 0, 0, 1), (1, 0, 1, -1), (0, 0, 1, 0), (0, 1, -1, 1), (0, 1, 0, 0), (-1, 1, 0, 1), (0, 1, 1, -1), (-1, 1, 1, 0)\}$. Applying Proposition 1.8, we obtain:

$$\mu_0(u, \chi_2, 3) = \tfrac{1}{26}(3\gamma_1 - 1) \geq 0; \quad \mu_{13}(u, \chi_2, 3) = \tfrac{1}{26}(-3\gamma_1 + 1) \geq 0.$$

It follows that there are no possible integer solutions for $(\nu_{2a}, \nu_{2b}, \nu_{13a}, \nu_{13b}, \nu_{13c}, \nu_{13d})$.

• $\chi(u^{13}) = 2\chi(2a) - \chi(2b)$ and $\chi(u^2) = m_1\chi(13a) + m_2\chi(13b) + m_3\chi(13c) + m_4\chi(13d)$ where $(m_1, m_2, m_3, m_4) \in \{(1,-1,0,1), (1,-1,1,0), (1,0,-1,1), (1,0,0,0), (0,0,0,1), (1,0,1,-1), (0,0,1,0), (0,1,-1,1), (0,1,0,0), (-1,1,0,1), (0,1,1,-1), (-1,1,1,0)\}$. Applying Proposition 1.8, we obtain:

$$\mu_0(u, \chi_2, *) = \tfrac{1}{26}(24\gamma_2 + 36) \geq 0; \quad \mu_{13}(u, \chi_2, *) = \tfrac{1}{26}(-24\gamma_2 + 16) \geq 0.$$

It follows that there are no possible integer solutions for $(\nu_{2a}, \nu_{2b}, \nu_{13a}, \nu_{13b}, \nu_{13c}, \nu_{13d})$.

• $\chi(u^{13}) = -\chi(2a) + 2\chi(2b)$ and $\chi(u^2) = m_1\chi(13a) + m_2\chi(13b) + m_3\chi(13c) + m_4\chi(13d)$ where $(m_1, m_2, m_3, m_4) \in \{(1,0,0,0), (0,0,0,1), (0,0,1,0), (0,1,0,0)\}$. Applying Proposition 1.8, we obtain:

$$\mu_0(u, \chi_2, *) = \tfrac{1}{26}(24\gamma_2 + 24) \geq 0; \quad \mu_{13}(u, \chi_2, *) = \tfrac{1}{26}(-24\gamma_2 + 28) \geq 0;$$
$$\mu_2(u, \chi_{23}, *) = \tfrac{1}{26}(\gamma_3 + 637) \geq 0; \quad \mu_5(u, \chi_{23}, *) = \tfrac{1}{26}(-\gamma_3 + 637) \geq 0.$$

It follows that there are no possible integer solutions for $(\nu_{2a}, \nu_{2b}, \nu_{13a}, \nu_{13b}, \nu_{13c}, \nu_{13d})$.

• $\chi(u^{13}) = -\chi(2a) + 2\chi(2b)$ and $\chi(u^2) = m_1\chi(13a) + m_2\chi(13b) + m_3\chi(13c) + m_4\chi(13d)$ where $(m_1, m_2, m_3, m_4) \in \{(1,-1,0,1), (1,-1,1,0), (1,0,-1,1), (1,0,1,-1), (0,1,-1,1), (-1,1,0,1), (0,1,1,-1), (-1,1,1,0)\}$. Applying Proposition 1.8, we obtain:

$$\mu_0(u, \chi_2, *) = \tfrac{1}{26}(24\gamma_2 + 24) \geq 0; \quad \mu_{13}(u, \chi_2, *) = \tfrac{1}{26}(-24\gamma_2 + 28) \geq 0.$$

It follows that there are no possible integer solutions for $(\nu_{2a}, \nu_{2b}, \nu_{13a}, \nu_{13b}, \nu_{13c}, \nu_{13d})$.

Case (vii). Let $u \in V(\mathbb{Z}G)$ where $|u| = 39$. There are 175 partial augmentation for units of order 3 and 12 partial augmentations for units of order 13. Therefore we need to consider $175 \cdot 12 = 2100$ cases for units of order 39. With the aid of LAGUNA ([26]) for the GAP system ([24]), we solved each case and it transpired that there are no solutions in each case.

Case (vii). Let $u \in V(\mathbb{Z}G)$ where $|u| = 65$. Using Propositions 1.5 & 1.6,

$$\nu_{5a} + \nu_{13a} + \nu_{13b} + \nu_{13c} + \nu_{13d} = 1.$$

Consider the cases $\chi(u^{13}) = \chi(5a)$ and $\chi(u^5) = m_1\chi(13a) + m_2\chi(13b) + m_3\chi(13c) + m_4\chi(13d)$ where $(m_1, m_2, m_3, m_4) \in \{(1,-1,0,1), (1,-1,1,0), (1,0,-1,1), (1,0,0,0), (0,0,0,1), (1,0,1,-1), (0,0,1,0), (0,1,-1,1), (0,1,0,0), (-1,1,0,1), (0,1,1,-1), (-1,1,1,0)\}$. Applying Proposition 1.8, we obtain:

$$\mu_0(u, \chi_2, *) = \tfrac{1}{65}(48\nu_{5a} + 30) \geq 0; \quad \mu_0(u, \chi_4, *) = \tfrac{1}{65}(-48\nu_{5a} + 35) \geq 0.$$

It follows that there are no possible integer solutions for $(\nu_{5a}, \nu_{13a}, \nu_{13b}, \nu_{13c}, \nu_{13d})$.

We shall now consider the prime graph of $G = PSL(4,3)$. G contains elements of order 6 and 10. Therefore $[2,3]$and $[2,5]$ are adjacent in $\pi(G)$ and consequently adjacent in $\pi(V(\mathbb{Z}G))$. Clearly

$\pi(G) = \pi(V(\mathbb{Z}G))$, since there are no torsion units of order 15, 26, 39 and 65 in $V(\mathbb{Z}G)$. This completes the proof.

3. **Proof of Theorem 1.2**

Let $G = PSL(5,2)$. Clearly $|G| = 9999360 = 2^{10}\cdot 3^2 \cdot 5 \cdot 7 \cdot 31$ and $exp(G) = 26040 = 2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 31$. Initially, for any torsion unit of $V(\mathbb{Z}G)$ of order k:

$$\nu_{2a} + \nu_{2b} + \nu_{3a} + \nu_{3b} + \nu_{4a} + \nu_{4b} + \nu_{4c} + \nu_{5a} + \nu_{6a} + \nu_{6b} + \nu_{7a} + \nu_{7b} + \nu_{8a} + \nu_{12a} +$$
$$\nu_{14a} + \nu_{14b} + \nu_{15a} + \nu_{15b} + \nu_{21a} + \nu_{21b} + \nu_{31a} + \nu_{31b} + \nu_{31c} + \nu_{31d} + \nu_{31e} + \nu_{31f} = 1.$$

In order to prove that the Zassenhaus Conjecture holds, we need to consider torsion units of $V(\mathbb{Z}G)$ of order 2, 3, 4, 5, 6, 7, 8, 12, 14, 15, 21, 31, 10, 24, 28, 35, 42, 62, 93, 155 and 217 (by Proposition 1.4). For the purpose of this paper and due to the complexity of certain orders, we shall consider elements of order 2, 3, 5, 7, 10, 31, 35, 62, 93, 155 and 217. We shall now consider each case separately.

Case (i). Let $u \in V(\mathbb{Z}G)$ where $|u| = 2$. Using Propositions 1.5 & 1.6,

$$\nu_{2a} + \nu_{2b} = 1.$$

Applying Proposition 1.8, we obtain the following system of inequalities:

$$\mu_0(u, \chi_2, *) = \tfrac{1}{2}(2\gamma + 30) \geq 0; \quad \mu_1(u, \chi_2, *) = \tfrac{1}{2}(-2\gamma + 30) \geq 0$$

where $\gamma = 7\nu_{2a} + 3\nu_{2b}$. It follows that $\gamma \in \{k \mid -15 \leq k \leq 15\}$ and the only possible integer solutions for (ν_{2a}, ν_{2b}) are listed in Theorem 1.2.

Case (ii). Let $u \in V(\mathbb{Z}G)$ where $|u| = 3$. Using Propositions 1.5 & 1.6,

$$\nu_{3a} + \nu_{3b} = 1.$$

Applying Proposition 1.8, we obtain the following system of inequalities:

$$\mu_0(u, \chi_2, *) = \tfrac{1}{3}(12\gamma_1 + 30) \geq 0; \quad \mu_1(u, \chi_2, *) = \tfrac{1}{3}(-6\gamma_1 + 30) \geq 0;$$
$$\mu_0(u, \chi_2, 2) = \tfrac{1}{3}(2\gamma_2 + 5) \geq 0; \quad \mu_1(u, \chi_2, 2) = \tfrac{1}{3}(-\gamma_2 + 5) \geq 0$$

where $\gamma_1 = \nu_{3a}$ and $\gamma_2 = 2\nu_{3a} - \nu_{3b}$. It follows that $\gamma_1 \in \{k \mid -2 \leq k \leq 5\}$ and the only possible integer solutions for (ν_{2a}, ν_{2b}) are listed in Theorem 1.2.

Case (iii). Let $u \in V(\mathbb{Z}G)$ where $|u| = 5$. By Proposition 1.5, $\nu_{kx} = 0$ for all

$kx \in \{2a, 2b, 3a, 3b, 4a, 4b, 4c, 6a, 6b, 7a, 7b, 8a, 12a, 14a, 14b, 15a, 15b, 21a, 21b, 31a, 31b, 31c, 31d, 31e, 31f\}$.

Therefore, u is rationally conjugated to some element $g \in G$ by Proposition 1.7.

Case (iv). Let $u \in V(\mathbb{Z}G)$ where $|u| = 7$. Using Propositions 1.5 & 1.6,

$$\nu_{7a} + \nu_{7b} = 1.$$

Applying Proposition 1.8, we obtain the following system of inequalities:

$$\mu_3(u, \chi_{14}, 2) = \tfrac{1}{7}(7\gamma_1 + 280) \geq 0; \quad \mu_1(u, \chi_{14}, 2) = \tfrac{1}{7}(-7\gamma_1 + 280) \geq 0;$$
$$\mu_1(u, \chi_2, 2) = \tfrac{1}{7}(\gamma_2 + 5) \geq 0; \quad \mu_3(u, \chi_2, 2) = \tfrac{1}{7}(\gamma_3 + 5) \geq 0$$

where $\gamma_1 = \nu_{7a} - \nu_{7b}$, $\gamma_2 = 2\nu_{7a} - 5\nu_{7b}$ and $\gamma_3 = -5\nu_{7a} + 2\nu_{7b}$. It follows that $\gamma_1 \in \{k \mid -40 \leq k \leq 40\}$ and the only possible integer solutions for (ν_{7a}, ν_{7b}) are $(0, 1)$ and $(1, 0)$. Therefore, u is rationally conjugated to some element $g \in G$ by Proposition 1.7.

Case (v). Let $u \in V(\mathbb{Z}G)$ where $|u| = 10$. Using Propositions 1.5 & 1.6,

$$\nu_{2a} + \nu_{2b} + \nu_{5a} = 1.$$

Let $\gamma_1 = 7\nu_{2a} + 3\nu_{2b}$, $\gamma_2 = 28\nu_{2a} + 12\nu_{2b} - \nu_{5a}$ and $\gamma_3 = 27\nu_{2a} - 5\nu_{2b}$. We shall now separately consider the following cases involving $\chi(u^n)$ for $n \in \{2, 5\}$:

- $\chi(u^5) = m_1\chi(2a) + m_2\chi(2b)$ and $\chi(u^2) = \chi(5a)$ where $(m_1, m_2) \in \{(1, 0), (0, 1)\}$. Applying Proposition 1.8, we obtain:

$$\mu_0(u, \chi_2, *) = \tfrac{1}{10}(8\gamma_1 + k_1) \geq 0; \quad \mu_5(u, \chi_2, *) = \tfrac{1}{10}(-8\gamma_1 + k_2) \geq 0;$$
$$\mu_0(u, \chi_3, *) = \tfrac{1}{10}(4\gamma_2 + k_3) \geq 0; \quad \mu_1(u, \chi_3, *) = \tfrac{1}{10}(\gamma_2 + k_4) \geq 0;$$
$$\mu_5(u, \chi_3, *) = \tfrac{1}{10}(-4\gamma_2 + k_5) \geq 0; \quad \mu_0(u, \chi_4, *) = \tfrac{1}{10}(4\gamma_3 + k_6) \geq 0;$$
$$\mu_2(u, \chi_4, *) = \tfrac{1}{10}(-\gamma_3 + k_7) \geq 0$$

where the values for k_i's and corresponding (m_1, m_2) values are as follows:

(m_1, m_2)	$(k_1, k_2, k_3, k_4, k_5, k_6, k_7)$
$(1, 0)$	$(44, 16, 148, 97, 92, 182, 182)$
$(0, 1)$	$(36, 24, 132, 113, 108, 150, 150)$

It follows that there are no possible integer solutions for $(\nu_{2a}, \nu_{2b}, \nu_{5a})$.

- $\chi(u^5) = m_1\chi(2a) + m_2\chi(2b)$ and $\chi(u^2) = \chi(5a)$ where $(m_1, m_2) \in \{(3, -2), (2, -1)\}$. Applying Proposition 1.8, we obtain:

$$\mu_1(u, \chi_2, *) = \tfrac{1}{10}(2\gamma_1) \geq 0; \quad \mu_5(u, \chi_2, *) = \tfrac{1}{10}(-8\gamma_1) \geq 0;$$
$$\mu_0(u, \chi_3, *) = \tfrac{1}{10}(4\gamma_2 + 180) \geq 0; \quad \mu_1(u, \chi_3, *) = \tfrac{1}{10}(\gamma_2 + 65) \geq 0;$$
$$\mu_5(u, \chi_3, *) = \tfrac{1}{10}(-4\gamma_2 + 60) \geq 0; \quad \mu_0(u, \chi_4, *) = \tfrac{1}{10}(4\gamma_3 + 246) \geq 0;$$
$$\mu_5(u, \chi_4, *) = \tfrac{1}{10}(-4\gamma_3 + 64) \geq 0$$

where the values for k_i's and corresponding (m_1, m_2) values are as follows:

(m_1, m_2)	$(k_1, k_2, k_3, k_4, k_5, k_6, k_7)$
$(3, -2)$	$(0, 0, 180, 65, 60, 246, 64)$
$(2, -1)$	$(8, 8, 164, 81, 76, 214, 96)$

It follows that there are no possible integer solutions for $(\nu_{2a}, \nu_{2b}, \nu_{5a})$.

- $\chi(u^5) = m_1\chi(2a) + m_2\chi(2b)$ and $\chi(u^2) = \chi(5a)$ where $(m_1, m_2) \in \{(-1, 2), (-2, 3)\}$. Applying Proposition 1.8, we obtain:

$$\mu_0(u, \chi_2, *) = \tfrac{1}{10}(8\gamma_1 + 28) \geq 0; \quad \mu_5(u, \chi_2, *) = \tfrac{1}{10}(-8\gamma_1 + 32) \geq 0;$$
$$\mu_0(u, \chi_3, *) = \tfrac{1}{10}(4\gamma_2 + 116) \geq 0; \quad \mu_1(u, \chi_3, *) = \tfrac{1}{10}(\gamma_2 + 129) \geq 0;$$
$$\mu_5(u, \chi_3, *) = \tfrac{1}{10}(-4\gamma_2 + 124) \geq 0; \quad \mu_0(u, \chi_4, *) = \tfrac{1}{10}(4\gamma_3 + 118) \geq 0;$$
$$\mu_2(u, \chi_4, *) = \tfrac{1}{10}(-\gamma_3 + 118) \geq 0$$

where the values for k_i's and corresponding (m_1, m_2) values are as follows:

(m_1, m_2)	$(k_1, k_2, k_3, k_4, k_5, k_6, k_7)$
$(-1, 2)$	$(28, 32, 116, 129, 124, 118, 118)$
$(-2, 3)$	$(20, 40, 100, 145, 140, 86, 86)$

It follows that there are no possible integer solutions for $(\nu_{2a}, \nu_{2b}, \nu_{5a})$.

- $\chi(u^5) = m_1\chi(2a) + m_2\chi(2b)$ and $\chi(u^2) = \chi(5a)$ where $(m_1, m_2) \in \{(-3, 4), (-4, 5)\}$. Applying Proposition 1.8, we obtain:

$$\mu_0(u, \chi_2, *) = \tfrac{1}{10}(8\gamma_1 + 4) \geq 0; \quad \mu_2(u, \chi_2, *) = \tfrac{1}{10}(-2\gamma_1 + 4) \geq 0;$$
$$\mu_0(u, \chi_3, *) = \tfrac{1}{10}(4\gamma_2 + 68) \geq 0; \quad \mu_1(u, \chi_3, *) = \tfrac{1}{10}(\gamma_2 + 177) \geq 0;$$
$$\mu_5(u, \chi_3, *) = \tfrac{1}{10}(-4\gamma_2 + 172) \geq 0; \quad \mu_0(u, \chi_4, *) = \tfrac{1}{10}(4\gamma_3 + 22) \geq 0;$$
$$\mu_2(u, \chi_4, *) = \tfrac{1}{10}(-\gamma_3 + 22) \geq 0$$

where the values for k_i's and corresponding (m_1, m_2) values are as follows:

(m_1, m_2)	$(k_1, k_2, k_3, k_4, k_5, k_6, k_7)$
$(-3, 4)$	$(12, 12, 84, 161, 156, 54, 54)$
$(-4, 5)$	$(4, 4, 68, 177, 172, 22, 22)$

It follows that there are no possible integer solutions for $(\nu_{2a}, \nu_{2b}, \nu_{5a})$.

Case (vi). Let $u \in V(\mathbb{Z}G)$ where $|u| = 31$. Using Propositions 1.5 & 1.6,

$$\nu_{31a} + \nu_{31b} + \nu_{31c} + \nu_{31d} + \nu_{31e} + \nu_{31f} = 1.$$

Applying Proposition 1.8, we obtain the following system of inequalities:

$$\begin{gathered}
\mu_1(u,\chi_7,*) = \tfrac{1}{31}(\gamma_1 + 315) \geq 0; \quad \mu_3(u,\chi_7,*) = \tfrac{1}{31}(\gamma_2 + 315) \geq 0;\\
\mu_5(u,\chi_7,*) = \tfrac{1}{31}(\gamma_3 + 315) \geq 0; \quad \mu_7(u,\chi_7,*) = \tfrac{1}{31}(\gamma_4 + 315) \geq 0;\\
\mu_{11}(u,\chi_7,*) = \tfrac{1}{31}(\gamma_5 + 315) \geq 0; \quad \mu_{15}(u,\chi_7,*) = \tfrac{1}{31}(\gamma_6 + 315) \geq 0;\\
\mu_1(u,\chi_4,2) = \tfrac{1}{31}(\gamma_7 + 10) \geq 0; \quad \mu_3(u,\chi_4,2) = \tfrac{1}{31}(\gamma_8 + 10) \geq 0;\\
\mu_5(u,\chi_4,2) = \tfrac{1}{31}(\gamma_9 + 10) \geq 0; \quad \mu_7(u,\chi_4,2) = \tfrac{1}{31}(\gamma_{10} + 10) \geq 0;\\
\mu_{11}(u,\chi_4,2) = \tfrac{1}{31}(\gamma_{11} + 10) \geq 0; \quad \mu_{15}(u,\chi_4,2) = \tfrac{1}{31}(\gamma_{12} + 10) \geq 0;\\
\mu_1(u,\chi_6,2) = \tfrac{1}{31}(\gamma_{13} + 24) \geq 0; \quad \mu_3(u,\chi_6,2) = \tfrac{1}{31}(\gamma_{14} + 24) \geq 0;\\
\mu_5(u,\chi_6,2) = \tfrac{1}{31}(\gamma_{15} + 24) \geq 0; \quad \mu_1(u,\chi_7,2) = \tfrac{1}{31}(\gamma_{16} + 40) \geq 0;\\
\mu_3(u,\chi_7,2) = \tfrac{1}{31}(\gamma_{17} + 40) \geq 0; \quad \mu_5(u,\chi_7,2) = \tfrac{1}{31}(\gamma_{18} + 40) \geq 0;\\
\mu_7(u,\chi_7,2) = \tfrac{1}{31}(\gamma_{19} + 40) \geq 0; \quad \mu_{11}(u,\chi_7,2) = \tfrac{1}{31}(\gamma_{20} + 40) \geq 0;\\
\mu_{15}(u,\chi_7,2) = \tfrac{1}{31}(\gamma_{21} + 40) \geq 0; \quad \mu_1(u,\chi_9,2) = \tfrac{1}{31}(\gamma_{22} + 40) \geq 0;\\
\mu_3(u,\chi_9,2) = \tfrac{1}{31}(\gamma_{23} + 40) \geq 0; \quad \mu_5(u,\chi_9,2) = \tfrac{1}{31}(\gamma_{24} + 40) \geq 0;\\
\mu_7(u,\chi_9,2) = \tfrac{1}{31}(\gamma_{25} + 40) \geq 0; \quad \mu_{11}(u,\chi_9,2) = \tfrac{1}{31}(\gamma_{26} + 40) \geq 0;\\
\mu_{15}(u,\chi_9,2) = \tfrac{1}{31}(\gamma_{27} + 40) \geq 0; \quad \mu_1(u,\chi_{11},2) = \tfrac{1}{31}(\gamma_{28} + 74) \geq 0;\\
\mu_3(u,\chi_{11},2) = \tfrac{1}{31}(\gamma_{29} + 74) \geq 0; \quad \mu_5(u,\chi_{11},2) = \tfrac{1}{31}(\gamma_{30} + 74) \geq 0;\\
\mu_1(u,\chi_{12},2) = \tfrac{1}{31}(\gamma_{31} + 160) \geq 0; \quad \mu_3(u,\chi_{12},2) = \tfrac{1}{31}(\gamma_{32} + 160) \geq 0;\\
\mu_5(u,\chi_{12},2) = \tfrac{1}{31}(\gamma_{33} + 160) \geq 0; \quad \mu_7(u,\chi_{12},2) = \tfrac{1}{31}(\gamma_{34} + 160) \geq 0;\\
\mu_{11}(u,\chi_{12},2) = \tfrac{1}{31}(\gamma_{35} + 160) \geq 0; \quad \mu_{15}(u,\chi_{12},2) = \tfrac{1}{31}(\gamma_{36} + 160) \geq 0;\\
\mu_1(u,\chi_{14},2) = \tfrac{1}{31}(\gamma_{37} + 280) \geq 0; \quad \mu_3(u,\chi_{14},2) = \tfrac{1}{31}(\gamma_{38} + 280) \geq 0;\\
\mu_5(u,\chi_{14},2) = \tfrac{1}{31}(\gamma_{39} + 280) \geq 0; \quad \mu_7(u,\chi_{14},2) = \tfrac{1}{31}(\gamma_{40} + 280) \geq 0;\\
\mu_{11}(u,\chi_{14},2) = \tfrac{1}{31}(\gamma_{41} + 280) \geq 0; \quad \mu_{15}(u,\chi_{14},2) = \tfrac{1}{31}(\gamma_{42} + 280) \geq 0
\end{gathered}$$

where $\gamma_1 = 26\nu_{31a} - 5\nu_{31b} - 5\nu_{31c} - 5\nu_{31d} - 5\nu_{31e} - 5\nu_{31f}$, $\gamma_2 = -5\nu_{31a} - 5\nu_{31b} - 5\nu_{31c} - 5\nu_{31d} - 5\nu_{31e} + 26\nu_{31f}$, $\gamma_3 = -5\nu_{31a} - 5\nu_{31b} + 26\nu_{31c} - 5\nu_{31d} - 5\nu_{31e} - 5\nu_{31f}$, $\gamma_4 = -5\nu_{31a} - 5\nu_{31b} - 5\nu_{31c} - 5\nu_{31d} + 26\nu_{31e} - 5\nu_{31f}$, $\gamma_5 = -5\nu_{31a} - 5\nu_{31b} - 5\nu_{31c} + 26\nu_{31d} - 5\nu_{31e} - 5\nu_{31f}$, $\gamma_6 = -5\nu_{31a} + 26\nu_{31b} - 5\nu_{31c} - 5\nu_{31d} - 5\nu_{31e} - 5\nu_{31f}$, $\gamma_7 = -10\nu_{31a} - 10\nu_{31b} - 10\nu_{31c} + 21\nu_{31d} + 21\nu_{31e} - 10\nu_{31f}$, $\gamma_8 = 21\nu_{31a} - 10\nu_{31b} - 10\nu_{31c} + 21\nu_{31d} - 10\nu_{31e} - 10\nu_{31f}$, $\gamma_9 = 21\nu_{31a} - 10\nu_{31b} - 10\nu_{31c} - 10\nu_{31d} - 10\nu_{31e} + 21\nu_{31f}$, $\gamma_{10} = -10\nu_{31a} + 21\nu_{31b} + 21\nu_{31c} - 10\nu_{31d} - 10\nu_{31e} - 10\nu_{31f}$, $\gamma_{11} = -10\nu_{31a} + 21\nu_{31b} - 10\nu_{31c} - 10\nu_{31d} + 21\nu_{31e} - 10\nu_{31f}$, $\gamma_{12} = -10\nu_{31a} - 10\nu_{31b} + 21\nu_{31c} - 10\nu_{31d} - 10\nu_{31e} + 21\nu_{31f}$, $\gamma_{13} = 7\nu_{31a} + 7\nu_{31b} + 7\nu_{31c} + 7\nu_{31d} - 24\nu_{31e} - 24\nu_{31f}$,

$\gamma_{14} = 7\nu_{31a}+7\nu_{31b}-24\nu_{31c}-24\nu_{31d}+7\nu_{31e}+7\nu_{31f}$, $\gamma_{15} = -24\nu_{31a}-24\nu_{31b}+7\nu_{31c}+7\nu_{31d}+7\nu_{31e}+7\nu_{31f}$, $\gamma_{16} = -9\nu_{31a} - 40\nu_{31b} + 22\nu_{31c} - 9\nu_{31d} + 22\nu_{31e} + 22\nu_{31f}$, $\gamma_{17} = -9\nu_{31a} + 22\nu_{31b} + 22\nu_{31c} + 22\nu_{31d} - 40\nu_{31e} - 9\nu_{31f}$, $\gamma_{18} = 22\nu_{31a} + 22\nu_{31b} - 9\nu_{31c} - 40\nu_{31d} + 22\nu_{31e} - 9\nu_{31f}$, $\gamma_{19} = 22\nu_{31a} - 9\nu_{31b} + 22\nu_{31c} + 22\nu_{31d} - 9\nu_{31e} - 40\nu_{31f}$, $\gamma_{20} = 22\nu_{31a} + 22\nu_{31b} - 40\nu_{31c} - 9\nu_{31d} - 9\nu_{31e} + 22\nu_{31f}$, $\gamma_{21} = -40\nu_{31a} - 9\nu_{31b} - 9\nu_{31c} + 22\nu_{31d} + 22\nu_{31e} + 22\nu_{31f}$, $\gamma_{22} = -9\nu_{31a} + 22\nu_{31b} - 9\nu_{31c} - 9\nu_{31d} - 9\nu_{31e} + 22\nu_{31f}$, $\gamma_{23} = -9\nu_{31a} - 9\nu_{31b} + 22\nu_{31c} - 9\nu_{31d} + 22\nu_{31e} - 9\nu_{31f}$, $\gamma_{24} = -9\nu_{31a} + 22\nu_{31b} - 9\nu_{31c} + 22\nu_{31d} - 9\nu_{31e} - 9\nu_{31f}$, $\gamma_{25} = -9\nu_{31a}-9\nu_{31b}-9\nu_{31c}+22\nu_{31d}-9\nu_{31e}+22\nu_{31f}$, $\gamma_{26} = 22\nu_{31a}-9\nu_{31b}+22\nu_{31c}-9\nu_{31d}-9\nu_{31e}-9\nu_{31f}$, $\gamma_{27} = 22\nu_{31a} - 9\nu_{31b} - 9\nu_{31c} - 9\nu_{31d} + 22\nu_{31e} - 9\nu_{31f}$, $\gamma_{28} = -12\nu_{31a} - 12\nu_{31b} + 19\nu_{31c} + 19\nu_{31d} - 12\nu_{31e} - 12\nu_{31f}$, $\gamma_{29} = 19\nu_{31a} + 19\nu_{31b} - 12\nu_{31c} - 12\nu_{31d} - 12\nu_{31e} - 12\nu_{31f}$, $\gamma_{30} = -12\nu_{31a} - 12\nu_{31b} - 12\nu_{31c} - 12\nu_{31d} + 19\nu_{31e} + 19\nu_{31f}$, $\gamma_{31} = -5\nu_{31a} - 36\nu_{31b} - 5\nu_{31c} + 26\nu_{31d} + 26\nu_{31e} - 5\nu_{31f}$, $\gamma_{32} = 26\nu_{31a}-5\nu_{31b}-5\nu_{31c}+26\nu_{31d}-36\nu_{31e}-5\nu_{31f}$, $\gamma_{33} = 26\nu_{31a}-5\nu_{31b}-5\nu_{31c}-36\nu_{31d}-5\nu_{31e}+26\nu_{31f}$, $\gamma_{34} = -5\nu_{31a} + 26\nu_{31b} + 26\nu_{31c} - 5\nu_{31d} - 5\nu_{31e} - 36\nu_{31f}$, $\gamma_{35} = -5\nu_{31a} + 26\nu_{31b} - 36\nu_{31c} - 5\nu_{31d} + 26\nu_{31e} - 5\nu_{31f}$, $\gamma_{36} = -36\nu_{31a} - 5\nu_{31b} + 26\nu_{31c} - 5\nu_{31d} - 5\nu_{31e} + 26\nu_{31f}$, $\gamma_{37} = 30\nu_{31a} + 30\nu_{31b} - \nu_{31c} - 32\nu_{31d} - 32\nu_{31e} - \nu_{31f}$, $\gamma_{38} = -32\nu_{31a} - \nu_{31b} - \nu_{31c} - 32\nu_{31d} + 30\nu_{31e} + 30\nu_{31f}$, $\gamma_{39} = -32\nu_{31a} - \nu_{31b} + 30\nu_{31c} + 30\nu_{31d} - \nu_{31e} - 32\nu_{31f}$, $\gamma_{40} = -\nu_{31a} - 32\nu_{31b} - 32\nu_{31c} - \nu_{31d} + 30\nu_{31e} + 30\nu_{31f}$, $\gamma_{41} = -\nu_{31a}-32\nu_{31b}+30\nu_{31c}+30\nu_{31d}-32\nu_{31e}-\nu_{31f}$, $\gamma_{42} = 30\nu_{31a}+30\nu_{31b}-32\nu_{31c}-\nu_{31d}-\nu_{31e}-32\nu_{31f}$. It follows that the only possible integer solutions for $(\nu_{5a}, \nu_{5b}, \nu_{5c}, \nu_{5d}, \nu_{5e}, \nu_{5f})$ are $(1,0,0,0,0,0)$, $(0,0,0,0,0,1)$, $(0,0,0,0,1,0)$, $(0,0,0,1,0,0)$, $(0,0,1,0,0,0)$, $(0,1,0,0,0,0)$. Therefore, u is rationally conjugated to some element $g \in G$ by Proposition 1.7.

Case (vii). Let $u \in V(\mathbb{Z}G)$ where $|u| = 35$. Using Propositions 1.5 & 1.6,

$$\nu_{5a} + \nu_{7a} + \nu_{7b} = 1.$$

Consider the cases $\chi(u^7) = \chi(5a)$ and $\chi(u^5) = m_1\chi(7a) + m_2\chi(7b)$ where $(m_1, m_2) \in \{(0,1),(1,0)\}$. Applying Proposition 1.8, we obtain:

$$\mu_0(u,\chi_2,2) = \tfrac{1}{35}(36\gamma + 14) \geq 0; \quad \mu_0(u,\chi_4,2) = \tfrac{1}{35}(-12\gamma + 7) \geq 0$$

where $\gamma = \nu_{7a} + \nu_{7b}$. It follows that there are no possible integer solutions for $(\nu_{5a}, \nu_{7a}, \nu_{7b})$.

Case $(viii)$. Let $u \in V(\mathbb{Z}G)$ where $|u| = 62$. Using Propositions 1.5 & 1.6,

$$\nu_{2a} + \nu_{2b} + \nu_{5a} = 1.$$

Let $\gamma_1 = 7\nu_{2a} + 3\nu_{2b}$ and $\gamma_2 = 14\nu_{2a} + 6\nu_{2b} - 5\nu_{31a} - 5\nu_{31b} - 5\nu_{31c} - 5\nu_{31d} - 5\nu_{31e} - 5\nu_{31f}$. We shall now separately consider the following cases involving $\chi(u^n)$ for $n \in \{2, 31\}$:

- $\chi(u^{31}) = 3\chi(2a) - 2\chi(2b)$ and $\chi(u^2) = m_1\chi(31a) + m_2\chi(31b) + m_3\chi(31c) + m_4\chi(31d) + m_5\chi(31e) + m_6\chi(31f)$ where $(m_1, m_2, m_3, m_4, m_5, m_6) \in \{(1,0,0,0,0,0), (0,1,0,0,0,0), (0,0,1,0,0,0),$

$(0,0,0,1,0,0),(0,0,0,0,1,0),(0,0,0,0,0,1)\}$. Applying Proposition 1.8, we obtain:

$$\mu_0(u,\chi_3,*) = \tfrac{1}{62}(120\gamma_1+184) \geq 0; \quad \mu_{31}(u,\chi_3,*) = \tfrac{1}{62}(-120\gamma_1+64) \geq 0.$$

It follows that there are no possible integer solutions for $(\nu_{2a},\nu_{2b},\nu_{31a},\nu_{31b},\nu_{31c},\nu_{31d},\nu_{31e},\nu_{31f})$.

- $\chi(u^{31}) = k_1\chi(2a) + k_2\chi(2b)$ and $\chi(u^2) = m_1\chi(31a) + m_2\chi(31b) + m_3\chi(31c) + m_4\chi(31d) + m_5\chi(31e) + m_6\chi(31f)$. Applying Proposition 1.8, we obtain:

$$\mu_0(u,\chi_2,*) = \tfrac{1}{62}(l_1\gamma_2+l_2) \geq 0; \quad \mu_{31}(u,\chi_2,*) = \tfrac{1}{62}(-l_1\gamma_2-l_2) \geq 0$$

where the values for l_i's, k_j's and corresponding $(m_1,m_2,m_3,m_4,m_5,m_6)$ values are as follows:

(k_1,k_2)	$(m_1,m_2,m_3,m_4,m_5,m_6)$	(l_1,l_2)	(k_1,k_2)	$(m_1,m_2,m_3,m_4,m_5,m_6)$	(l_1,l_2)
$(1,0)$	$(1,0,0,0,0,0)$	$(15,7)$	$(1,0)$	$(0,1,0,0,0,0)$	$(15,7)$
$(1,0)$	$(0,0,1,0,0,0)$	$(15,7)$	$(1,0)$	$(0,0,0,1,0,0)$	$(15,7)$
$(1,0)$	$(0,0,0,0,1,0)$	$(15,7)$	$(1,0)$	$(0,0,0,0,0,1)$	$(15,7)$
$(0,1)$	$(1,0,0,0,0,0)$	$(5,1)$	$(0,1)$	$(0,1,0,0,0,0)$	$(5,1)$
$(0,1)$	$(0,0,1,0,0,0)$	$(5,1)$	$(0,1)$	$(0,0,0,1,0,0)$	$(5,1)$
$(0,1)$	$(0,0,0,0,1,0)$	$(5,1)$	$(0,1)$	$(0,0,0,0,0,1)$	$(5,1)$
$(2,-1)$	$(1,0,0,0,0,0)$	$(15,11)$	$(2,-1)$	$(0,1,0,0,0,0)$	$(15,11)$
$(2,-1)$	$(0,0,1,0,0,0)$	$(15,11)$	$(2,-1)$	$(0,0,0,1,0,0)$	$(15,11)$
$(2,-1)$	$(0,0,0,0,1,0)$	$(15,11)$	$(2,-1)$	$(0,0,0,0,0,1)$	$(15,11)$
$(-1,2)$	$(1,0,0,0,0,0)$	$(15,-1)$	$(-1,2)$	$(0,1,0,0,0,0)$	$(15,-1)$
$(-1,2)$	$(0,0,1,0,0,0)$	$(15,-1)$	$(-1,2)$	$(0,0,0,1,0,0)$	$(15,-1)$
$(-1,2)$	$(0,0,0,0,1,0)$	$(15,-1)$	$(-1,2)$	$(0,0,0,0,0,1)$	$(15,-1)$
$(-2,3)$	$(1,0,0,0,0,0)$	$(3,-1)$	$(-2,3)$	$(0,1,0,0,0,0)$	$(3,-1)$
$(-2,3)$	$(0,0,1,0,0,0)$	$(3,-1)$	$(-2,3)$	$(0,0,0,1,0,0)$	$(3,-1)$
$(-2,3)$	$(0,0,0,0,1,0)$	$(3,-1)$	$(-2,3)$	$(0,0,0,0,0,1)$	$(3,-1)$
$(-3,4)$	$(1,0,0,0,0,0)$	$(5,-3)$	$(-3,4)$	$(0,1,0,0,0,0)$	$(5,-3)$
$(-3,4)$	$(0,0,1,0,0,0)$	$(5,-3)$	$(-3,4)$	$(0,0,0,1,0,0)$	$(5,-3)$
$(-3,4)$	$(0,0,0,0,1,0)$	$(5,-3)$	$(-3,4)$	$(0,0,0,0,0,1)$	$(5,-3)$
$(-4,5)$	$(1,0,0,0,0,0)$	$(15,-13)$	$(-4,5)$	$(0,1,0,0,0,0)$	$(15,-13)$
$(-4,5)$	$(0,0,1,0,0,0)$	$(15,-13)$	$(-4,5)$	$(0,0,0,1,0,0)$	$(15,-13)$
$(-4,5)$	$(0,0,0,0,1,0)$	$(15,-13)$	$(-4,5)$	$(0,0,0,0,0,1)$	$(15,-13)$

It follows that there are no possible integer solutions for $(\nu_{2a},\nu_{2b},\nu_{31a},\nu_{31b},\nu_{31c},\nu_{31d},\nu_{31e},\nu_{31f})$.

Case (ix). Let $u \in V(\mathbb{Z}G)$ where $|u| = 93$. Using Propositions 1.5 & 1.6,

$$\nu_{3a}+\nu_{3b}+\nu_{31a}+\nu_{31b}+\nu_{31c}+\nu_{31d}+\nu_{31e}+\nu_{31f} = 1.$$

Let $\gamma = 6\nu_{3a} - \nu_{31a} - \nu_{31b} - \nu_{31c} - \nu_{31d} - \nu_{31e} - \nu_{31f}$. We shall now separately consider the following cases involving $\chi(u^n)$ for $n \in \{3, 31\}$:

- $\chi(u^{31}) = \chi(3a)$ and $\chi(u^2) = m_1\chi(31a)+m_2\chi(31b)+m_3\chi(31c)+m_4\chi(31d)+m_5\chi(31e)+m_6\chi(31f)$ where $(m_1, m_2, m_3, m_4, m_5, m_6) \in \{(1,0,0,0,0,0), (0,1,0,0,0,0), (0,0,1,0,0,0), (0,0,0,1,0,0), (0,0,0,0,1,0), (0,0,0,0,0,1)\}$. Applying Proposition 1.8, we obtain:

$$\mu_0(u, \chi_2, *) = \tfrac{1}{93}(60\gamma + 12) \geq 0; \quad \mu_{31}(u, \chi_2, *) = \tfrac{1}{93}(-30\gamma - 6) \geq 0.$$

It follows that there are no possible integer solutions for $(\nu_{3a}, \nu_{3b}, \nu_{31a}, \nu_{31b}, \nu_{31c}, \nu_{31d}, \nu_{31e}, \nu_{31f})$.

- $\chi(u^{31}) = \chi(3b)$ and $\chi(u^2) = m_1\chi(31a)+m_2\chi(31b)+m_3\chi(31c)+m_4\chi(31d)+m_5\chi(31e)+m_6\chi(31f)$ where $(m_1, m_2, m_3, m_4, m_5, m_6) \in \{(1,0,0,0,0,0), (0,1,0,0,0,0), (0,0,1,0,0,0), (0,0,0,1,0,0), (0,0,0,0,1,0), (0,0,0,0,0,1)\}$. Applying Proposition 1.8, we obtain:

$$\mu_0(u, \chi_2, *) = \tfrac{1}{93}(60\gamma) \geq 0; \quad \mu_{31}(u, \chi_2, *) = \tfrac{1}{93}(-30\gamma) \geq 0.$$

It follows that there are no possible integer solutions for $(\nu_{3a}, \nu_{3b}, \nu_{31a}, \nu_{31b}, \nu_{31c}, \nu_{31d}, \nu_{31e}, \nu_{31f})$.

- $\chi(u^{31}) = 2\chi(3a)-\chi(3b)$ and $\chi(u^2) = m_1\chi(31a)+m_2\chi(31b)+m_3\chi(31c)+m_4\chi(31d)+m_5\chi(31e)+m_6\chi(31f)$ where $(m_1, m_2, m_3, m_4, m_5, m_6) \in \{(1,0,0,0,0,0), (0,1,0,0,0,0), (0,0,1,0,0,0), (0,0,0,1,0,0), (0,0,0,0,1,0), (0,0,0,0,0,1)\}$. Applying Proposition 1.8, we obtain:

$$\mu_0(u, \chi_2, *) = \tfrac{1}{93}(60\gamma + 24) \geq 0; \quad \mu_{31}(u, \chi_2, *) = \tfrac{1}{93}(-30\gamma - 12) \geq 0.$$

It follows that there are no possible integer solutions for $(\nu_{3a}, \nu_{3b}, \nu_{31a}, \nu_{31b}, \nu_{31c}, \nu_{31d}, \nu_{31e}, \nu_{31f})$.

Case (x). Let $u \in V(\mathbb{Z}G)$ where $|u| = 155$. Using Propositions 1.5 & 1.6,

$$\nu_{5a} + \nu_{31a} + \nu_{31b} + \nu_{31c} + \nu_{31d} + \nu_{31e} + \nu_{31f} = 1.$$

Consider the cases $\chi(u^{13}) = \chi(5a)$ and $\chi(u^5) = m_1\chi(31a) + m_2\chi(31b) + m_3\chi(31c) + m_4\chi(31d) + m_5\chi(31e)+m_6\chi(31f)$ where $(m_1, m_2, m_3, m_4, m_5, m_6) \in \{(1,0,0,0,0,0), (0,1,0,0,0,0), (0,0,1,0,0,0), (0,0,0,1,0,0), (0,0,0,0,1,0), (0,0,0,0,0,1)\}$. Let $\gamma_1 = 5\nu_{31a} + 5\nu_{31b} + 5\nu_{31c} + 5\nu_{31d} - 26\nu_{31e} + 5\nu_{31f}$, $\gamma_2 = 5\nu_{31a} + 5\nu_{31b} + 5\nu_{31c} - 26\nu_{31d} + 5\nu_{31e} + 5\nu_{31f}$, $\gamma_3 = 26\nu_{31a} - 5\nu_{31b} - 5\nu_{31c} - 5\nu_{31d} - 5\nu_{31e} - 5\nu_{31f}$, $\gamma_4 = 5\nu_{31a} + 5\nu_{31b} - 26\nu_{31c} + 5\nu_{31d} + 5\nu_{31e} + 5\nu_{31f}$, $\gamma_5 = 5\nu_{31a} - 26\nu_{31b} + 5\nu_{31c} + 5\nu_{31d} + 5\nu_{31e} + 5\nu_{31f}$

and $\gamma_6 = \nu_{31a} + \nu_{31b} + \nu_{31c} + \nu_{31d} + \nu_{31e} + \nu_{31f}$. Applying Proposition 1.8, we obtain:

$$\mu_1(u, \chi_2, 2) = \tfrac{1}{155}(\gamma_1 + k_1) \geq 0; \quad \mu_{35}(u, \chi_2, 2) = \tfrac{1}{155}(-4\gamma_1 + k_2) \geq 0;$$
$$\mu_3(u, \chi_2, 2) = \tfrac{1}{155}(\gamma_2 + k_3) \geq 0; \quad \mu_{55}(u, \chi_2, 2) = \tfrac{1}{155}(-4\gamma_2 + k_4) \geq 0;$$
$$\mu_5(u, \chi_2, 2) = \tfrac{1}{155}(4\gamma_3 + k_5) \geq 0; \quad \mu_9(u, \chi_2, 2) = \tfrac{1}{155}(-\gamma_3 + k_6) \geq 0;$$
$$\mu_7(u, \chi_2, 2) = \tfrac{1}{155}(\gamma_4 + k_7) \geq 0; \quad \mu_{25}(u, \chi_2, 2) = \tfrac{1}{155}(-4\gamma_4 + k_8) \geq 0;$$
$$\mu_{11}(u, \chi_2, 2) = \tfrac{1}{155}(\gamma_5 + k_9) \geq 0; \quad \mu_{75}(u, \chi_2, 2) = \tfrac{1}{155}(-4\gamma_4 + k_{10}) \geq 0;$$
$$\mu_1(u, \chi_2, *) = \tfrac{1}{155}(-\gamma_6 + 31) \geq 0$$

where the values for k_i's and corresponding $(m_1, m_2, m_3, m_4, m_5, m_6)$ values are as follows:

$(m_1, m_2, m_3, m_4, m_5, m_6)$	$(k_1, k_2, k_3, k_4, k_5, k_6, k_7, k_8, k_9, k_{10})$
$(1, 0, 0, 0, 0, 0)$	$(31, 31, 0, 0, 0, 0, 0, 0, 0, 0)$
$(0, 1, 0, 0, 0, 0)$	$(0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$
$(0, 0, 1, 0, 0, 0)$	$(0, 0, 0, 0, 31, 31, 0, 0, 0,)$
$(0, 0, 0, 1, 0, 0)$	$(0, 0, 0, 0, 0, 0, 0, 0, 31, 31)$
$(0, 0, 0, 0, 1, 0)$	$(0, 0, 0, 0, 0, 0, 31, 31, 0,)$
$(0, 0, 0, 0, 0, 1)$	$(0, 0, 31, 31, 0, 0, 0, 0, 0, 0)$

It follows that there are no possible integer solutions for $(\nu_{5a}, \nu_{31a}, \nu_{31b}, \nu_{31c}, \nu_{31d}, \nu_{31e}, \nu_{31f})$.

Case (xi). Let $u \in V(\mathbb{Z}G)$ where $|u| = 217$. Using Propositions 1.5 & 1.6,

$$\nu_{5a} + \nu_{31a} + \nu_{31b} + \nu_{31c} + \nu_{31d} + \nu_{31e} + \nu_{31f} = 1.$$

Consider the cases $\chi(u^{31}) = k_1\chi(7a) + k_2\chi(7b)$ and $\chi(u^5) = m_1\chi(31a) + m_2\chi(31b) + m_3\chi(31c) + m_4\chi(31d) + m_5\chi(31e) + m_6\chi(31f)$ where $(k_1, k_2) \in \{(1, 0), (0, 1)\}$ and $(m_1, m_2, m_3, m_4, m_5, m_6) \in \{(1, 0, 0, 0, 0, 0), (0, 1, 0, 0, 0, 0), (0, 0, 1, 0, 0, 0), (0, 0, 0, 1, 0, 0), (0, 0, 0, 0, 1, 0), (0, 0, 0, 0, 0, 1)\}$. Applying Proposition 1.8, we obtain:

$$\mu_0(u, \chi_2, *) = \tfrac{1}{217}(180\gamma + 12) \geq 0; \quad \mu_{31}(u, \chi_2, *) = \tfrac{1}{217}(-30\gamma + 30\nu_{31f} - 2) \geq 0$$

where $\gamma = 2\nu_{7a} + 2\nu_{7b} - \nu_{31a} - \nu_{31b} - \nu_{31c} - \nu_{31d} - \nu_{31e} - \nu_{31f}$. It follows that there are no possible integer solutions for $(\nu_{7a}, \nu_{7a}, \nu_{31a}, \nu_{31b}, \nu_{31c}, \nu_{31d}, \nu_{31e}, \nu_{31f})$.

We shall now consider the prime graph of $G = PSL(5, 2)$. G contains elements of order 6, 14, 15, 21. Therefore $[2, 3]$, $[2, 7]$, $[3, 5]$ and $[3, 7]$ are adjacent in $\pi(G)$ and consequently adjacent in $\pi(V(\mathbb{Z}G))$. Clearly $\pi(G) = \pi(V(\mathbb{Z}G))$, since there are no torsion units of order 10, 35, 62, 93, 155 and 217 in $V(\mathbb{Z}G)$. This completes the proof.

Table 1.

Possible solutions for $(\nu_{3a}, \nu_{3b}, \nu_{3c}, \nu_{3d})$ when $\lvert u\rvert = 3$
(4,-1,-1,-1),(3,-1,-1,0),(2,-1,-1,1),(1,-1,-1,2),(0,-1,-1,3),(-1,-1,-1,4),(-2,-1,-1,5),(4,-1,0,-2),(3,-1,0,-1),
(2,-1,0,0),(1,-1,0,1),(0,-1,0,2),(-1,-1,0,3),(-2,-1,0,4),(4,-1,1,-3),(3,-1,1,-2),(2,-1,1,-1),(1,-1,1,0),
(0,-1,1,1),(-1,-1,1,2),(-2,-1,1,3),(4,-1,2,-4),(3,-1,2,-3),(2,-1,2,-2),(1,-1,2,-1),(0,-1,2,0),(-1,-1,2,1),
(-2,-1,2,2),(4,-1,3,-5),(3,-1,3,-4),(2,-1,3,-3),(1,-1,3,-2),(0,-1,3,-1),(-1,-1,3,0),(-2,-1,3,1),(4,0,-1,-2),
(3,0,-1,-1),(2,0,-1,0),(1,0,-1,1),(0,0,-1,2),(-1,0,-1,3),(-2,0,-1,4),(4,0,0,-3),(3,0,0,-2),(2,0,0,-1),
(1,0,0,0),(0,0,0,1),(-1,0,0,2),(-2,0,0,3),(4,0,1,-4),(3,0,1,-3),(2,0,1,-2),(1,0,1,-1),(0,0,1,0),
(-1,0,1,1),(-2,0,1,2),(4,0,2,-5),(3,0,2,-4),(2,0,2,-3),(1,0,2,-2),(0,0,2,-1),(-1,0,2,0),(-2,0,2,1),
(4,0,3,-6),(3,0,3,-5),(2,0,3,-4),(1,0,3,-3),(0,0,3,-2),(-1,0,3,-1),(-2,0,3,0),(4,1,-1,-3),(3,1,-1,-2),
(2,1,-1,-1),(1,1,-1,0),(0,1,-1,1),(-1,1,-1,2),(-2,1,-1,3),(4,1,0,-4),(3,1,0,-3),(2,1,0,-2),(1,1,0,-1),
(0,1,0,0),(-1,1,0,1),(-2,1,0,2),(4,1,1,-5),(3,1,1,-4),(2,1,1,-3),(1,1,1,-2),(0,1,1,-1),(-1,1,1,0),
(-2,1,1,1),(4,1,2,-6),(3,1,2,-5),(2,1,2,-4),(1,1,2,-3),(0,1,2,-2),(-1,1,2,-1),(-2,1,2,0),(4,1,3,-7),
(3,1,3,-6),(2,1,3,-5),(1,1,3,-4),(0,1,3,-3),(-1,1,3,-2),(-2,1,3,-1),(4,2,-1,-4),(3,2,-1,-3),(2,2,-1,-2),
(1,2,-1,-1),(0,2,-1,0),(-1,2,-1,1),(-2,2,-1,2),(4,2,0,-5),(3,2,0,-4),(2,2,0,-3),(1,2,0,-2),(0,2,0,-1),
(-1,2,0,0),(-2,2,0,1),(4,2,1,-6),(3,2,1,-5),(2,2,1,-4),(1,2,1,-3),(0,2,1,-2),(-1,2,1,-1),(-2,2,1,0),
(4,2,2,-7),(3,2,2,-6),(2,2,2,-5),(1,2,2,-4),(0,2,2,-3),(-1,2,2,-2),(-2,2,2,-1),(4,2,3,-8),(3,2,3,-7),
(2,2,3,-6),(1,2,3,-5),(0,2,3,-4),(-1,2,3,-3),(-2,2,3,-2),(4,3,-1,-5),(3,3,-1,-4),(2,3,-1,-3),(1,3,-1,-2),
(0,3,-1,-1),(-1,3,-1,0),(-2,3,-1,1),(4,3,0,-6),(3,3,0,-5),(2,3,0,-4),(1,3,0,-3),(0,3,0,-2),(-1,3,0,-1),
(-2,3,0,0),(4,3,1,-7),(3,3,1,-6),(2,3,1,-5),(1,3,1,-4),(0,3,1,-3),(-1,3,1,-2),(-2,3,1,-1),(4,3,2,-8),
(3,3,2,-7),(2,3,2,-6),(1,3,2,-5),(0,3,2,-4),(-1,3,2,-3),(-2,3,2,-2),(4,3,3,-9),(3,3,3,-8),(2,3,3,-7),
(1,3,3,-6),(0,3,3,-5),(-1,3,3,-4),(-2,3,3,-3)

References

[1] V. Artamonov and A. Bovdi, Integral group rings: groups of invertible elements and classical K-theory, (Algebra. Topology. Geometry, **27** (Russian), Itogi Nauki i Tekhniki, Akad. Nauk SSSR Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., 1989 3–43.), (Russian) Translated in *J. Soviet Math.*, **57** (1991) 2931–2958.

[2] A. Bachle and L. Margolisv, Rational conjugacy of torsion units in integral group rings of non-solvable groups, *arXiv:1305.7419*.

[3] V. Bovdi, A. Grishkov and A. Konovalov, Kimmerle conjecture for the Held and O'Nan sporadic simple groups, *Sci. Math. Jpn.*, **69** (2009) 353–361.

[4] V. Bovdi and M. Hertweck, Zassenhaus conjecture for central extensions of S_5, *J. Group Theory*, **11** (2008) 63–74.

[5] V. Bovdi, C. Höfert and W. Kimmerle, On the first Zassenhaus conjecture for integral group rings, *Publ. Math. Debrecen*, **65** (2004) 291–303.

[6] V. Bovdi, E. Jespers and A. Konovalov, Torsion units in integral group rings of Janko simple groups, *Math. Comp.*, **80** (2011) 593–615.

[7] V. Bovdi and A. Konovalov, *Integral group ring of the first Mathieu simple group*, Groups St. Andrews 2005., London Math. Soc. Lecture Note Ser., Cambridge Univ. Press, Cambridge, **1** (2007) 237–245.

[8] V. Bovdi, A. Konovalov, Integral group ring of the Mathieu simple group M_{23}, *Comm. Algebra*, **36** no. 7 (2008) 2670–2680.

[9] V. Bovdi, A. Konovalov, Integral group ring of Rudvalis simple group (English, with English and Ukrainian summaries, Ukraïn. Mat. Zh., 61(1), 2009, 3–13), *Ukrainian Math. J.*, **61** no. 1 (2009) 1–13.

[10] V. Bovdi, A. Konovalov, Torsion units in integral group ring of Higman-Sims simple group, *Studia Sci. Math. Hungar.*, **47** no. 1 (2010) 1–11.

[11] V. Bovdi, A. Konovalov, Integral group ring of the McLaughlin simple group, *Algebra Discrete Math.*, **2** (2007) 43–53.

[12] V. Bovdi, A. Konovalov, Integral group ring of the Mathieu simple group M_{24}, *J. Algebra Appl.*, **1** (2012) 10 pages.

[13] V. Bovdi, A. Konovalov. S. Linton, Torsion units in integral group ring of the Mathieu simple group M_{22}, *LMS J. Comput. Math.*, **11** (2008) 28–39.

[14] V. Bovdi, A. Konovalov. S. Linton, Torsion units in integral group rings of Conway simple groups M_{22}, *Internat. J. Algebra Comput.*, **21** no. 4 (2011) 615–634.

[15] V. Bovdi, A. Konovalov. E. Marcos, Integral group ring of the Suzuki sporadic simple group, *Publ. Math. Debrecen*, **72** no. 3-4 (2008) 487–503.

[16] V. Bovdi, A. Konovalov. S. Siciliano, Integral group ring of the Mathieu simple group M_{12}, *Rend. Circ. Mat. Palermo (2)*, **56** no. 1 (2007) 125–136.

[17] J. Cohn, D. Livingstone, On the structure of group algebras. I, *Canad. J. Math.*, **17** (1965) 583–593.

[18] M. Caicedo, L. Margolis, Á. del Río, Zassenhaus conjecture for cyclic-by-abelian groups, *J. Lond. Math. Soc. (2)*, **88** (2013) 65–78.

[19] J. Gildea, Zassenhaus conjecture for integral group ring of simple linear groups, *J. Algebra Appl.*, **12** (2013) 10 pages.

[20] M. Hertweck, On the torsion units of some integral group rings, *Algebra Colloq.*, **13** (2006) 329–348.

[21] M. Hertweck, Zassenhaus conjecture for A_6, *Proc. Indian Acad. Sci. Math. Sci.*, **118** (2008) 189–195.

[22] M. Hertweck,*Partial augmentations and Brauer character values of torsion units in group rings*, 2007, http://arxiv.org/abs/math/0612429.

[23] C. Höfert and W. Kimmerle, On torsion units of integral group rings of groups of small order (Groups, rings and group rings, Chapman & Hall/CRC, Boca Raton, FL), *Lect. Notes Pure Appl. Math.*, **248** (2006) 243–252.

[24] The GAP Group, GAP- Groups, Algorithms and Programming, version 4.4, 2006 http:/www.gap-system.org.

[25] W. Kimmerle,*On the prime graph of the unit group of integral group rings of finite groups*, Groups, rings and algebras, 215–228, Contemp. Math., **420**, Amer. Math. Soc., Providence, RI, 2006.

[26] V. Bovdi, A. Konovalov, R. Rossmanith and C. Schneider, LAGUNA — Lie AlGebras and UNits of group Algebras, Version 3.5.0; 2009, http://www.cs.st-andrews.ac.uk/~alexk/laguna.htm.

[27] I. Luther, I. Passi, Zassenhaus conjecture for A_5, *Proc. Indian Acad. Sci. Math. Sci.*, **99** (1989) 1–5.

[28] I. Luther and P. Trama, Zassenhaus conjecture for S_5, *Comm. Algebra*, **19** no. 8 (1991) 2353–2362.

[29] Z. Marciniak, J. Ritter, S. Sehgal and A. Weiss, Torsion units in integral group rings of some metabelian groups. II, *J. Number Theory*, **25** (1987) 340–352.

[30] Mini-Workshop, Arithmetik von Gruppenringen, Oberwolfach Rep. 4 (2007), no. 4, $3209-3239$, Abstracts from the mini-workshop held November $25-$ December 1, 2007, Organized by E. Jespers, Z. Marciniak, G. Nebe and W. Kimmerle, Oberwolfach Reports. Vol. 4, no. 4.

[31] A. Rosa, Torsion units of integral group ring of the simple group $S_4(4)$, *Miskolc Math. Notes*, **16** (2015) 443–452.

[32] K. Roggenkamp and L. Scott, Isomorphisms of p-adic group rings, *Ann. of Math. (2)*, **126** (1987) 593–647.

[33] M. Salim, Torsion units in the integral group ring of the alternating group of degree 6, *Comm. Algebra*, **35** (2007) 4198–4204.

[34] M. Salim, Kimmerle's conjecture for integral group rings of some alternating groups, *Acta Math. Acad. Paedagog. Nyházi. (N.S.)*, **27** (2011) 9–22.

[35] M. Salim, The prime graph conjecture for integral group rings of some alternating groups, *Int. J. Group Theory*, **2** (2013) 175–185.

[36] A. Weiss, Rigidity of p-adic p-torsion, *Ann. of Math. (2)*, **127** (1988) 317–332.

[37] H. Zassenhaus, *On the torsion units of finite group rings*, Studies in mathematics (in honor of A. Almeida Costa) (Portuguese), Instituto de Alta, Lisbon, 1974, 119–126.

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