www.theoryofgroups.ir

International Journal of Group Theory
ISSN (print): 2251-7650, ISSN (on-line): 2251-7669
Vol. 6 No. 2 (2017), pp. 17–20.

www.ui.ac.ir

A NOTE ON FINITE GROUPS WITH THE INDICES OF SOME MAXIMAL SUBGROUPS BEING PRIMES*

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Communicated by Mark L. Lewis

Abstract. The Theorem 12 in [A note on p-nilpotence and solvability of finite groups, J. Algebra 321 (2009) 1555–1560.] investigated the non-abelian simple groups in which some maximal subgroups have primes indices. In this note we show that this result can be applied to prove that the finite groups in which every non-nilpotent maximal subgroup has prime index are solvable.

1. Introduction

It is known that a theorem of Huppert shows that a finite group G is supersolvable if and only if the index of every maximal subgroup of G is prime. In [3], we investigated the non-abelian simple groups in which some maximal subgroups have primes indices. We had the following result:

Theorem 1.1. [3, Theorem 12] *Let $\pi_t(G)$ be the set of indices of maximal subgroups of a non-abelian simple group G, then there exists at most one prime number in $\pi_t(G)$. Moreover, if there exists a prime number $p \in \pi_t(G)$, then*
(1) *p must be the largest prime divisor of $|G|$,*
(2) *p must be the smallest number in $\pi_t(G)$.*

In [3], as an application of [3, Theorem 12], we proved the result:

Theorem 1.2. [3, Theorem 13] *Let M be a maximal subgroup of G such that M is isomorphic to a direct product of some isomorphic simple groups. Assume that there exists a subgroup $1 \neq N \leq M$*

MSC(2010): Primary: 20D05; Secondary: 20D10.

Keywords: Non-abelian simple group, the index of maximal subgroup, solvable group.

Received: 07 August 2015, Accepted: 06 January 2016.

*This work was supported by NSFC (Grant 11201403).

such that $N \trianglelefteq G$ and for every maximal subgroup K of G which does not contain N we all have $|G : K|$ is a prime. Then G is solvable.

As a further study, the main goal of this note is to give another application of [3, Theorem 12]. Our result is as follows, the proof of which is given in Section 3.

Theorem 1.3. *Let G be a finite group. If every non-nilpotent maximal subgroup of G has prime index then G is solvable.*

Remark 1.4. *(1) For convenience, we call a group in which every non-nilpotent maximal subgroup has prime index an NNMPI-group. The alternating group A_4 implies that an NNMPI-group might not be supersolvable. Moreover, A_4 shows that an NNMPI-group might not be 2-supersolvable.*

(2) Let G be an NNMPI-group and p any odd prime divisor of $|G|$. Can we ensure that G is p-supersolvable? The answer is negative. For example, let $G = \langle a, b, c \,|\, a^3 = b^3 = c^4 = [a, b] = 1, a^c = b^{-1}, b^c = a\rangle$. It is easy to see that G is not 3-supersolvable since $\langle a, b\rangle$ is the unique minimal normal subgroup of G but the order of $\langle a, b\rangle$ is 9.

(3) The alternating group A_4 shows that an NNMPI-group might not have a Sylow tower of supersolvable type.

2. Preliminaries

Here we first list two lemmas that are needed in the proof of Theorem 1.3.

Lemma 2.1. [2] *A non-abelian simple group with a nilpotent maximal subgroup can only be a $\mathrm{PSL}(2, p)$, where p is a Fermat or Mersenne prime with $p \geq 17$.*

Lemma 2.2. [2, Theorem 1] *Suppose that G is a finite non-solvable group with a nilpotent maximal subgroup M. If $Z(G) = 1$ then M is a Sylow 2-subgroup of G.*

3. Proof of Theorem 1.3

Let G be a counterexample of minimal order.

If G has no nilpotent maximal subgroups. Then the index of every maximal subgroup of G is prime by the hypothesis. It follows that G is supersolvable, a contradiction. Therefore, G has at least one nilpotent maximal subgroup. Moreover, since both nilpotent groups and minimal non-nilpotent groups are solvable, one has that G has at least one non-nilpotent maximal subgroup.

We divide it into two cases to discuss:

Case I: Assume that G is a non-abelian simple group.

Since G has at least one nilpotent maximal subgroup, by Lemma 2.1, one has $G \cong \mathrm{PSL}(2, p)$, where p is a Fermat or Mersenne prime with $p \geq 17$.

Let K be a non-nilpotent maximal subgroup of G. By the hypothesis, one has $|G : K| = q$ for some prime q. Let $\pi_t(G)$ be the set of indices of maximal subgroups of G. By [3, Theorem 12], we know that q is the smallest number in $\pi_t(G)$. However, since $G \cong \mathrm{PSL}(2, p)$ with $p \geq 17$, by [1, Table 5.2.A], we know that the smallest number in $\pi_t(\mathrm{PSL}(2,p))$ is $p+1$. Note that p is a Fermat or Mersenne prime with $p \geq 17$. Then $p+1$ cannot be a prime. It follows that $p+1 \neq q$, a contradiction.

Case II: Assume that G is not a non-abelian simple group.

Let N be a minimal normal subgroup of G. Then $1 < N < G$. It is easy to see that G/N is also a group in which the index of every non-nilpotent maximal subgroup is prime. Since $|G/N| < |G|$, by the minimality of G, one has that G/N is solvable. It implies that N is non-solvable since G is non-solvable. Then N is a non-solvable characteristically simple group.

Let $N \cong N_1 \times N_2 \times \cdots \times N_t$, where $N_i \cong N_j$ is a non-abelian simple group for every $1 \leq i < j \leq t$. Let l be an odd prime divisor of $|N|$ and L be a Sylow l-subgroup of N. By Frattini argument, one has $G = NN_G(L)$.

Note that $Z(G) = 1$.

Otherwise, if $Z(G) \neq 1$ then by the minimality of G we have that $G/Z(G)$ is solvable, which implies that G is solvable, a contradiction. Thus $Z(G) = 1$.

By Lemma 2.2, one has that all nilpotent maximal subgroups of G are Sylow 2-subgroups of G. It is clear that L is not normal in N since N is a non-solvable characteristically simple group. Thus L is not normal in G. It follows that $N_G(L) < G$. Let M be a maximal subgroup of G such that $N_G(L) \leq M$. Then $G = NM$. Since all nilpotent maximal subgroups of G are Sylow 2-subgroups of G and l is an odd prime, one has that M is non-nilpotent. Then $|G : M|$ is a prime by the hypothesis. It follows that $|N : N \cap M|$ is a prime. Assume that $|N : N \cap M| = w$, where w is a prime. It implies that $N \cap M$ is maximal in N. Note that $w \neq 2$ since N has no normal maximal subgroups. Moreover, one has that $w \neq l$ since $L \leq N_N(L) = N \cap N_G(L) \leq N \cap M$. Since $N \cap M < N$, there exists a N_i for some $1 \leq i \leq t$ such that $N_i \nleq N \cap M$. Then $N = N_i(N \cap M)$. It follows that $|N_i : N_i \cap M| = |N_i : N_i \cap (N \cap M)| = |N_i(N \cap M) : N \cap M| = |N : N \cap M| = w$. Thus N_i has a maximal subgroup $N_i \cap M$ of index w.

Let W be a Sylow w-subgroup of N. By Frattini argument, one has $G = NN_G(W)$. Arguing as above, there also exists a non-nilpotent maximal subgroup H of G such that $N_G(W) \leq H$. Then $G = NH$. By the hypothesis, we assume that $|G : H| = r$, where r is a prime. It follows that $|N : N \cap H| = r$. Thus $N \cap H$ is a maximal subgroup of N. There exists a N_j for some $1 \leq j \leq t$ such that $N_j \nleq N \cap H$. Then $N = N_j(N \cap H)$. It follows that $|N_j : N_j \cap H| = |N_j : N_j \cap (N \cap H)| = |N_j(N \cap H) : N \cap H| = |N : N \cap H| = r$. Note that $W \leq N_N(W) = N \cap N_G(W) \leq N \cap H$. Then $r \neq w$.

Since $N_i \cong N_j$, it follows that N_i has two maximal subgroups of distinct primes indices. By [3, Theorem 12], this is a contradiction.

Final conclusion.

By Case I and Case II we know that the counterexample does not exist. Hence G is solvable. □

Acknowledgments

The author is grateful to the referee for valuable suggestions.

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