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QUASIRECOGNITION BY PRIME GRAPH OF $U_3(q)$ WHERE $2 < q = p^\alpha < 100$

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Abstract. Let G be a finite group and let $\Gamma(G)$ be the prime graph of G. Assume $2 < q = p^\alpha < 100$. We determine finite groups G such that $\Gamma(G) = \Gamma(U_3(q))$ and prove that if $q \neq 3, 5, 9, 17$, then $U_3(q)$ is quasirecognizable by prime graph, i.e. if G is a finite group with the same prime graph as the finite simple group $U_3(q)$, then G has a unique non-Abelian composition factor isomorphic to $U_3(q)$. As a consequence of our results, we prove that the simple groups $U_3(8)$ and $U_3(11)$ are $4-$recognizable and $2-$recognizable by prime graph, respectively. In fact, the group $U_3(8)$ is the first example which is a $4-$recognizable by prime graph.

1. Introduction

If n is an integer, then we denote by $\pi(n)$ the set of all prime divisors of n. If G is a finite group, then $\pi(|G|)$ is denoted by $\pi(G)$. We construct the *prime graph* of G, which is denoted by $\Gamma(G)$, as follows: the vertex set is $\pi(G)$ and two distinct vertices p and p' are joined by an edge if and only if G has an element of order pp' (we write $p \sim p'$). Let $s(G)$ be the number of connected components of $\Gamma(G)$ and let $\pi_1, \pi_2, \ldots, \pi_{s(G)}$ be the connected components of $\Gamma(G)$. If $2 \in \pi(G)$, then we always suppose $2 \in \pi_1$.

The *spectrum* of a finite group G, which is denoted by $\pi_e(G)$ is the set of its element orders. It is clear that the set $\pi_e(G)$ is closed and partially ordered by divisibility, and hence it is uniquely determined by $\mu(G)$, the subset of its maximal elements.

A subset X of the vertices of a graph is called an independent set if the induced subgraph on X has no edge. Let G be a finite group and $r \in \pi(G)$. We denote by $\rho(G)$, some independent set of

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vertices in $\Gamma(G)$ with the maximal number of elements. Also some independent set of vertices in $\Gamma(G)$ containing r with the maximal number of elements is denoted by $\rho(r, G)$. Also let $t(G) = |\rho(G)|$ and $t(r, G) = |\rho(r, G)|$.

Let G be a non-Abelian finite simple group. The numbers $t(G)$ and $t(r, G)$ have been determined in [38] and [40]. Also for every finite non-Abelian simple group, we use of these reference for adjacency of vertices in a prime graph of the group.

A finite group G is called *recognizable by spectrum*, if every finite group H with $\pi_e(G) = \pi_e(H)$ is isomorphic to G. A finite simple non-Abelian group P is called *quasirecognizable by spectrum*, if each finite group H with $\pi_e(P) = \pi_e(H)$ has a unique non-Abelian composition factor isomorphic to P [3].

A finite group G is called *recognizable by prime graph*, if every finite group H with $\Gamma(G) = \Gamma(H)$ is isomorphic to G. A finite simple non-Abelian group P is called *quasirecognizable by prime graph*, if each finite group H with $\Gamma(P) = \Gamma(H)$ has a unique nonabelian composition factor isomorphic to P [16].

We denote by $k(\Gamma(G))$ the number of isomorphism classes of the finite groups H satisfying $\Gamma(G) = \Gamma(H)$. A finite group G is called *$n-$recognizable by prime graph* if $k(\Gamma(G)) = n$ [19].

We note that quasirecognition by prime graph implies quasirecognition by spectrum, but the converse is not true in general. Also quasirecognition by prime graph is in general harder to establish than quasirecognition by spectrum, since some methods fail in the former case.

The structure of finite groups G such that $\Gamma(G)$ is not connected has been determined by Gruenberg and Kegel (1975). Moreover it has been proved that $s(G) \leq 6$ and all the simple groups G such that $\Gamma(G)$ is not connected have been described in [12, 28, 41].

Finite groups G satisfying$\Gamma(G) = \Gamma(H)$ have been determined, where H is one of the following groups: a sporadic simple group [9], a CIT simple group [15], $PSL(2, q)$, where $q = p^\alpha < 100$ [14], $PSL(2, p)$, where $p > 3$ is a prime [17], $G_2(7)$, ${}^2G_2(q)$, where $q = 3^{2n+1} > 3$, $(n > 0)$ [16, 42], $PSL(2, q)$ [19, 20], $L_{16}(2)$ [21], $B_p(3)$, where p is an odd prime [34].

Also, the quasirecognizability of the following simple non-Abelian groups by their prime graphs have been obtained: Alternating group A_p where p and $p-2$ are primes [18], $L_9(2)$ [22], $L_{10}(2)$ [23], ${}^2F_4(q)$, where $q = 2^{2m+1}$ for some $m \geq 1$ [1], ${}^2D_p(3)$, where $p = 2^n + 1 \geq 5$ is a prime [4], $C_n(2)$, where $n \neq 3$ is odd [6], $L_n(2)$ and $U_n(2)$, where $n \geq 17$ [25].
In [7], there is the following problem:
Problem: If $m \geq 3$ is a natural number, is there a finite simple group G such that $k(\Gamma(G)) = m$?

In this paper, we give a positive answer to this problem for $m = 4$. In fact, we prove that the group $U_3(8)$ is $4-$recognizable by prime graph, and therefore this group is the first group which has this property. Also we determine finite groups G such that $\Gamma(G) = \Gamma(U_3(q))$, where $2 < q = p^\alpha < 100$ is a prime power. As a consequence of our results, we prove that if $q \neq 3, 5, 9, 17$, then $U_3(q)$ is quasirecognizable by prime graph and the simple group $U_3(11)$ is $2-$recognizable by prime graph. In fact, the main theorem of our paper is as follow:

Main Theorem. Let $q = p^\alpha$ be a prime power, $U = U_3(q)$, where $2 < q = p^\alpha < 100$ and G be a finite group satisfying $\Gamma(G) = \Gamma(U)$. Then G is one of the groups in 2nd column of Table 2 and Table 3 ($\overline{G}$ means $G/O_\pi(G)$).

We denote by (a, b) the greatest common divisor of positive integers a and b. If G is a finite group, then we denote by P_q a Sylow $q-$subgroup of G. Let π be a set of prime numbers and let G be a finite group. Then G has unique largest normal $\pi-$ subgroup, which is denote by $O_\pi(G)$ and called the $\pi-$ radical of G. In fact, $O_\pi(G)$ contains every normal $\pi-$subgroup of G. All further unexplained notations are standard and refer to [5].

2. Preliminary Results

We first quote some remarks and lemmas that are used in deducing the main theorem of this paper.

Remark 2.1. *Let G be a finite group and K be a normal subgroup of G. If $p \sim q$in $\Gamma(G/K)$, then $p \sim q$ in $\Gamma(G)$. In fact if $xK \in G/K$ has order pq, then there is a power of x which has order pq.*

Remark 2.2. *We know that $\mu(L_2(q))$=$\{p, (q-1)/d, (q+1)/d\}$, where $q = p^\alpha$ and $d = (2, q-1)$,* [11] *(page. 213). By* [32]*, $\Gamma(U_3(q))$ has two connected components: $\pi_1(G) = \pi(p(q^2-1))$ and $\pi_2(G) = \pi((q^3+1)/(q+1)(3, q+1))$. Also the set of element orders $U_3(q)$, can be found in* [2] *and* [30]*; we have:*

$$\mu(U_3(q)) = \begin{cases} \{q+1, \frac{1}{3}p(q+1), \frac{1}{3}(q^2-1), \frac{1}{3}(q^2-q+1)\} & if \quad d=3, \\ \{p(q+1), (q^2-1), (q^2-q+1)\} & if \quad d=1, \end{cases}$$

where $q = p^\alpha$ is odd number and $d = (3, q+1)$, and

$$\mu(U_3(2^n)) = \begin{cases} \{4, 2^n+1, \frac{2}{3}(2^n+1), \frac{1}{3}(2^{2n}-1), \frac{1}{3}(2^{2n}-2^n+1)\} & if \quad d=3, \\ \{4, 2(2^n+1), 2^{2n}-1, 2^{2n}-2^n+1\} & if \quad d=1, \end{cases}$$

where $d = (3, 2^n+1)$.

Remark 2.3. *Let p be a prime number and $(a, p) = 1$. Let $k \geq 1$ be the smallest positive integer such that $a^k \equiv 1$ (mod p). Then k is called the order of a with respect to p and we assume that $ord_p(a) = k$. Obviously by the Fermat little theorem $ord_p(a) \mid (p-1)$. Also, if $a^n \equiv 1$ (mod p), then $ord_p(a) \mid n$.*

Lemma 2.4. [35] *Let G be a nonsolvable complement of a Frobenius group. Then G has a normal subgroup G_0=$SL(2,5) \times Z$, such that $|G : G_0| \leq 2$, $\pi(Z) \cap \{2, 3, 5\} = \emptyset$ and the Sylow subgroups of Z are cyclic.*

Lemma 2.5. [37] *Let G be a finite Frobenius group with kernel K and complement C. Then*

(1) *K is nilpotent,*

(2) *The Sylow $p-$subgroups of C are cyclic if $p > 2$ and cyclic or generalized quaternion if $p = 2$.*

Using [41, Theorem A], we can conclude that the following lemma:

Lemma 2.6. *A finite group G with disconnected prime graph $\Gamma(G)$ satisfies one of the following conditions:*

(1) *$s(G) = 2$, $G = KC$ is a Frobenius group with kernel K and complement C, and the two connected components of $\Gamma(G)$ are $\pi(K)$ and $\pi(C)$. Moreover, K is nilpotent, and hence $\pi(K)$ is a complete graph. If C is solvable then $\Gamma(C)$ is complete; otherwise, $2, 3, 5 \in \pi(G)$ and $\pi(C)$ can be obtained from the complete graph with vertex set $\pi(C)$ by removing the edge $3 \sim 5$.*

(2) *$s(G) = 2$ and G is a 2-Frobenius group, i.e, $G = ABC$, where A and AB are normal subgroups of G, B is a normal subgroup of BC, and AB and BC are Frobenius groups. The two connected components of $\Gamma(G)$ are complete graphs $\Gamma(AC)$ and $\Gamma(B)$.*

(3) *There exists a finite non-Abelian simple group S such that $S \leq \overline{G} = G/K \leq Aut(S)$, where K is a nilpotent normal subgroup of G; furthermore K and $\overline{G}/S$ are trivial or π_1-groups, $s(S) \geq s(G)$, and for every $2 \leq i \leq s(G)$, there exists $2 \leq j \leq s(S)$ such that $\pi_i(G) = \pi_j(S)$.*

Remark 2.7. *A 2-Frobenius group is solvable and the above lemma implies that $t(G) = 1$ or 2 for a solvable group.*

Lemma 2.8. [39] *Let G be a finite group with $t(G) \geq 3$ and $t(2, G) \geq 2$. Then the following hold:*

(1) *There exists a finite non-Abelian simple group S such that $S \leq \overline{G} = G/K \leq Aut(S)$ for a maximal normal soluble subgroup K of G.*

(2) *For every independent subset ρ of $\pi(G)$ with $|\rho| \geq 3$ at most one prime in ρ divides the product $|K|.|\overline{G}/S|$. In particular, $t(S) \geq t(G) - 1$.*

(3) *One of the following holds:*

(a) *every prime $r \in \pi(G)$ nonadjacent to 2 in $\Gamma(G)$ does not divide the product $|K|.|\overline{G}/S|$; in particular, $t(2, S) \geq t(2, G)$;*

(b) *there exists a prime $r \in \pi(K)$ nonadjacent to 2 in $\Gamma(G)$; in which case $t(G) = 3$, $t(2, G) = 2$, and $S \cong A_7$ or $A_1(q)$ for some odd q.*

Remark 2.9. *Note that the condition of Lemma 2.8, implies an insolubility of G , and so by the Feit-Thompson Theorem, it is not necessary to assume in the hypotheses of above theorem, G is of even order.*

Lemma 2.10. [34] *Let G be a finite group such that $s(G) \geq 2$ and K be a normal π_1-subgroup of G. Let S be a finite simple group such that $S \leq G/K$ and S is not a π_1-group. If $K \neq 1$, and S contains a Frobenius subgroup with kernel F and a cyclic complement C such that $(|F|, |K|) = 1$, then $r|C| \in \pi_e(G)$, for every prime divisor r of $|K|$.*

Lemma 2.11. [26] *Let G be a group with disconnected prime graph, such that $t(G) \geq 3$. Also let K be the maximal normal solvable subgroup of G. Then K is a nilpotent π_1- group.*

Lemma 2.12. [43] *There are a total of 1972 isomorphism types of finite non-Abelian simple groups G such that all prime divisors of $|G|$ do not exceed 1000.*

The groups from Lemma 2.12, are listed in Tables $1 - 4$ of [43].

Let$t > 1$ and n be natural numbers and let $\varepsilon \in \{+, -\}$. If there exists a prime that divides $t^n - (\varepsilon 1)^n$

and does not divide $t^i - (\varepsilon 1)^i$ for $1 \leq i < n$, then we denote this prime by $t_{[\varepsilon n]}$ and call it a *primitive divisor* of $t^n - (\varepsilon 1)^n$. A primitive divisor need not be exist, nor be unique. The following lemma generalizes Zsigmondy's theorem:

Lemma 2.13. [33] *Let $t, n > 1$ be natural numbers. Then, for all $\varepsilon \in \{+, -\}$, there exists a primitive divisor $t_{[\varepsilon n]}$ of $t^n - (\varepsilon 1)^n$ except in the following cases:*

(1) $\varepsilon = +$, $n = 6$, $t = 2$;
(2) $\varepsilon = +$, $n = 2$ *and* $t = 2^l - 1$ *for some* $l \geq 2$;
(3) $\varepsilon = -$, $n = 3$, $t = 2$;
(4) $\varepsilon = -$, $n = 2$ *and* $t = 2^l + 1$ *for some* $l \geq 0$.

In the next lemma we write $L_n^{\varepsilon}(q)$, where $\varepsilon \in \{+, -\}$, $L_n^+(q) = L_n(q)$ and $L_n^-(q) = U_n(q)$.

Lemma 2.14. [33] *Suppose that $L_n^{\varepsilon}(q)$ is a simple group for some natural numbers n and q and that the primitive prime divisor $r = q_{[\varepsilon n]}$ of $q^n - (\varepsilon 1)^n$ exists. Then $L_n^{\varepsilon}(q)$ contains a Frobenius subgroup whose kernel is of order r and cyclic complement is of order n. Moreover, if n is odd or q is even, then such a Frobenius subgroup exists in $SL_n^{\varepsilon}(q)$.*

Lemma 2.15. *Let $q = p^{\alpha}$ be a prime power and $U = U_3(q)$, where $2 < q < 100$. If S is a non-Abelian simple group such that $\Gamma(S)$ is a subgraph of $\Gamma(U)$ and there exists $i \geq 2$ such that $\pi_2(U) = \pi_i(S)$, then S is one of the groups in the 2nd column of Table 1.*
In Table 1, X is one of the non-Abelian simple groups $U_3(q)$ such that $2 < q = p^{\alpha} < 100$ is a prime power and $q \neq 3, 5, 19, 23$.

Proof. For every finite non-Abelian simple group, we use [38] and [40] for adjacency of vertices in a prime graph of the group. If $q \neq 67, 71, 79, 81$ and 83, then $\pi(U_3(q)) \subseteq \{2, 3, ..., 997\}$. So the result follows from Lemma 2.12. Now suppose that $q = 67$ and S is a non-Abelian simple group such that $\{2, 4423\} \subseteq \pi(S) \subseteq \{2, 3, 11, 17, 67, 4423\}$. According to the classification of the finite simple groups we consider the following cases:
case 1. S is isomorphic to an alternating group. Since $4423 \mid |S|$ and $5 \nmid |S|$, we get a contradiction.
case 2. S is isomorphic to a sporadic simple group. Since $4423 \nmid |S|$, we get a contradiction.
case 3. S is isomorphic to a simple group of Lie type. We use the list of orders of these groups given in [5]. If S is defined over the field F_q where $q = p^m$, then since $p \mid |S|$ we must have $p = 2, 3, 11, 17, 67$ or 4423. We note that $\pi(S)$ contains all prime divisors of $q^2 - 1$, except ${}^2B_2(q)$, where $q = 2^{2m+1}$.
If $p = 4423$, then $7 \in \pi(4423^2 - 1) \subseteq \pi(q^2 - 1)$, which is a contradiction. So $p \neq 4423$. Similarly, we can see $p \neq 11$.
If $p = 2$, then max $\{ord_r(2) \mid r \in \{3, 11, 17, 67, 4423\}\} = 737$ and we have $2m \leq 737$, for otherwise $2^{2m} - 1 = q^2 - 1$ would be divisible by a prime not in $\pi(S)$ by Lemma 2.13. Hence $m \leq 368$. We observe that m is odd, since otherwise $2^m \equiv \pm 1 \pmod 5$ and 5 divides $q^2 - 1 = (2^m - 1)(2^m + 1)$, which is a contradiction. Since $2^3 - 1 = 7$, then $3 \nmid m$. The relations $2^2 - 1 = 3$, $2^4 + 1 = 17$, $11 \mid 2^{10} - 1$ and Lemma 2.13, imply that $m \in \{1, 33\}$. Since $3 \mid 33$, then $m = 1$ and we don't obtain a possibility

in this case.
If $p = 3$, then $\max\{ord_r(3) \mid r \in \{2, 11, 17, 67, 4423\}\} = 4422$. As the above, m does not exceed 2211 and is odd. Furthermore, $3 \nmid m$, since otherwise $13 \in \pi(3^3 - 1) \subseteq \pi(3^m - 1)$. It is easy to see $m \in \{1, 8, 11, 2211\}$. Since $2 \mid 8$, $3 \mid 2211$ and $23 \mid 3^{11} - 1$, then $m = 1$ and there is no possibility in this case.
If $p = 17$, then $\max\{ord_r(17) \mid r \in \{2, 3, 11, 67, 4423\}\} = 1474$. So similar to the previous case, there is no possibility in this case as well.
If $p = 67$, then $\max\{ord_r(67) \mid r \in \{2, 3, 11, 17, 4423\}\} = 6$. So $m \leq 3$. Since $5 \mid 67^2+1$ and $7 \mid 67^3-1$, then $m = 1$ and the only possibility is $S \cong U_3(67)$. Similarly, we get our results for $q = 71, 79, 81$ and 83. □

Table 1.

U	S
$U_3(3)$	$L_2(7)$, $L_2(8)$, $U_3(3)$
$U_3(5)$	$L_2(7)$, $L_2(8)$, $U_3(3)$, A_7, $L_2(49)$, $L_3(4)$, $U_3(5)$, $U_4(3)$
$U_3(19)$	$L_2(7)$, $L_2(8)$, $U_3(3)$, A_7, $L_2(49)$, $L_3(4)$, $U_3(5)$, $U_4(3)$, A_8, A_9, J_2, $S_4(7)$, $S_6(2)$, $O_8^+(2)$, $U_3(19)$
$U_3(23)$	$L_3(3)$, $U_3(23)$
X	X

Corollary 2.16. *Let $U = U_3(q)$, where $2 < q < 100$ and S be a finite simple group such that $\Gamma(S) = \Gamma(U)$. Then S is isomorphic to U.*

Proof. Straightforward from Lemma 2.15. □

In the following, we give the structure of the group of outer automorphisms of the group $U_n(q)$.

Lemma 2.17. [27] *Let $n \geq 3$, and $q = p^f$. Then $Out(U_n(q)) \cong \mathbb{Z}_{(n,\ q+1)} : \mathbb{Z}_{2f}$.*

By the above Lemma, we conclude that $Out(U_3(q)) \cong \mathbb{Z}_{(3,\ q+1)} : \mathbb{Z}_{2f}$.

Lemma 2.18. [29] *Let $U_3(q) < G \leq Aut(U_3(q))$, then $\Gamma(G)$is not connected if and only if $G/U_3(q) \cong \langle\gamma\rangle$, where γ is a filed automorphism.*

By the above lemma, if $(3, q + 1) = 3$, then $U_3(q).3$ has a connected prime graph. So $\Gamma(U_3(q).3)$ is not equal to $\Gamma(U_3(q))$.

Lemma 2.19. [36] *Let G be a finite group and K a nontrivial normal p-subgroup, for some prime p, and set $U = G/K$. Suppose that U contains an element x of order m coprime to p such that $\langle\varphi|_{\langle x\rangle}, 1|_{\langle x\rangle}\rangle > 0$ for every Brauer character φ of (an absolutely irreducible representation of) U in characteristic p. Then G contains elements of order pm.*

3. **Proof of the Main Theorem**

We now prove the theorem stated in the introduction. Let G be a finite group such that $\Gamma(G) = \Gamma(U_3(q))$, where $2 < q = p^\alpha < 100$. By [38], we have $3 \le t(G) \le 4$ and $t(2, G) \ge 2$, except for $q = 3, 9, 17$. So at first, we consider these cases separately.

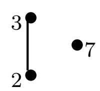

Fig. 1. $\Gamma(U_3(3))$.

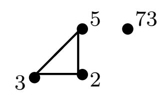

Fig. 2. $\Gamma(U_3(9))$.

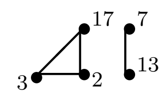

Fig. 3. $\Gamma(U_3(17))$.

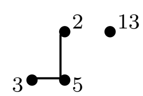

Fig. 4. $\Gamma(U_3(4))$.

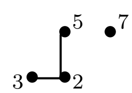

Fig. 5. $\Gamma(U_3(5))$.

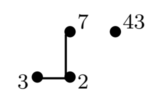

Fig. 6. $\Gamma(U_3(7))$.

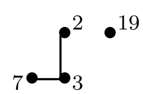

Fig. 7. $\Gamma(U_3(8))$.

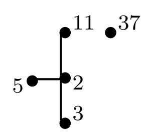

Fig. 8. $\Gamma(U_3(11))$.

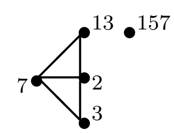

Fig. 9. $\Gamma(U_3(13))$.

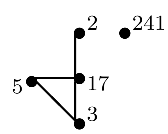

Fig. 10. $\Gamma(U_3(16))$.

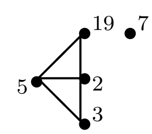

Fig. 11. $\Gamma(U_3(19))$.

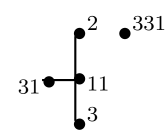

Fig. 12. $\Gamma(U_3(32))$.

Table 2.

U	G or $\overline{G} = G/O_\pi(G)$
$U_3(3)$	G is solvable Frobenius or 2-Frobenius group, or $\overline{G} \cong L_2(7)$, $L_2(7).2$, $L_2(8)$, $L_2(8).3$, $U_3(3)$ or $U_3(3).2$, where $\pi \subseteq \{2, 3\}$
$U_3(9)$	G is solvable Frobenius or 2-Frobenius group, or $\overline{G} \cong U_3(9)$, $U_3(9).2$ or $U_3(9).4$, where $\pi \subseteq \{2, 3, 5\}$
$U_3(17)$	G is solvable Frobenius or 2-Frobenius group, or $\overline{G} \cong U_3(17)$ or $U_3(17).2$, where $\pi \subseteq \{2, 3, 17\}$

Table 3.

U	G or $\overline{G} = G/O_{\pi}(G)$
$U_3(4)$	$\overline{G} \cong U_3(4)$, where $\pi \subseteq \{5\}$
$U_3(5)$	$\overline{G} \cong A_7$ or $A_7.2$, where $\pi \subseteq \{2,3\}$, $G \cong L_2(49).2_1$ or $L_2(49).2_3$, $\overline{G} \cong L_3(4)$, $L_3(4).2_1$, $L_3(4).2_3$ or $L_3(4).6$, where $\pi \subseteq \{2,3\}$, $\overline{G} \cong U_3(5)$ or $U_3(5).2$, where $\pi \subseteq \{2\}$, $\overline{G} \cong U_4(3)$, $U_4(3).2_2$ or $U_4(3).2_3$, , where $\pi \subseteq \{2,3\}$
$U_3(7)$	$\overline{G} \cong U_3(7)$ or $U_3(7).2$, where $\pi \subseteq \{2\}$
$U_3(8)$	$G \cong U_3(8)$, $U_3(8).3_1$, $U_3(8).3_3$, or $U_3(8).6$
$U_3(11)$	$G \cong U_3(11)$ or $U_3(11).2$
$U_3(13)$	$\overline{G} \cong U_3(13)$ or $U_3(13).2$, where $\pi \subseteq \{2,7\}$
$U_3(16)$	$\overline{G} \leq U_3(16).\mathbb{Z}_8$, where $\pi \subseteq \{17\}$
$U_3(19)$	$\overline{G} \cong U_3(19)$ or $U_3(19).2$, where $\pi \subseteq \{2,5\}$
$U_3(23)$	$\overline{G} \cong U_3(23)$ or $U_3(23).2$, where $\pi \subseteq \{2\}$
$U_3(25)$	$\overline{G} \leq U_3(25).\mathbb{Z}_4$, where $\pi \subseteq \{2,13\}$
$U_3(27)$	$\overline{G} \leq U_3(27).\mathbb{Z}_6$, where $\pi \subseteq \{2,3,5\}$
$U_3(29)$	$\overline{G} \cong U_3(29)$ or $U_3(29).2$, where $\pi \subseteq \{2,5\}$
$U_3(31)$	$\overline{G} \cong U_3(31)$ or $U_3(31).2$, where $\pi \subseteq \{2\}$
$U_3(32)$	$\overline{G} \leq U_3(32).(\mathbb{Z}_3 : \mathbb{Z}_{10})$, where $\pi \subseteq \{11\}$
$U_3(37)$	$\overline{G} \cong U_3(37)$ or $U_3(37).2$, where $\pi \subseteq \{2,19\}$
$U_3(41)$	$\overline{G} \cong U_3(41)$ or $U_3(41).2$, where $\pi \subseteq \{2,7\}$
$U_3(43)$	$\overline{G} \cong U_3(43)$ or $U_3(43).2$, where $\pi \subseteq \{2,11\}$
$U_3(47)$	$\overline{G} \cong U_3(47)$ or $U_3(47).2$, where $\pi \subseteq \{2\}$
$U_3(49)$	$\overline{G} \leq U_3(49).\mathbb{Z}_4$, where $\pi \subseteq \{2,5\}$
$U_3(53)$	$\overline{G} \cong U_3(53)$ or $U_3(53).2$, where $\pi \subseteq \{2,3\}$
$U_3(59)$	$\overline{G} \cong U_3(59)$ or $U_3(59).2$, where $\pi \subseteq \{2,5\}$
$U_3(61)$	$\overline{G} \cong U_3(61)$ or $U_3(61).2$, where $\pi \subseteq \{2,31\}$
$U_3(64)$	$\overline{G} \leq U_3(64).\mathbb{Z}_{12}$, where $\pi \subseteq \{5,13\}$
$U_3(67)$	$\overline{G} \cong U_3(67)$ or$U_3(67).2$, where $\pi \subseteq \{2,17\}$
$U_3(71)$	$\overline{G} \cong U_3(71)$ or$U_3(71).2$, where $\pi \subseteq \{2,3\}$
$U_3(73)$	$\overline{G} \cong U_3(73)$ or$U_3(73).2$, where $\pi \subseteq \{2,37\}$
$U_3(79)$	$\overline{G} \cong U_3(79)$ or$U_3(79).2$, where $\pi \subseteq \{2,5\}$
$U_3(81)$	$\overline{G} \leq U_3(81).\mathbb{Z}_8$, where $\pi \subseteq \{2,3,41\}$
$U_3(83)$	$\overline{G} \cong U_3(83)$ or$U_3(83).2$, where $\pi \subseteq \{2,7\}$
$U_3(89)$	$\overline{G} \cong U_3(89)$ or$U_3(89).2$, where $\pi \subseteq \{2,3,5,89\}$
$U_3(97)$	$\overline{G} \cong U_3(97)$ or$U_3(97).2$, where $\pi \subseteq \{2,7\}$

The Case $U = U_3(3)$

The prime graph of the simple group $U_3(3)$ is shown in Fig. 1. By assumption $\Gamma(G) = \Gamma(U_3(3))$. So $s(G) = 2$ and we can apply Lemma 2.6. At first, suppose that G is a Frobenius or 2-Frobenius group. If G is nonsolvable, then G is a Frobenius group by Remark 2.7. Hence the Frobenius complement of G has a normal subgroup $SL_2(5) \times Z$ with index at most 2 by Lemma 2.4. Since $5 \notin \pi(G)$, we get a contradiction. Therefore G is solvable Frobenius or 2-Frobenius group. Now suppose that there exists a finite non-Abelian simple group S such that $S \leq \overline{G} = G/K \leq Aut(S)$, where K is a π_1- group and nilpotent normal subgroup of G. It follows from Lemma 2.15, that $S \cong L_2(7), L_2(8)$ or $U_3(3)$. If $S \cong L_2(7)$, then $L_2(7) \leq \overline{G} = G/K \leq Aut(L_2(7))$. We know that $Out(L_2(7)) \cong \mathbb{Z}_2$ by [5]. Hence $\overline{G}/L_2(7) \leq \mathbb{Z}_2$ and so $G/K \cong L_2(7)$ or $L_2(7).2$. Therefore $G/O_\pi(G) \cong L_2(7)$ or $L_2(7).2$, where $\pi \subseteq \{2, 3\}$. Furthermore, if $G/O_\pi(G) \cong L_2(7)$, then $O_\pi(G) \neq 1$, otherwise $2 \sim 3$ in $\Gamma(L_2(7))$, which is a contradiction. Now suppose that $S \cong L_2(8)$. We note that $Out(L_2(8)) \cong \mathbb{Z}_3$. Hence $\overline{G}/L_2(8) \leq \mathbb{Z}_3$. So $G/O_\pi(G) \cong L_2(8)$ or $L_2(8).3$, where $\pi \subseteq \{2, 3\}$. Furthermore, if $G/O_\pi(G) \cong L_2(8)$, then $O_\pi(G) \neq 1$, otherwise $2 \sim 3$ in $\Gamma(L_2(8))$, which is a contradiction. Finally, let $S \cong U_3(3)$. Since $Out(U_3(3)) \cong \mathbb{Z}_2$ by Lemma 2.17, then $\overline{G}/U_3(3) \leq \mathbb{Z}_2$. Therefore $G/O_\pi(G) \cong U_3(3)$ or $U_3(3).2$, where $\pi \subseteq \{2, 3\}$.

The Case $U = U_3(9)$

The prime graph of the simple group $U_3(9)$ is shown in Fig. 2. By assumption $\Gamma(G) = \Gamma(U_3(9))$. So $s(G) = 2$ and and we can apply Lemma 2.6. At first, suppose that G is a Frobenius or 2-Frobenius group. If G is nonsolvable, then G is a Frobenius group by Remark 2.7, and Frobenius complement of G has a normal subgroup $SL_2(5) \times Z$ with index at most 2 by Lemma 2.4. Since $15 \notin SL_2(5)$, we get a contradiction. So G is solvable Frobenius or 2-Frobenius group. Now suppose that there exists a finite non-Abelian simple group S such that $S \leq \overline{G} = G/K \leq Aut(S)$, where K is a π_1- group and nilpotent normal subgroup of G. It follows from Lemma 2.15, that $S \cong U_3(9)$. Since $Out(U_3(9)) \cong \mathbb{Z}_4$ by Lemma 2.17, then $G/K \leq \mathbb{Z}_4$. By [5], it is easy to check $\Gamma(U_3(9)) = \Gamma(U_3(9).2) = \Gamma(U_3(9).4)$. Therefore $G/O_\pi(G)$ is isomorphic to $U_3(9)$, $U_3(9).2$ or $U_3(9).4$, where $\pi \subseteq \{2, 3, 5\}$.

The Case $U = U_3(17)$

The prime graph of the simple group $U_3(17)$ is shown in Fig. 3. By $\Gamma(U_3(17))$ we have $s(G) = 2$ and we can apply Lemma 2.6. At first, suppose that G is a Frobenius or 2-Frobenius group. If G is nonsolvable, then G is a Frobenius group by Remark 2.7, and Frobenius complement of G has a normal subgroup $SL_2(5) \times Z$ with index at most 2 by Lemma 2.4. Since $5 \notin \pi(G)$, we get a contradiction. Thus G is solvable Frobenius or 2-Frobenius group. Now suppose that there exists a finite non-Abelian simple group S such that $S \leq \overline{G} = G/K \leq Aut(S)$, where K is a π_1-group and nilpotent normal subgroup of G. It follows from Lemma 2.15, that $S \cong U_3(17)$. Since $Out(U_3(17)) \cong \mathbb{Z}_3 : \mathbb{Z}_2$ by Lemma 2.17, then $G/K \leq \mathbb{Z}_3 : \mathbb{Z}_2$. Since $U_3(17).3$ has a connected prime graph by Lemma 2.18, then $\Gamma(U_3(17).3)$ is not subgraph of $\Gamma(G)$. So we have $G/K \cong U_3(17)$ or $U_3(17).2$. Then $G/O_\pi(G)$ is isomorphic to $U_3(17)$, $U_3(17).2$, where $\pi \subseteq \{2, 3, 17\}$.

If $q \neq 3, 9, 17$, then $t(G) \geq 3$ and $t(2, G) \geq 2$. By Lemma 2.8 and Lemma 2.11, there exists a finite non-Abelian simple group S such that $S \leq \overline{G} = G/K \leq Aut(S)$ for the maximal normal solvable subgroup K of G, and K is nilpotent π_1-group.

The Case $U = U_3(4)$

The prime graph of the simple group $U_3(4)$ is shown in Fig. 4. It follows from Lemma 2.15, that $S \cong U_3(4)$. We know that $Out(U_3(4)) \cong \mathbb{Z}_4$. Therefore $\overline{G}/U_3(4) \leq \mathbb{Z}_4$. Since $U_3(4).2$ and $U_3(4).4$ have an element of order 6, then $\Gamma(U_3(4).2)$ and $\Gamma(U_3(4).4)$ are not subgraphs of $\Gamma(G)$. So we have $\overline{G} = G/K \cong U_3(4)$. We know that $U_3(4)$ contains a Frobenius subgroup with Frobenius kernel of order 13 and Frobenius complement of order 3 by Lemma 2.14. If $2 \in \pi(K)$, then by Lemma 2.10, $2 \sim 3$ in $\Gamma(G)$, which is a contradiction. Since $S \ncong A_7$, $A_1(q)$ and $2 \not\sim 3$, then by Lemma 2.8, we have $3 \nmid |K||\overline{G}/U_3(4)|$. So $3 \notin \pi(K)$. Therefore, $G/O_5(G) \cong U_3(4)$.

The Case $U = U_3(5)$

The prime graph of the simple group $U_3(5)$ is shown in Fig. 5. It follows from Lemma 2.15, that $S \cong L_2(7)$, $L_2(8)$, $U_3(3)$, A_7, $L_2(49)$, $L_3(4)$, $U_3(5)$, $U_4(3)$.

If $S \cong L_2(7)$, then $L_2(7) \leq \overline{G} = G/K \leq Aut(L_2(7))$. We know that $L_2(7)$ contains a Frobenius subgroup with Frobenius kernel of order 7 and Frobenius complement of order 3 by [5]. If $5 \in \pi(K)$, then by Lemma 2.10, $3 \sim 5$ in $\Gamma(G)$, which is a contradiction. So $5 \in \pi(G)$ and $5 \notin \pi(K)$. Since $G/K \leq Aut(L_2(7))$, then $5 \mid |Aut(L_2(7))|$, which is a contradiction. Similarly S can not be isomorphic to $L_2(8)$, $U_3(3)$.

If $S \cong A_7$, then $\overline{G}/A_7 \leq Out(A_7) \cong \mathbb{Z}_2$. Hence $G/K \cong A_7$ or $A_7.2$. By [5], $L_2(7)$ is a maximal subgroup of A_7 and $L_2(7)$ contains a Frobenius subgroup with Frobenius kernel of order 7 and Frobenius complement of order 3. If $5 \in \pi(K)$, then by Lemma 2.10, $3 \sim 5$ in $\Gamma(G)$, which is a contradiction. Therefore $G/O_\pi(G) \cong A_7$ or $A_7.2$, where $\pi \subseteq \{2, 3\}$.

If $S \cong L_2(49)$, then $\overline{G}/L_2(49) \leq Out(L_2(49)) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. By [5], $\Gamma(L_2(49).2_2)$ has an element of order 14. So $\Gamma(L_2(49).2_2)$ can not be a subgraph of $\Gamma(G)$. But $\Gamma(L_2(49).2_1)$ and $\Gamma(L_2(49).2_3)$ are equal to $\Gamma(G)$. Thus $G/K \cong L_2(49)$, $L_2(49).2_1$ or $L_2(49).2_3$. If $2 \in \pi(K)$, then let $P \in Syl_2(K)$ and $Q \in Syl_7(G)$. Since K is a nilpotent group, then P *char* K. On the other hand, $K \trianglelefteq G$, we conclude that $P \trianglelefteq G$. We know $2 \not\sim 7$ in $\Gamma(G)$, So Q acts fixed point freely on P. Thus PQ is a Frobenius group, with kernel P and complement Frobenius Q. Therefore Q is cyclic by Lemma 2.5, which is a contradiction, since $L_2(49)$ has no element of order 49. By the same method we can show that 3 and 5 are not belong to $\pi(K)$. It is easy to check $\Gamma(G) \neq \Gamma(L_2(49))$. Therefore in this case $K = 1$ and $G \cong L_2(49).2_1$ or $L_2(49).2_3$.

If $S \cong L_3(4)$, then $\overline{G}/L_3(4) \leq Out(L_3(4)) \cong \mathbb{Z}_2 \times S_3$. We know that $L_3(4).3$ and $L_3(4).2_2 \cong L_3(4).2_2' \cong L_3(4).2_2''$ have elements of order 21 and 14, respectively, by [5]. Then $\Gamma(L_3(4).3)$ and $\Gamma(L_3(4).2_2) = \Gamma(L_3(4).2_2') = \Gamma(L_3(4).2_2'')$ are not subgraphs of $\Gamma(G)$. Also we note that $L_3(4).2_3 \cong L_3(4).2_3' \cong L_3(4).2_3''$. Similar to case $S \cong A_7$, we have $5 \notin \pi(K)$. Hence $G/O_\pi(G) \cong L_3(4)$, $L_3(4).2_1$,

$L_3(4).6$ or $L_3(4).2_3$, where $\pi \subseteq \{2,3\}$.
If $S \cong U_3(5)$, then $\overline{G}/U_3(5) \leq Out(U_3(5)) \cong \mathbb{Z}_3 : \mathbb{Z}_2$. Since $U_3(5).3$ has element of order 21, then $\Gamma(U_3(5).3)$ is not subgraph of $\Gamma(G)$. We note that $U_3(5).2 \cong U_3(5).2' \cong U_3(5).2''$. Also we note that $U_3(5)$ contains a Frobenius subgroup with Frobenius kernel of order 7 and Frobenius complement of order 3 by Lemma 2.14. If $5 \in \pi(K)$, then by Lemma 2.10, $3 \sim 5$ in $\Gamma(G)$, which is a contradiction. If $3 \in \pi(K)$, then let $P \in Syl_3(K)$ and $Q \in Syl_5(G)$. Since $3 \not\sim 5$ in $\Gamma(G)$, then Q acts fixed point freely on P. Thus PQ is a Frobenius group, with Frobenius kernel P and Frobenius complement Q. Therefore Q is cyclic by Lemma 2.5. This is a contradiction, since $U_3(5)$ has no element of order 5^3. Therefore in this case we have $G/O_2(G) \cong U_3(5)$ or $U_3(5).2$.
Finally If $S \cong U_4(3)$, then $\overline{G}/U_4(3) \leq Out(U_4(3)) \cong D_8$. Since $U_4(3).2_1$ and $U_4(3).4$ have elements of order 14 and 28, respectively. Therefore $\Gamma(U_4(3).2_1)$ and $\Gamma(U_4(3).4)$ are not subgraphs of $\Gamma(G)$. By [5], $U_4(3).2_2 \cong U_4(3).2'_2$ and $U_4(3).2_3 \cong U_4(3).2'_3$. Since $U_3(3)$ is a maximal subgroup of $U_4(3)$, then $5 \notin \pi(K)$. Therefore $G/O_\pi(G) \cong U_4(3)$, $U_4(3).2_2$, or $U_4(3).2_3$, where $\pi \subseteq \{2,3\}$.

The Case $U = U_3(7)$

The prime graph of the simple group $U_3(7)$ is shown in Fig. 6. It follows from Lemma 2.15, that $S \cong U_3(7)$. We know that $Out(U_3(7)) \cong \mathbb{Z}_2$. Therefore $\overline{G}/U_3(7) \leq \mathbb{Z}_2$. Hence $G/K \cong U_3(7)$ or $U_3(7).2$. We know that $U_3(7)$ contains a Frobenius subgroup with Frobenius kernel of order 43 and Frobenius complement of order 3 by Lemma 2.14. If $7 \in \pi(K)$, then by Lemma 2.10, $3 \sim 7$ in $\Gamma(G)$, which is a contradiction. If $3 \in \pi(K)$, then let $P \in Syl_3(K)$ and $Q \in Syl_7(G)$. Since $3 \not\sim 7$ in $\Gamma(G)$, then Q acts fixed point freely on P. Thus PQ is a Frobenius group, with Frobenius kernel P and Frobenius complement Q. Therefore Q is cyclic by Lemma 2.5. This is a contradiction, since $U_3(7)$ has no element of order 7^3, by Remark 2.2. Therefore, $G/O_2(G) \cong U_3(7)$ or $U_3(7).2$.

The Case $U = U_3(8)$

The prime graph of the simple group $U_3(8)$ is shown in Fig. 7. It follows from Lemma 2.15, that $S \cong U_3(8)$. Since $Out(U_3(8)) \cong \mathbb{Z}_3 : \mathbb{Z}_6 \cong \mathbb{Z}_3 \times S_3$. Therefore $\overline{G}/U_3(8) \leq \mathbb{Z}_3 \times S_3$. By [5], the extensions of $U_3(8)$ by a diagonal, a field or a diagonal-field automorphism are $U_3(8).3_1$, $U_3(8).3_2$, $U_3(8).3_3 \cong U_3(8).3_{3'}$, $U_3(8).2 \cong U_3(8).2' \cong U_3(8).2''$ and $U_3(8).6 \cong U_3(8).6' \cong U_3(8).6''$. By [5], we know that $U_3(8).2 \cong U_3(8).2' \cong U_3(8).2''$ and $U_3(8).3_2$ have elements of order 14 and 57, respectively. Then $\Gamma(U_3(8).2) = \Gamma(U_3(8).2') = \Gamma(U_3(8).2'')$ and $\Gamma(U_3(8).3_2)$ are not subgraphs of $\Gamma(G)$. It is easy to check $\Gamma(G) = \Gamma(U_3(8).3_1) = \Gamma(U_3(8).3_3) = \Gamma(U_3(8).6)$. Hence $G/K \cong U_3(8)$, $U_3(8).3_1$, $U_3(8).3_3$ or $U_3(8).6$, where K is a nilpotent $\{2,3,7\}-$ group. Since $S \ncong A_7$, $A_1(q)$ and $2 \not\sim 7$ in $\Gamma(G)$, then by Lemma 2.8, we have $7 \nmid |K||\overline{G}/U_3(8)|$. So $7 \notin \pi(K)$. We may assume that K is an elementary Abelian p-group for $p \in \{2,3\}$.
If $2 \in \pi(K)$, then Let $x \in G/K$, $X = \langle x \rangle$, $o(x) = 7$ and $z = exp(2\pi i/7)$. Now by using [13] about the irreducible characters of $U_3(8)$ (mod 2), we can see that $\langle \varphi_1|_X, 1|_X \rangle = (1 + 1 \times 6)/7 = 1 > 0$, $\langle \varphi_2|_X, 1|_X \rangle = (8 + 2(-(z + z^{-1}) - 2(z^2 + z^{-2})) + 2(-(z^2 + z^{-2}) - 2(z^4 + z^{-4})) + 2(-(z^4 + z^{-4}) - 2(z^8 + z^{-8})))/7 = (8 - 2(\sum_{i=1}^{6} z^i) - 4(\sum_{i=1}^{6} z^i))/7 = (8 - 2(-1) - 4(-1))/7 = 2 > 0$, similarly

$\langle\varphi_3|_X, 1|_X\rangle = \langle\varphi_4|_X, 1|_X\rangle = 2 > 0$, $\langle\varphi_5|_X, 1|_X\rangle = (9 + 2(z + z^{-1}) + 2(z^2 + z^{-2}) + 2(z^4 + z^{-4}))/7$ $= (9 + 2(\sum_{i=1}^{6} z^i))/7 = (9 + 2(-1))/7 = 1 > 0$, similarly $\langle\varphi_i|_X, 1|_X\rangle = 1 > 0$ for $i = 6, 7, 8, 9, 10$, $\langle\varphi_{11}|_X, 1|_X\rangle = \langle\varphi_{12}|_X, 1|_X\rangle = (27 + 2(-1) + 2(-1) + 2(-1))/7 = (27 - 6)/7 = 3 > 0$, $\langle\varphi_{13}|_X, 1|_X\rangle =$ $(64+2(-1+(z+z^{-1}))+2(-1+(z^2+z^{-2}))+2(-1+(z^4+z^{-4})))/7 = (64-6+2(-1))/7 = 8 > 0$, similarly $\langle\varphi_{14}|_X, 1|_X\rangle = \langle\varphi_{15}|_X, 1|_X\rangle = 8 > 0$, $\langle\varphi_{16}|_X, 1|_X\rangle = (72+2(z+z^{-1})+2(z^2+z^{-2})+2(z^4+z^{-4}))/7$ $= (72 + 2(-1))/7 = 10 > 0$, similarly $\langle\varphi_j|_X, 1|_X\rangle = 10 > 0$ for $j = 16, 17, ..., 21$ and $\langle\varphi_{22}|_X, 1|_X\rangle =$ $(512 + 2(1) + 2(1) + 2(1))/7 = (512 + 6)/7 = 74 > 0$. Therefore for every irreducible character ϕ of $U_3(8)$ (mod 2) we show that $\langle\varphi|_X, 1|_X\rangle = \sum_{x\in X}\varphi(x)/|X| > 0$. Now by using Lemma 2.19, it follows that $14 \in \pi_e(G)$. Then $2 \sim 7$ in $\Gamma(G)$, which is a contradiction.

If $3 \in \pi(K)$, then Let $x \in G/K$, $X = \langle x\rangle$, $o(x) = 19$ and $z = exp(2\pi i/19)$. Now by using [13] about the irreducible characters of $U_3(8)$ (mod 3), we have $\langle\varphi_1|_X, 1|_X\rangle = (1 + 1 \times 18)/19 = 1 > 0$, $\langle\varphi_2|_X, 1|_X\rangle = (56 + 3(-1) + 3(-1) + 3(-1) + 3(-1) + 3(-1) + 3(-1))/19 = (56 - 18)/19 = 2 > 0$, $\langle\varphi_i|_X, 1|_X\rangle = (133+0\times 18)/19 = 7 > 0$ for $i = 3, 4, 5$, $\langle\varphi_j|_X, 1|_X\rangle = (513+0\times 18)/19 = 27 > 0$ for $j =$ 6, 7, 8, $\langle\varphi_9|_X, 1|_X\rangle = (567+(-(z+z^7+z^{11})-(z^{-1}+z^{-7}+z^{-11})-(z^2+z^{14}+z^{22})-(z^{-2}+z^{-14}+z^{-22})-$ $(z^4 + z^{28} + z^{44}) - (z^{-4} + z^{-28} + z^{-44})) \times 3)/19 = (567 - (\sum_{i=1}^{18} z^i) \times 3)/19 = (567 - (-1) \times 3)/19 =$ $570/19 = 30 > 0$, similarly $\langle\varphi_i|_X, 1|_X\rangle = 30 > 0$ for $i = 10, 11, ..., 14$. Now by using Lemma 2.19, it follows that $57 \in \pi_e(G)$. Then $3 \sim 19$ in $\Gamma(G)$, which is a contradiction. Hence $K = 1$ and $G \cong U_3(8)$, $U_3(8).3_1$, $U_3(8).3_3$ or $U_3(8).6$. Therefore $k(\Gamma(U_3(8)) = 4$.

The proof of the other cases are similar and for convenience, we do some of them, namely $U_3(11)$, $U_3(13)$, $U_3(16)$, $U_3(19)$ and $U_3(32)$.

The Case $U = U_3(11)$

The prime graph of the simple group $U_3(11)$ is shown in Fig. 8. It follows from Lemma 2.15, that $S \cong U_3(11)$. We know that $Out(U_3(11)) \cong \mathbb{Z}_3 : \mathbb{Z}_2$. Therefore $\overline{G}/U_3(11) \leq \mathbb{Z}_3 : \mathbb{Z}_2$. Since $U_3(11).3$ has a connected prime graph by Lemma 2.18, then $\Gamma(U_3(11).3)$ is not subgraph of $\Gamma(G)$. So we have $G/K = U_3(11)$ or $U_3(11).2$, where K is a nilpotent $\{2, 3, 5, 11\}-$ group. We know that $U_3(11)$ contains a Frobenius subgroup with Frobenius kernel of order 37 and Frobenius complement of order 3 by Lemma 2.14. If $5 \in \pi(K)$, then by Lemma 2.10, $3 \sim 5$ in $\Gamma(G)$, which is a contradiction. Similarly $11 \notin \pi(K)$. If $3 \in \pi(K)$, then let $P \in Syl_3(K)$ and $Q \in Syl_{11}(G)$. We know that $3 \nsim 11$ in $\Gamma(G)$. So Q acts fixed point freely on P. Thus PQ is a Frobenius group, with Frobenius kernel P and Frobenius complement Q. Therefore Q is cyclic by Lemma 2.5. This is a contradiction, since $U_3(11)$ has no element of order 11^3. If $2 \in \pi(K)$, then Let $x \in G/K$, $X = \langle x\rangle$, $o(x) = 37$ and $z = exp(2\pi i/37)$. Now by using [13] about the irreducible characters of $U_3(11)$ (mod 2), we have $\langle\varphi_1|_X, 1|_X\rangle = (1 + 1 \times 36)/37 = 1 > 0$, $\langle\varphi_2|_X, 1|_X\rangle =$ $(110 + 12 \times 3 \times (-1))/37 = 74/37 = 2 > 0$, $\langle\varphi_i|_X, 1|_X\rangle = (370 + 36 \times 0)/37 = 10 > 0$ for $i = 3, 4, 5$, $\langle\varphi_6|_X, 1|_X\rangle = (1110 + 36 \times 0)/37 = 30 > 0$, $\langle\varphi_7|_X, 1|_X\rangle = \langle\varphi_8|_X, 1|_X\rangle = (1332 + 36 \times 0)/37 = 36 > 0$, $\langle\varphi_i|_X, 1|_X\rangle = (1440 + (-(z + z^{10} + z^{26}) - (z^{-1} + z^{-10} + z^{-26})- (z^5 + z^{50} + z^{130}) - (z^{-5} + z^{-50} +$ $z^{-130})- (z^7 + z^{70} + z^{182}) - (z^{-7} + z^{-70} + z^{-182})- (z^{14} + z^{140} + z^{364}) - (z^{-14} + z^{-140} + z^{-364})-$ $(z^{17} + z^{170} + z^{442}) - (z^{-17} + z^{-170} + z^{-442})- (z^{21} + z^{210} + z^{546}) - (z^{-21} + z^{-210} + z^{-546})) \times 3)/37$

$= (1440 - (\sum_{i=1}^{36} z^i) \times 3)/37 = (1440 - (-1) \times 3)/37 = 1443/37 = 39 > 0$ for $i = 9, 11, ..., 20$. Now by using Lemma 2.19, it follows that $74 \in \pi_e(G)$. Then $2 \sim 37$ in $\Gamma(G)$, which is a contradiction. Hence $K = 1$, $G \cong U_3(11)$ or $U_3(11).2$. Therefore $k(\Gamma(U_3(11))) = 2$.

The Case $U = U_3(13)$

The prime graph of the simple group $U_3(13)$ is shown in Fig. 9. It follows from Lemma 2.15, that $S \cong U_3(13)$. We know that $Out(U_3(13)) \cong \mathbb{Z}_2$. Therefore $\overline{G}/U_3(13) \leq \mathbb{Z}_2$. Thus $G/K = U_3(13)$ or $U_3(13).2$. We know that $U_3(13)$ contains a Frobenius subgroup with Frobenius kernel of order 157 and Frobenius complement of order 3 by Lemma 2.14. If $13 \in \pi(K)$, then by Lemma 2.10, $3 \sim 13$ in $\Gamma(G)$, which is a contradiction. If $3 \in \pi(K)$, then let $P \in Syl_3(K)$ and $Q \in Syl_{13}(G)$. We know that $3 \not\sim 13$ in $\Gamma(G)$. So Q acts fixed point freely on P. Thus PQ is a Frobenius group, with Frobenius kernel P and Frobenius complement Q. Therefore Q is cyclic by Lemma 2.5. This is a contradiction, since $U_3(13)$ has no element of order 13^3. Therefore, $G/O_\pi(G) \cong U_3(13)$ or $U_3(13).2$, where $\pi \subseteq \{2, 7\}$.

The Case $U = U_3(16)$

The prime graph of the simple group $U_3(16)$ is shown in Fig. 10. It follows from Lemma 2.15, that $S \cong U_3(16)$. We know that $Out(U_3(16)) \cong \mathbb{Z}_8$. Therefore $\overline{G}/U_3(16) \leq \mathbb{Z}_8$ and so $G/K \leq U_3(16).\mathbb{Z}_8$. We know that $U_3(16)$ contains a Frobenius subgroup with Frobenius kernel of order 241 and Frobenius complement of order 3 by Lemma 2.14. If $2 \in \pi(K)$, then by Lemma 2.10, $2 \sim 3$ in $\Gamma(G)$, which is a contradiction. Since $S \ncong A_7$, $A_1(q)$ and $2 \not\sim 3$ in $\Gamma(G)$, then by Lemma 2.8, we have $3 \nmid |K||\overline{G}/U_3(16)|$. So $3 \notin \pi(K)$. Similarly $5 \notin \pi(K)$. Therefore $G/O_{17}(G) \leq U_3(16).\mathbb{Z}_8$.

The Case $U = U_3(19)$

The prime graph of the simple group $U_3(19)$ is shown in Fig. 11. It follows from Lemma 2.15, that $S \cong L_2(7)$, $L_2(8)$, $U_3(3)$, A_7, $L_2(49)$, $L_3(4)$, $U_3(5)$, $U_4(3)$, A_8, A_9, J_2, $S_4(7)$, $S_6(2)$, $O_8^+(2)$ or $U_3(19)$. If $S \cong L_2(7)$, then $L_2(7) \leq \overline{G} = G/K \leq Aut(L_2(7))$. We know that $L_2(7)$ contains a Frobenius subgroup with Frobenius kernel of order 7 and Frobenius complement of order 3 by [5]. Now if $19 \in \pi(K)$, then by Lemma 2.10, $3 \sim 19$ in $\Gamma(G)$, which is a contradiction. So $19 \in \pi(G)$ and $19 \notin \pi(K)$. Since $G/K \leq Aut(L_2(7))$, then $19 \mid |Aut(L_2(7))|$, which is a contradiction. If $S \cong L_2(49)$, then $L_2(49) \leq \overline{G} = G/K \leq Aut(L_2(49))$. We know that $L_2(49)$ contains a Frobenius subgroup with Frobenius kernel of order 49 and Frobenius complement of order 24 by [10]. Now if $19 \in \pi(K)$, then by Lemma 2.10, $3 \sim 19$ in $\Gamma(G)$, which is a contradiction. So $19 \in \pi(G)$ and $19 \notin \pi(K)$. Since $G/K \leq Aut(L_2(49))$, then $19 \mid |Aut(L_2(49))|$, which is a contradiction. Similarly S can not be isomorphic to $L_2(8)$, $U_3(3)$, A_7, $L_3(4)$, $U_3(5)$, $U_4(3)$, A_8, A_9, J_2, $S_4(7)$, $S_6(2)$ and $O_8^+(2)$. Now suppose that $S \cong U_3(19)$. We know that $Out(U_3(19)) \cong \mathbb{Z}_2$. Therefore $\overline{G}/U_3(19) \leq \mathbb{Z}_2$ and so $G/K = U_3(19)$ or $U_3(19).2$. We know that $U_3(19)$ contains a Frobenius subgroup with Frobenius kernel of order 7 and Frobenius complement of order 3 by Lemma 2.14. If $19 \in \pi(K)$, then by Lemma 2.10, $3 \sim 19$ in $\Gamma(G)$, which is a contradiction. If $3 \in \pi(K)$, then let $P \in Syl_3(K)$ and $Q \in Syl_{19}(G)$. We know that $3 \not\sim 19$ in $\Gamma(G)$. So Q acts fixed point freely on P. Thus PQ is a Frobenius group, with Frobenius kernel P and Frobenius complement Q. Therefore Q is cyclic by Lemma 2.5. This is a contradiction, since $U_3(19)$ has no element of order 19^3. Therefore, $G/O_\pi(G) \cong U_3(19)$ or $U_3(19).2$, where $\pi \subseteq \{2,5\}$.

The Case $U = U_3(32)$

The prime graph of the simple group $U_3(32)$ is shown in Fig. 12. It follows from Lemma 2.15, that $S \cong U_3(32)$. We know that $Out(U_3(32)) \cong \mathbb{Z}_3 : \mathbb{Z}_{10}$. Hence $\overline{G}/U_3(32) \leq \mathbb{Z}_3 : \mathbb{Z}_{10}$ and so $G/K \leq U_3(32).(\mathbb{Z}_3 : \mathbb{Z}_{10})$. We know that $U_3(32)$ contains a Frobenius subgroup with Frobenius kernel of order 331 and Frobenius complement of order 3 by Lemma 2.14. If $2 \in \pi(K)$, then by Lemma 2.10, $2 \sim 3$ in $\Gamma(G)$, which is a contradiction. Similarly $31 \notin \pi(K)$. Since $S \ncong A_7$, $A_1(q)$ and $2 \nsim 3$ in $\Gamma(G)$, then by Lemma 2.8, we have $3 \nmid |K||\overline{G}/U_3(32)|$. So $3 \notin \pi(K)$. Therefore $G/O_{11}(G) \leq U_3(32).(\mathbb{Z}_3 : \mathbb{Z}_{10})$.

By the main theorem and definition of quasirecognizability and $n-$recognizability by prime graph, we can conclude the following corollaries:

Corollary 3.1. *The finite simple group* $U_3(q)$ *for* $2 < q = p^\alpha \neq 3, 5, 9, 17 < 100$ *is quasirecognizable by prime graph.*

Corollary 3.2. *The finite simple groups* $U_3(8)$ *and* $U_3(11)$ *are* $4-$*recognizable and* $2-$*recognizable by prime graph, respectively.*

By [31], we know that the simple groups $U_3(3)$, $U_3(5)$, $U_3(7)$ are not recognizable by spectrum, and so they are not recognizable by prime graph. It seems that if $p \geq 11$ is a prime number, then the simple group $U_3(p)$ is $2-$recognizable by prime graph. We pose the following problem:

Problem 1: Is the simple group $U_3(p)$, where $p \geq 11$ is a prime number, $2-$recognizable by prime graph?

In this paper we find a first example of finite group G such that $k(\Gamma(G)) = 4$, namely $G = U_3(8)$. Until recently, no examples of groups with $k(\Gamma(G)) \notin \{1, 2, \infty\}$ were known. So we posed the following question, too.

Problem 2: Is there an integer n such that, for any group G, either $k(\Gamma(G)) \leq n$ or not finite?

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