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# ONE-PRIME POWER HYPOTHESIS FOR CONJUGACY CLASS SIZES

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Abstract. A finite group $G$ satisfies the one-prime power hypothesis for conjugacy class sizes if any two conjugacy class sizes $m$ and $n$ are either equal or have common divisor a prime power. Taeri conjectured that an insoluble group satisfying this condition is isomorphic to $S \times A$ where $A$ is abelian and $S \cong PSL_2(q)$ for $q \in \{4, 8\}$. We confirm this conjecture.

## 1. Introduction

To determine structural information about a finite group $G$ given the set of conjugacy class sizes of $G$ is an ongoing line of research, see [4] for an overview. How the arithmetic data given by the set of conjugacy class sizes is encoded varies, but one representation is via the bipartite graph $B(X)$. Let $X$ be a set of positive integers and let $X^* = X \setminus 1$ ($X$ may or may not contain the element 1). If $x \in X$ we denote the set of prime divisors of $x$ by $\pi(x)$ and let $\rho(X) = \bigcup_{x \in X} \pi(x)$.

**Definition 1.1.** [7] *The vertex set of $B(X)$ is given by the disjoint union of $X^*$ and $\rho(X)$. There is an edge between $p \in \rho(X)$ and $x \in X^*$ if $p$ divides $x$, i.e. if $p \in \pi(x)$.*

In our context we let $X$ be the set of conjugacy class sizes of a finite group $G$, and in this case we denote $B(X)$ by $B(G)$. In [14] Taeri investigates the case when $B(G)$ is a cycle, or contains no cycle of length 4. In particular, he proves the following.

**Theorem 1.2.** [14] *Let $G$ be a finite group and $Z(G)$ the centre of $G$. Suppose $G/Z(G)$ is simple, then $B(G)$ has no cycle of length 4 if and only if $G \cong A \times S$, where $A$ is abelian, and $S \cong PSL_2(q)$ for $q \in \{4, 8\}$.*

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Taeri goes on to conjecture that the same conclusion holds if the assumption is just that $G$ is finite and insoluble. In this paper we confirm Taeri's conjecture.

**Main Theorem.** If $G$ is a finite insoluble group, then $B(G)$ has no cycle of length 4 if and only if $G \cong A \times S$, where $A$ is abelian and $S \cong PSL_2(q)$ for $q \in \{4, 8\}$.

As Taeri comments, $B(G)$ having no cycle of length 4 is equivalent to $G$ satisfying the *one-prime power hypothesis*, that is, if $m$ and $n$ are distinct non-trivial conjugacy class sizes of $G$ then either $m$ and $n$ are coprime or their greatest common divisor is a prime power. This is similar to the one-prime hypothesis introduced by Lewis to study character degrees [11]. We use this terminology.

Throughout the paper $G$ will be assumed to be a finite group. Most of the notation used will be standard. In particular, $Z(G)$ is the centre of $G$, the maximal normal soluble subgroup of $G$ is denoted by $S(G)$, the maximal normal $p$-subgroup of $G$ is denoted $O_p(G)$ and the Fitting and second Fitting subgroups are denoted by $F(G)$ and $F_2(G)$ respectively. The conjugacy class size of an element $x \in G$ will be denoted by $|x^G|$ and shall be called the *index* of $x \in G$. We say an element has *mixed index* if its index is not a prime power. The greatest common divisor of two numbers $m$ and $n$ shall be denoted by $(m, n)$ and $p$ will always be prime.

## 2. Preliminary Remarks

We begin by making some preliminary remarks.

**Lemma 2.1.** *Suppose $N$ is a normal subgroup of a group $G$.*

*(i) Let $x \in N$, then $|x^N|$ divides $|x^G|$.*

*(ii) Let $\bar{x} \in G/N = \bar{G}$, then $|\bar{x}^{\bar{G}}|$ divides $|x^G|$.*

Let $C_G(x)$ be the centraliser of an element $x$ in $G$. Then $C_G(x)$ is said to be minimal if $C_G(y) \leq C_G(x)$ for some $y \in G$ implies $C_G(y) = C_G(x)$. The following lemma is well-known.

**Lemma 2.2.** *Suppose $x$ is a $p$-element with minimal centraliser. Then $C_G(x) = P_0 \times A$, where $P_0$ is a $p$-group and $A$ is abelian.*

We have the following lemma.

**Lemma 2.3.** *Assume $G$ satisfies the one-prime power hypothesis and there exists $x, y \in G$ with $C_G(x) < C_G(y)$. Then $|y^G|$ is a prime power.*

*Proof.* Let $|x^G| = m$ and $|y^G| = n$, then $(m, n) = n$ and hence $n$ is a prime power, i.e. any non-minimal centraliser has prime power index. □

The following result will prove useful.

**Proposition 2.4.** [3, Theorem 1] *All elements of prime power index in $G$ lie in $F_2(G)$.*

Recall, $G$ is called an $F$-group if whenever $x$ and $y$ are non-central elements of $G$ satisfying $C_G(x) \leq C_G(y)$, then $C_G(x) = C_G(y)$. Rebmann has classified $F$-groups [13].

**Lemma 2.5.** *(i) Suppose $G$ satisfies the one-prime power hypothesis and $F(G)$, the Fitting subgroup of $G$, is central. Then $G$ is an $F$-group.*

*(ii)*[14] *Suppose $G$ is an insoluble $F$-group that satisfies the one-prime power hypothesis. Then $G \cong S \times A$ where $S \cong PSL_2(q)$ for $q \in \{4, 8\}$ and $A$ is abelian.*

*Proof.* (i) As $F(G)$ is central so is $F_2(G)$ and thus $G$ has no elements of prime power index by Proposition 2.4. Applying Lemma 2.3 gives that $G$ is an $F$-group.
(ii) This is a combination of [14, Lemma 4] and [14, Theorem 1]. □

Consider the following property. Let $G$ be a finite non-abelian group with proper normal subgroup $N$ and suppose all the conjugacy class sizes outside of $N$ have equal sizes. Isaacs proved that in this situation then either $G/N$ is cyclic, or else every non-identity element of $G/N$ has prime order [8]. We combine this result with Proposition 2.4 and a result of Qian to give the following lemma.

**Lemma 2.6.** *Suppose $G$ is a finite group with at most one conjugacy class size that is not a prime power. Then either $G$ is soluble or $G/F_2(G) \cong PSL_2(4)$.*

*Proof.* By Proposition 2.4 all elements outside of $F_2(G)$ have the same conjugacy class size. Applying [8] gives that $G/F_2(G)$ is a non-soluble group with all elements of prime order. The result follows from [12]. □

This lemma leads us to ask the following question. Suppose $G$ is a finite group with at most one conjugacy class that is not a prime power, does it follow that $G$ is soluble?

Groups in which all elements have prime power order are well studied and are called CP-groups. Delgado and Wu have given a full description of locally finite CP-groups, the following considers the special case when the Fitting subgroup is trivial.

**Theorem 2.7.** [5] *Let $G$ be a finite CP-group with trivial Fitting subgroup. Then either $G$ is simple and isomorphic to one of $PSL_2(q)$ where $q \in \{4, 7, 8, 9, 17\}$, $PSL_3(4)$, $Sz(8)$, $Sz(32)$ or $G$ is isomorphic to $M_{10}$.*

The following observation is useful.

**Lemma 2.8.** *Suppose $G$ satisfies the one-prime power hypothesis and that $N$ is a normal subgroup of $G$. If $\bar{x} \in \bar{G} = G/N$ has mixed index in $\bar{G}$, then $|x^G| = |(xn)^G|$ for all $n \in N$.*

*Proof.* Note that $|\bar{x}^{\bar{G}}|$ divides both $|x^G|$ and $|(xn)^G|$. So, by the one-prime power hypothesis, the result follows. □

## 3. Main Result

The property of satisfying the one-prime power hypothesis does not (clearly) restrict to normal subgroups (however we know of no examples where this is not the case). We do have the following.

**Lemma 3.1.** *Suppose $G$ satisfies the one-prime power hypothesis and $r$ is a prime dividing $|G|$. If $N$ is a normal $r$-complement in $G$ then $N$ also satisfies the one-prime power hypothesis.*

*Proof.* Suppose not, then there exist $x, y \in N$ with $|x^N| \neq |y^N|$ and distinct primes $p$ and $q$ with $pq$ dividing both $|x^N|$ and $|y^N|$. As $G$ satisfies the one-prime power hypothesis this forces $|x^G| = |y^G|$. However note that $\frac{|x^G|}{|x^N|}$ divides $|G/N|$ and is thus a power of $r$, and similarly for $y$, so $|x^G| \neq |y^G|$, a contradiction. □

We first consider the case where there is only one mixed index.

**Proposition 3.2.** *Suppose $G$ satisfies the one-prime power hypothesis and all elements of mixed index have index $m$. Then $G$ is soluble.*

*Proof.* By Lemma 2.6 we can assume $G/F_2(G)$ is isomorphic to $PSL_2(4)$. Furthermore, if there exists a prime power index, say $r^a$ with $r$ not dividing $m$ then $G$ is quasi-Frobenius and hence soluble by [10]. So we can assume otherwise.

Let $\bar{G} = G/F_2(G)$. Since $\bar{G}$ has elements of index 12, 15 and 20 we see that $m$ is divisible by 60. Let $x \in G$ with $\bar{x}$ of order 2. Then $|\bar{x}^{\bar{G}}| = 15$. But in $G$ the index of $x$ has to be $m$, so we see that $F_2(G)$ has to have a non-central 2-subgroup. We can argue similarly to show $F_2(G)$ has to have non-central 3 and 5 subgroups.

Suppose $x, y \in F_2(G)$, that $x$ and $y$ commute and have coprime orders. Suppose further that $|x^G| = p^a$ and $|y^G| = q^b$. If $p \neq q$ then $|(xy)^G|$ is divisible by just two different primes and so cannot equal $m$, a contradiction.

So assume $x, y \in F_2(G)$ with $|x^G| = p^a$, $|y^G| = q^b$ and $p \neq q$. Given that the indices of $x$ and $y$ are prime powers we can assume that each of $x$ and $y$ have prime power orders. Assume first that the orders of $x$ and $y$ are coprime. $C_G(x)$ contains a Sylow $r$-subgroup of $G$ for each prime $r \neq p$. If $y$ is not a $p$-element it, or some conjugate of it, is in $C_G(x)$ which contradicts the above assertion. So $y$ is a $p$-element and $x$ is $q$-element. Let $r$ be a prime distinct from $p$ and $q$ and dividing the order $G/F_2(G)$.

Both $C_G(x)$ and $C_G(y)$ can be assumed to contain a Sylow $r$-subgroup of $G$. Let $u$ be an $r$-element of mixed index, there is one because $r$ divides the order of $G/F_2(G)$. Taking conjugates we can assume $x, y \in C_G(u)$. By Lemma 2.2, $C_G(u) = R_0 \times A$ where $A$ is an abelian $r'$-subgroup which must contain both $x$ and $y$, a contradiction as $x$ and $y$ do not commute. So if $x, y \in F_2(G)$ with $|x^G| = p^a$ and $|y^G| = q^b$ with $p \neq q$ then $x$ and $y$ are both $l$-elements for some prime $l$. If there is an $l'$-element of prime power index then we can apply the previous argument. So every $l'$-element has mixed index. So $G$ satisfies the hypothesis that every $l'$-element of $G$ has the same index, using [2], we get $G$ is soluble. We end this paragraph by noting that if the proposition is not true then there is a prime $p$ so that every element, $x$, of prime power index has $|x^G| = p^a$ for some $a$.

Note that if $M$ is the subgroup generated by all the elements of prime power index then $M \subseteq F_2(G)$ and every element not in $M$ has index $m$. As $G/M$ is not soluble it is isomorphic to $PSL_2(4)$ and so $M = F_2(G)$.

Let $t$ be a prime such that $t \neq p$. Any element of prime power index contains a Sylow $t$-subgroup of $G$ in its centraliser and so centralises $O_t(G)$. Now $O_t(G) \subseteq Z(F_2(G))$. As $F_2(G)$ is metanilpotent if $P$ is the Sylow $p$-subgroup of $F_2(G)$ then $PF$ is normal in $F_2(G)$. But $PF = PU$ where $U$ is the product of $O_t$ for all $t \neq p$. So $U$ is central in $F_2(G)$ and hence $PF = P \times U$ and $P$ is normal in $G$.

There exist $p$-elements of mixed index otherwise all $p$-elements of $G$ have $p$-power index and $G = P \times H$ for $H$ some $p'$-subgroup of $G$, by [3], but such a group cannot satisfy the conditions of the proposition. Assume that there exists a $p$-element $x$ of mixed index in $F_2(G)$ so $x \in P$. Then $C_G(x) = P_0 \times A_0$ where $P_0$ is a $p$-group and $A_0$ is an abelian $p'$-group. Let $m = p^e m_0$ where $(m_0, p) = 1$, then $[G : A_0] = p^f m_0$ for some $f$. Also $A_0$ cannot be central in $G$ otherwise there would be no $p'$-elements of mixed index which is false. Then $A_0 \subseteq C_G(P)$, by an application of Thompson's Lemma [6, 5.3.4]. As $x \in P$, $A_0$ is the Hall $p'$-subgroup of $C_G(P) = Z(P) \times A_0$. So $A_0$ is a normal abelian $p'$-subgroup of $G$. Furthermore, $A_0$ is central in $F_2(G)$ as it commutes with all elements that generate $F_2$ and since it is not central it follows that $m = 60$ and thus $p$ is a divisor of 60. So, there exists a $p$-element, say $y$, of mixed index not in $F_2(G)$. Then $C_G(y) = P_1 \times A_1$ and, again by [6, 5.3.4], $A_1$ centralises $P$ but $|A_1| = |A_0|$ as $x$ and $y$ have the same index. This implies that $C_G(A_0) > F_2(G)$ so $A_0$ is central in $G$, a contradiction.

The last case to consider is that there are no elements of mixed index in $P$. That means that all the $p$-elements of $F_2(G)$ have index a power of $p$. By [3] it follows that $F_2(G) = P \times A$ where $A$ has order prime to $p$ and $A$ is normal in $G$ and central in $F_2(G)$. As $A$ is not central we see that $p = 5$. Let $y$ be a $p$-element of mixed index not in $F_2(G)$. Then $C_G(y) = P_1 \times A_1$ and $A_1$ centralises $P$ by [6, 5.3.4]. As $A_1$ is a subgroup of $A$ it centralises $P$ and $y$ but $P$ and $y$ generate the Sylow $p$-subgroup of $G$ and hence $A_1$ is in the centre of $G$. Then no $p'$ element can have mixed index which is false as there are both 2 and 3 elements of mixed index. $\square$

We are now ready to prove the main theorem.

**Theorem 3.3.** *Suppose $G$ is insoluble and satisfies the one-prime power hypothesis. Then $G \cong PSL_2(q) \times A$ for $q \in \{4, 8\}$ where $A$ is abelian.*

*Proof.* We suppose the result is not true and take $G$ to be a counterexample of minimal order.

(i) Case 1: Suppose $\bar{G} = G/F_2(G)$ has elements of mixed order.

Let such an element be $\bar{u}$. Then we can assume $\bar{u}$ has order divisible by precisely two primes, $p$ and $q$ say, and further we can assume $u$ similarly has order divisible by two primes $p$ and $q$. We write $u = xy$ where $x$ and $y$ commute and $x$ has $p$-power order and $y$ has $q$-power order. As $u$ is not an element of $F_2(G)$ it follows that $u$ has mixed index, and as $\bar{u}$ has mixed order we also know that both $x$ and $y$ do not lie in $F_2(G)$ and thus also have mixed index. As $C_G(x)$ is minimal it follows from

Lemma 2.2 that $C_G(x) = P_0 \times A$ where $P_0$ is a $p$-group and $A$ is abelian. A similar statement holds for $C_G(y)$ and thus we obtain that $C_G(u) = C_G(x) = C_G(y)$ and is abelian. Now there exists $z$ an element of mixed index different to $|u^G|$ otherwise all elements of $G/F_2(G)$ would be of prime power order [8]. If $|z^G|$ is coprime to $p$ then $z$ centralises a Sylow $p$-subgroup and a conjugate of $z$ lies in $C_G(x)$, but then the index of $z$ divides the index of $x$, a contradiction. Thus both $p$ and $q$ divide $|z^G|$. So we have shown that there are only two mixed indices of elements of $G$ and these are given by $|x^G|$ and $|z^G|$. Thus, by the one-prime power hypothesis there exist a pair of primes $r$ and $s$ say with $r$ dividing $|x^G|$ and $s$ dividing $|z^G|$ but the product $rs$ does not divide any conjugacy class size in $G$. Thus, by [9, Prop. 5.1], $G$ has a normal $r$-complement (say), call this complement $N$. Then $N$ satisfies the one-prime power hypothesis by Lemma 3.1. If $N$ is soluble so is $G$, so we can assume $N$ is insoluble. Thus, by induction, $N \cong S \times A$ where $A$ is abelian and $S$ is one of the simple groups $PSL_2(q)$ for $q$ equal to 4 or 8. Note $A$ must be central in $G$ as otherwise $G$ does not satisfy the one-prime power hypothesis. However, if $A$ is central in $G$ all $r$-elements have $r$-power index as the outer automorphism groups of these two simple groups have no elements of order $r$. Thus the Sylow $r$-subgroup is a direct factor of $G$ by [3, Theorem A]. As $G$ satisfies the one-prime power hypothesis, this forces the Sylow $r$-subgroup to be central. Thus, $G/Z(G) \cong S$, and all elements of the quotient are of prime power order, a contradiction.

ii) Case 2: Assume all elements of $G/F_2(G)$ have prime power order.

We can assume we have at least one mixed index by Proposition 2.4. If we have precisely one then $G$ is soluble by Proposition 3.2. So we can assume there exist elements of mixed index which are not equal.

Let $\bar{G} = G/F_2(G)$. Let $\bar{x}$ be a $p$-element in $\bar{G}$. As $C_{\bar{G}}(\bar{x})$ is a $p$-group it follows that $|\bar{G}|/|\bar{G}|_p$ divides $|\bar{x}|^{\bar{G}}$ where $|\bar{G}|_p$ denotes the $p$-part of $|\bar{G}|$. A similar statement holds for all elements of $\bar{G}$.

If $|\bar{G}|$ were divisible by more that 3 primes this would force all elements outside of $F_2(G)$ to have the same conjugacy class size in $G$, a contradiction. Thus we can assume $|\bar{G}|$ is divisible by exactly 3 primes. Assume that $p$, $q$, $r$ are the primes that divide the order of $G/F_2(G)$ and there is an element of index divisible by $pqr$. But every element not in $F_2(G)$ has index divisible by at least two of $p, q$ or $r$ so all elements would have the same index which we are assuming is not the case. So we must have that $|x|^G$ is coprime to $p$ and likewise for other elements.

Now, consider $O_t(G) \neq 1$, there exists an element $x \in G \setminus F_2(G)$ such that $|x^G|$ and $t$ are coprime. This follows from the argument above if $t$ divides the order of $|\bar{G}|$. If not, note that the indices of any two elements $y, z \in G \setminus F_2(G)$ already have a prime in common that also divides $|\bar{G}|$. Thus $O_t(G) \leq C_G(x)$. Let $n \in F_2(G)$, then by Lemma 2.8, it follows that $O_t(G) \leq C_G(xn)$ and thus $O_t(G) \leq C_G(n)$. So, $C_G(O_t(G))$ is a normal subgroup of $G$ containing $F_2(G)$. Since $F(G)$ is a direct product of $O_t(G)$ for all $t$, $F(G)$ is central in $F_2(G)$. It follows that $F(G) = F_2(G) = S(G)$.

As $\bar{G}$ has trivial Fitting subgroup it follows from Theorem 2.7 that $\bar{G}$ is a simple group which comes from a known list or is isomorphic to $M_{10}$. However $M_{10}$ has order 720 and an element with index 90, see [1], which contradicts the discussion above. Thus we can assume that $\bar{G}$ is simple. Note that $O_t(G)$, for any $t$, centralises some element not in $S(G)$ so $C_G(O_t(G))$ is a normal subgroup of

$G$ strictly containing $F_2(G)$. But as $\bar{G}$ is simple, $O_t(G)$ is central but then so is $F(G)$. But then, by Lemma 2.5, we have that $G \cong PSL_2(q) \times A$ for $q \in \{4, 8\}$ and $A$ abelian, as required. □

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