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CENTRALIZERS IN SIMPLE LOCALLY FINITE GROUPS

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Dedicated to the memories of Narain Gupta and James Wiegold

Abstract. This is a survey article on centralizers of finite subgroups in locally finite, simple groups or LFS-groups as we will call them. We mention some of the open problems about centralizers of subgroups in LFS-groups and applications of the known information about the centralizers of subgroups to the structure of the locally finite group. We also prove the following: Let G be a countably infinite non-linear LFS-group with a Kegel sequence $\mathcal{K} = \{(G_i, N_i) \mid i \in \mathbf{N}\}$. If there exists an upper bound for $\{|N_i| \mid i \in \mathbf{N}\}$, then for any finite semisimple subgroup F in G the subgroup $C_G(F)$ has elements of order p_i for infinitely many distinct prime p_i. In particular $C_G(F)$ is an infinite group. This answers Hartley's question provided that there exists a bound on $\{|N_i| \mid i \in \mathbf{N}\}$.

1. Brief History

In 1954 World Mathematical Congress R. Brauer indicated the importance of the centralizers of involutions in the classification of the finite simple groups. He asked whether it is possible to detect the finite simple group from the structure of the centralizers of its involutions. Then it became a program in the classification of the finite simple groups. There were two types of questions:

1) Given the finite simple group G, find the structure of $C_G(i)$ for all involutions $i \in G$.

2) Find the structure of the simple group G when the group $H = C_G(i)$ is known for an involution $i \in G$.

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One may ask similar questions for the locally finite, simple groups or LFS-groups as we will call them. One may generalize these questions from the centralizers of involutions to the centralizers of arbitrary elements or subgroups.

1′) Given an infinite LFS-group, find the structure of the centralizers of elements (or subgroups) in particular, centralizers of involutions.

2′) Given the structure of the centralizer of an element in a LFS-group G, find the structure of G.

In order to study the centralizers of elements in LFS-groups one needs to use the information for the centralizers of elements in finite simple groups. In this case, one of the indispensable tool is to use the notion of Kegel covers.

2. Kegel Covers and Centralizers

Recall that a *Kegel cover* $\mathcal{K}$ of a locally finite group G is a set $\mathcal{K} = \{(H_i, M_i) \mid i \in I\}$ such that for all $i \in I$, the group H_i is a finite subgroup of G, M_i is a maximal normal subgroup of H_i and for each finite subgroup K of G, there exists $i \in I$ such that $K \leq H_i$ and $K \cap M_i = 1$. The simple groups H_i/M_i are called *Kegel factors* of $\mathcal{K}$. Kegel proved that every infinite simple locally finite group has such a Kegel cover [14], [15, Lemma 4.5]. In the case of G has countably infinite order, we have an increasing chain of finite subgroups $G_1 \leq G_2 \leq G_3 \leq \ldots$ such that $G = \bigcup_{i \in \mathbf{N}} G_i$, and $G_i \cap M_{i+1} = 1$, and G_i/M_i is a finite simple group. In this case $\mathcal{K} = \{(G_i, M_i) \mid i \in \mathbf{N}\}$ is called a *Kegel sequence* of G.

For the study of centralizers of elements in infinite LFS-groups one of the obstacle is the following: We know the structure of the centralizers of subgroups or elements X in the finite simple section H_i/M_i and in general $C_{H_i/M_i}(X)$ is not isomorphic or equal to $C_{H_i}(X)M_i/M_i$, for this reason the information about the centralizers in the finite simple group, in general does not transform directly to the information on $C_G(X) = \bigcup_{i \in I} C_{H_i}(X)$. For the LFS-groups which have a certain type of Kegel cover, it is possible to transfer, the structure of centralizers of elements in simple finite groups to the structure in the centralizers of elements in LFS-groups. One obtains quite nice, similar structures in infinite LFS-groups as in the case of finite simple groups. One can rediscover the importance of the Jordan-Hölder theorem for finite groups and the extension of this to the groups which has this property locally. For this, recall that, the class of all locally finite groups having a series of finite length in which there are at most n, non-abelian simple factors and the rest are locally soluble is denoted by $\mathfrak{F}_n$.

Lemma 2.1. [10, Lemma 2.3] *If all finitely generated subgroups of a locally finite group lie in $\mathfrak{F}_n$, then the group also lies in $\mathfrak{F}_n$.*

The examples of LFS-groups which do not have a certain kind of Kegel cover may have quite different types of structures in centralizers of elements. Therefore we need to assume that our LFS-group has a certain type of Kegel cover which enables us to get the required information for this class of groups.

Let $\mathcal{L}$ denote the class of all LFS-groups. Let $\mathcal{L}_1$ denote the class of LFS-groups G satisfying:

For a finite subgroup F of order n in G, there exists a Kegel cover $\mathcal{K} = \{(G_i, N_i) \mid i \in \mathbf{I}\}$ of G such that, if π is the set of prime divisors of n,

(i) $O_{\pi'}(N_i)$ is soluble.

(ii) $N_i/O_{\pi'}(N_i)$ is hypercentral in G_i.

(iii) G_i/N_i is either an alternating group or a simple group of Lie type over a field of characteristic not in π.

Let $\mathcal{L}_2 = \mathcal{L} \setminus \mathcal{L}_1$. Then $\mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_2$.

In the next theorem B. Hartley and the author proved that for a non-linear LFS-group G and an element $x \in G$ of order n, such that $G \in \mathcal{L}_1$, the structure of centralizers are similar to the case in finite simple groups.

Theorem 2.2. [10, Theorem B$'$] *Let G be a non-linear LFS-group, and x be an element of G of order n. Suppose that there exists a Kegel Sequence $\mathcal{K} = \{(G_i, N_i) \mid i \in \mathbf{N}\}$ of G such that, if π is the set of prime divisors of n, then*

(i) $O_{\pi'}(N_i)$ is soluble.

(ii) $N_i/O_{\pi'}(N_i)$ is hypercentral in G_i.

(iii) G_i/N_i is either an alternating group or a simple group of Lie type over a field of characteristic not in π.

Then $C_G(x)$ belongs to $\mathfrak{F}_{n+[\frac{4}{n}]}$ and involves a non-linear simple group.

In the special case in which every finite subset of elements lies in a finite simple group we have the following.

Theorem 2.3. *(*[10, Theorem B]*) Suppose that every finite set of elements of G lies in a finite simple subgroup, and suppose that G is a non-linear LFS-group. Then there exists a prime p with the following property.*

Let n be any natural number not divisible by p, let g be any element of order n in G, and let $r(n) = n + [\frac{4}{n}]$. Then $C_G(g)$ has a series of finite length at most $2r(n) + 1$ in which each factor is either non-abelian simple or soluble. The number of non-abelian simple factors is at most $r(n)$ and at least one of them is non-linear. The derived length of each soluble factor is at most 6, and there are at most $r(n) + 1$ of them.

For simplicity, we will mention the class $\mathcal{L}_{1a}$ which denotes the class of LFS-groups in which for a finite subgroup F of order n there exists a Kegel sequence (G_i, N_i) with $N_i = 1$ for all $i \in \mathbf{N}$ and when G_i is a simple group of Lie type over a field of characteristic p, then p is not in π, the set of prime divisors of n.

Clearly $\mathcal{L}_{1a} \subset \mathcal{L}_1$. In general there is a standard technique that the results in the class $\mathcal{L}_{1a}$ can be transferred to the class $\mathcal{L}_1$ see the proof of Theorem 2.2 and [18, Theorem 5]. The class of simple linear locally finite groups is in the class $\mathcal{L}_{1a}$ see [15, Theorem 4.6].

Let V be a vector space which has finite or infinite dimension over a field K. An invertible linear transformation $g : V \to V$ is called *finitary* if $\dim(g-1)V < \infty$. The set of all finitary linear maps on V generates the finitary linear, general linear group $FGL(V)$ and the subgroups of $FGL(V)$ are called the *finitary linear groups*. An arbitrary group G is called a finitary linear group, if it has a finitary representation on a vector space V over a field K. If the field K is a locally finite field, then the finitary linear groups are locally finite groups. The class of finitary linear LFS-groups is in $\mathcal{L}_1$, by the classification of J. I. Hall in [8, Page 165] and [7].

But the involvement in these classes are proper $\mathcal{L} \supsetneqq \mathcal{L}_1 \supsetneqq \mathcal{L}_{1a}$. In particular for the groups in $\mathcal{L}_{1a}$ we have $M_i = 1$, for all $i \in I$. Then G has a local system consisting of finite simple subgroups. But not all simple locally finite groups has such a type of Kegel cover see [24], [13] and Theorem 2.4. For the structure of the centralizers of elements in the groups in class $\mathcal{L}_1$ and the ones, in class $\mathcal{L}_2$ the behavior are quite different. For this, consider the infinite simple groups constructed by Meierfrankenfeld in [21]. In fact, this construction of infinite non-linear LFS-groups answers negatively many conjectures about centralizers of elements in LFS-groups.

Theorem 2.4. *(Meierfrankenfeld [21]) Let Π be a non-empty set of primes. Then there exists a non-linear, locally finite, simple group G such that*

(a) The centralizer of every non-trivial Π-element has a locally soluble Π-subgroup of finite index.

(b) There exists an element whose centralizer is a locally soluble Π-group.

Observe that the above groups are defined for any subset of the set of prime numbers. If we choose the set Π to contain all primes and only one prime p respectively, then it is a consequence of the above theorem that:

Corollary 2.5. [21, Theorem A] *(a) There exists a non-linear LFS-group such that the centralizer of every non-trivial element is locally soluble-by-finite.*

(b) Let p be a prime. Then there exists a non-linear LFS-group with an element whose centralizer is a p-group.

Then by this construction there are non-linear LFS-groups such that the centralizer of every non-trivial element is a locally soluble by finite group i.e. has a locally soluble subgroup of finite index. On the other hand, we proved in Theorem 2.2 that, if a non-linear LFS-group $G \in \mathcal{L}_1$, then there are infinitely many elements whose centralizers involve an infinite simple group. Recall that a group X involves an infinite simple group if there exist subgroups A and B of X such that $A \lhd B$ and B/A is isomorphic to an infinite simple group. Therefore we may use this information about the centralizers of elements to decide whether such groups are in class $\mathcal{L}_1$ or not. For example, the groups in Theorem 2.4 do not belong to $\mathcal{L}_1$. This property answers the question of type $2'$, namely whether a group G is in $\mathcal{L}_1$ or not, whenever we know the structure of the centralizers of elements or subgroups.

We asked the following question to B. Hartley as mentioned in [8, Question 3.8].

Question 2.6. *Does there exist a non-linear LFS-group in which the centralizer of every non-trivial element is almost soluble?*

Observe that by [8, Lemma 3.7], we cannot expect $C_G(x)$ is soluble. But observe also that for the almost locally soluble case the examples exist by Theorem 2.4.

Infinite simple locally finite groups are studied in two classes:

(i) Infinite linear LFS-groups.

(ii) Infinite non-linear LFS-groups.

The infinite linear ones are classified in [1], [4], [12], [22], they are the simple groups of Lie type over a locally finite field. A group is called a **non-linear group** if it does not have a faithful representation on a finite dimensional vector space V over a field F. For an infinite set Ω, the group $Alt(\Omega)$ forms a natural example of a non-linear LFS-group.

The following question is asked by Otto H. Kegel in [20, Question 5.18]. Let G be an infinite locally finite simple group. Is the order $|C_G(g)|$ infinite for every element $g \in G$. An affirmative answer is given in [10].

Theorem 2.7. *(Hartley -Kuzucuoğlu) In an infinite LFS-group, the centralizer of every element is infinite.*

Then a natural generalization of this is the following: Is the centralizer of every finite subgroup in an infinite simple locally finite group infinite?

The answer is no, even for abelian subgroups, as the following easy example shows. One can see that in $PSL(2, F)$ where F is an algebraically closed, locally finite field of odd characteristic, the subgroup

$$A = \langle \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} Z, \quad \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix} Z \mid \lambda^2 = -1 \rangle$$

is an abelian subgroup of order 4 and $C_{PSL(2,F)}(A) = A$. In fact, it is easy to see that for an infinite simple linear locally finite group G one can always find, finite subgroups with finite centralizer in G. Then as was pointed out in [8, Lemma 3.11], for a locally finite group G with $Z(G) = 1$ having a finite subgroup H with $C_G(H)$ finite is equivalent to having a finite subgroup $F \geq H$ such that $C_G(F) = 1$. For simple non-linear LFS-groups the question of whether centralizers of finite subgroups are infinite or not is still open, namely:

Question 2.8. *(Hartley) Is the centralizer of every finite subgroup in a simple non-linear locally finite group infinite?*

Stronger question:

Question 2.9. *Does the centralizer of every element in non-linear LFS-groups contain infinite abelian subgroups which has elements of order p_i for infinitely many distinct prime p_i.*

The answer to this question in this generality is negative by Theorem 2.4. The question of whether the centralizer of every involution in an infinite LFS-group involves an infinite simple group is answered negatively also by Theorem 2.4. On the other hand if we assume that the non-linear LFS-group belongs to class $\mathcal{L}_1$, then the answer to the above question for the involutions is affirmative. This can be extracted from [3, Theorem]. In fact we proved in [17, Theorem 1] that, centralizer of every element of odd order in a non-linear LFS-group involves a non-linear LFS-group provided that every element lies in a finite simple group.

On the other hand the structure of centralizers of elements in simple locally finite groups gives information about the structure of proper subgroups and internal structure of subgroup lattice of the group. We have applied our results also for minimal non-$\wp$-groups. Let $\wp$ be a group theoretical property. A group G is called a minimal non-$\wp$ group if G does not have the property $\wp$ but every proper subgroup has the property $\wp$. For minimal non-$\wp$ groups one needs to understand whether such a group could be a simple group or not, or one needs to classify those locally finite simple minimal non-$\wp$ groups. For these type of questions, if the centralizers of elements do not have the property $\wp$, then we may conclude that such a group cannot be a minimal non-$\wp$ group. This technique is used to show that there exists no simple locally finite minimal non-FC-group [19] and there exists no simple locally finite barely transitive group [11]. Recall that a group H is an FC-group if for every element $x \in H$ the index $|H : C_H(x)|$ is finite. The group G is a minimal non-FC-group, if it is not an FC-group, but every proper subgroup of G is an FC-group. The group X is called a *barely transitive group* if it has a subgroup H of infinite index in X and $\bigcap_{g \in X} H^g = 1$ and for every proper subgroup K of X, the index $|K : K \cap H| < \infty$.

In this respect for infinite LFS-groups, centralizers of elements are a natural source to find proper subgroups. Therefore the information of the structure of the centralizers of elements plays an important rôle. The question by Kleidman and Wilson in [16]: "Classify all simple locally finite minimal almost locally soluble groups" can be considered in this fashion. They proved that, every infinite simple locally finite minimal non-almost locally soluble group is isomorphic to $PSL(2, \mathbf{F})$ or $Sz(\mathbf{F})$ for some locally finite field $\mathbf{F}$ which has no infinite proper subfields. In particular such groups are linear.

In this spirit we will ask the following:

Question 2.10. *Classify all infinite simple locally finite groups in which the centralizer of an involution is almost locally soluble.*

The examples of Meierfrankenfeld in Theorem 2.4 show that, there are infinitely many non-isomorphic non-linear LFS-groups such that in these groups centralizer of every involution is almost locally soluble. So there are non-linear LFS-groups in which centralizer of every involution (element) is almost locally soluble. In order to get rid of non-linear LFS-groups of this type, we restrict the question for those groups $G \in \mathcal{L}_1$. Then one can extract the following theorem from [3, Theorem 4].

Theorem 2.11. *Let G be an infinite simple locally finite group in which centralizer of an involution has a locally soluble subgroup of finite index. If G has a Kegel cover $\mathcal{K}$ such that $(H_i, M_i) \in \mathcal{K}$, $M_i/O_{2'}(M_i)$ is hypercentral in $H_i/O_{2'}(M_i)$. Then G is isomorphic to one of the following:*

(i) *$PSL(2, \mathbf{K})$ where $\mathbf{K}$ is an infinite locally finite field of arbitrary characteristic, $PSL(3, \mathbf{F})$, $PSU(3, \mathbf{F})$ and $Sz(\mathbf{F})$ where $\mathbf{F}$ is an infinite locally finite field of characteristic 2. In this case, centralizers of involutions are soluble.*

(ii) *There exist involutions $i, j \in G$ such that $C_G(i)$ is locally soluble by finite and $C_G(j)$ involves an infinite simple group if and only if $G \cong PSp(4, \mathbf{F})$ and the characteristic of $\mathbf{F}$ is 2.*

Corollary 2.12. *If G is as in the above Theorem and if we assume that centralizer of every involution has a locally soluble subgroup of finite index, then G is as in **(i)** in the above Theorem.*

One of the other question in LFS-groups about the centralizers which is also discussed in [2] is the following:

Question 2.13. *Let G be a LFS-group. Is it true that, if $C_G(F)$ is a linear group for a finite subgroup F of G, then G is a linear group?*

Yet again if we assume that G is in the class $\mathcal{L}_{1a}$, then we have a positive answer for involutions by a Corollary of [3, Theorem 4].

Corollary 2.14. *Let G be an infinite simple locally finite group such that every finite subset lies in a finite simple subgroup. Let $i \in G$ be an involution. Then the following statements hold.*

(1) *$C_G(i)$ is linear if and only if G is linear.*
(2) *If $C_G(i)$ involves a finite non-abelian simple group for some involution $i \in G$, then $C_G(i)$ involves an infinite simple group.*

If $G \in \mathcal{L}_{1a}$ and for every element $x \in G$ the group $C_G(x)$ is linear, then G is linear can be extracted from [10, Theorem B] for abelian semisimple subgroups of odd order see [18, Theorem 1].

One may ask the above questions about centralizers of subgroups in LFS-groups for the fixed point subgroups of finite subgroups of the automorphism groups. For the structure of the centralizers of elements and fixed points of automorphism see Hartley's survey in [8] and the paper [9].

Recall that an element in a simple group of Lie type is *semisimple* if its order and the characteristic of the field is relatively prime. In the alternating groups all elements are semisimple.

Definition 2.15. *Let G be a simple group of Lie type. A finite subgroup A of G is called a **totally semisimple** subgroup (the name suggested by A. E. Zalesski) if every element of A is a semisimple element in G.*

For the centralizers of finite abelian semisimple subgroups in infinite LFS-group G we have the following [18, Theorem 2].

Theorem 2.16. *Suppose that G is infinite non-linear and every finite set of elements of G lies in a finite simple group. Then*

(i) There exist infinitely many finite abelian semisimple subgroups F of G and local systems L of G consisting of simple subgroups such that F is abelian totally semisimple in every member of L.

(ii) There exists a function f from natural numbers to natural numbers independent of G such that $C = C_G(F)$ has a series of finite length in which at most $f(|F|)$ factors are simple non-abelian groups for any F as in (i). Furthermore C involves a non-linear simple group. In particular $C_G(F)$ is an infinite group.

Definition 2.17. *Let G be a countably infinite simple locally finite group and F be a finite subgroup of G. The group F is called a $\mathcal{K}$-semisimple subgroup of G, if G has a Kegel sequence $\mathcal{K} = \{(G_i, M_i) : i \in \mathbb{N}\}$ such that $(|M_i|, |F|) = 1$, M_i are soluble for all i and if G_i/M_i is a linear group over a field of characteristic p_i, then $(p_i, |F|) = 1$.*

In the following Theorem the finite subgroups are not necessarily abelian. Then the proof technique in Theorem 2.16 and in Theorem 2.18 are quite different. For details see [6].

Theorem 2.18. *([6] Ersoy-Kuzucuoğlu) Let G be a non-linear simple locally finite group which has a Kegel sequence $\mathcal{K} = \{(G_i, 1) : \ i \in \mathbf{N}\}$ consisting of finite simple subgroups. Then for any finite $\mathcal{K}$-semisimple subgroup F, the centralizer $C_G(F)$ is an infinite group.*

Moreover $C_G(F)$ has an infinite abelian subgroup A isomorphic to the restricted direct product of Z_{p_i} for infinitely many distinct prime p_i.

By using the above Theorem from the information on the structure of the centralizers of elements we may decide whether such a simple group is in class $\mathcal{L}_1$ or not.

Question 2.19. *Is it true that in all LFS-groups centralizers of elements are in class $\Im_n$ for some $n \in \mathbf{N}$.*

As we have mentioned in Page 2 one needs a control on the index $|C_{G/N}(FN/N) : C_G(F)N/N|$. The following lemma gives such a control which is a generalization of [2, Theorem 4.5] from cyclic subgroups to arbitrary subgroups.

Lemma 2.20. *Let G be a finite group and $N \trianglelefteq G$. Let F be a subgroup generated by the set $\{a_1, a_2, \dots a_k\}$. Then*

$$|C_{G/N}(FN/N) : C_G(F)N/N| \leq |C_N(F)||N|^{k-1}$$

Proof. Let $\{\bar{g}_1, \bar{g}_2, \dots, \bar{g}_m\}$ be a set of cosets of $C_G(F)N/N$ in $C_{G/N}(FN/N)$ and let $\{g_1, , \dots g_m\} \subseteq G$ be a set of coset representatives in G of the above cosets. Let $\{h_1, h_2, , \dots h_n\} \subseteq N$ be the set of coset representatives of $C_N(F)$ in N.

Let

$$M = \{ \ ([a_1, h_i g_j], [a_2, h_i g_j], \dots, [a_k, h_i g_j]) \ : \ i = 1, \dots n, \ \ j = 1, \dots m \ \ \}$$

We first observe that $|M| = mn$. Indeed if x and y are two elements of M, then

$x = ([a_1, h_i g_j], [a_2, h_i g_j], \dots, [a_k, h_i g_j])$ and $y = ([a_1, h_s g_t], [a_2, h_s g_t], \dots, [a_k, h_s g_t])$. If $x = y$, then $[a_r, h_i g_j] = [a_r, h_s g_t]$ for all $r = 1, \dots, k$. Hence $(h_i g_j)(h_s g_t)^{-1} \in C_G(a_r)$ for all $r = 1, \dots, t$. Since a_r's generate the group F for $r = 1, \dots, k$, we have $(h_i g_j)(h_s g_t)^{-1} = h_i(g_j g_t^{-1})h_s^{-1} \in C_G(F)$. It follows that $(\bar{g}_j \bar{g}_t^{-1}) \in C_G(F)N/N$. Hence $j = t$. Then we have $[a_r, h_i] = [a_r, h_s]$ for all $r = 1, \dots, k$. It follows that $h_i h_s^{-1} \in C_N(a_r)$ for all $r = 1, \dots, k$ and so $h_i h_s^{-1} \in C_N(F)$. Hence we obtain $h_i = h_s$. So whenever $i \neq s$ or $j \neq t$ we have $x \neq y$. Hence $|M| = mn$.

It is clear that, for any r and for any i and j, the element $[a_r, h_i g_j] \in N$. Then define a map

$$\theta : M \quad \to \underbrace{N \times N \ldots \times N}_{k-times}$$

$([a_1, h_i g_j], [a_2, h_i g_j], \ldots, [a_k, h_i g_j]) \to [a_1, h_i g_j], [a_2, h_i g_j], \ldots, [a_k, h_i g_j]$.

The map θ is one-to-one. Then $mn \leq |N|^k$ and so

$$|C_{G/N}(FN/N) : C_G(F)N/N| = m \leq \frac{|N|^k}{n} = \frac{|N|^k}{|N : C_N(F)|} = |C_N(F)| \; |N|^{k-1}.$$

Remark. Observe that if F is cyclic and $C_N(F) = 1$, then $C_{G/N}(FN/N) = C_G(F)N/N$.

We call a subgroup F in a LFS-group G with a Kegel cover $\mathcal{K} = \{(G_i, N_i) \mid i \in \mathbf{I}\}$ *semisimple* if FN_i/N_i is a totally semisimple subgroup in the finite simple group G_i/N_i for all $i \in \mathbf{I}$.

The following Lemma will extend most of the results for the centralizers of finite subgroups in LFS-groups.

Proposition 2.21. *Let G be a countably infinite non-linear LFS-group with a Kegel sequence $\mathcal{K} = \{(G_i, N_i) \mid \ i \in \mathbf{N}\ \}$. If there exists an upper bound for $\{|N_i| \mid \ i \in \mathbf{N}\ \}$, then for any finite semisimple subgroup F in G the subgroup $C_G(F)$ has elements of order p_i for infinitely many distinct prime p_i. In particular $C_G(F)$ is an infinite group.*

Proof. If FN_i/N_i is a semisimple subgroup in G_i/N_i, then one can extract from the proof of [6, Theorem 1.2] that $C_{G_i/N_i}(FN_i/N_i)$ has elements of order p_i and $p_i \neq p_j$ when $i \neq j$. Since by Proposition 2.20 we have a bound for the index $|C_{G_i/N_i}(FN_i/N_i) : C_{G_i}N_i/N_i|$, for i sufficiently large, the elements of order $p_i \in C_{G_i/N_i}(FN_i/N_i)$ become elements of $C_{G_i}(F)N_i/N_i$, hence there exist elements of order p_i in $C_G(F)$ for infinitely many distinct prime p_i.

References

[1] V. V. Belyaev, Locally finite Chevalley groups, Studies in group theory, *Akad. Nauk SSSR, Ural. Nauchn. Tsentr, Sverdlovsk*, **150** (1984) 39–50.

[2] V. V. Belyaev and B. Hartley, Centralizers of finite nilpotent subgroups in locally finite groups, *Algebra and Logic*, **35** (1996) 217–228.

[3] A. Berkman, M. Kuzucuoğlu and E. Özyurt, Centralizers of involutions in locally finite simple groups, *Rend. Semin. Mat. Univ. Padova*, **118** (2007) 189–196.

[4] A. V. Borovik, The Embedding of finite Chevalley groups and periodic linear groups, *Sibirsk. Mat. Zh.*, **24** (1983) 26–35, English Translation: *Siberian Math. J.*, **24** (1983) 843–851.

[5] R. Brauer and K. A. Fowler, On groups of even order, *Ann. of Math. (2)*, **62** (1955) 565–583.

[6] K. Ersoy and M. Kuzucuoğlu, Centralizers of subgroups in simple locally finite groups, *J. Group Theory*, **15** no. 1 (2012) 9-22.

[7] J. I. Hall, Periodic simple groups of finitary linear transformation, *Ann. of Math. (2)*, **163** (2006) 445-498.

[8] B. Hartley and et. al. (eds), *Finite and Locally finite groups*, NATO ASI series, **471**, Kluwer Academic Publishers Group, Dordrecht, (1995).

[9] B. Hartley, A general Brauer-Fowler Theorem and centralizers in locally finite groups, *Pacific J. Math.*, **152** (1992) 101–117.

[10] B. Hartley and M. Kuzucuoğlu, Centralizers of elements in locally finite simple groups, *Proc. London Math. Soc. (3)*, **62** (1991) 301–324.

[11] B. Hartley and M. Kuzucuoğlu, Non-simplicity of locally finite barely transitive permutation groups, *Proc. Edinburg Math. Soc*, **40** (1997) 483–490.

[12] B. Hartley and G. Shute, Monomorphism and direct limits of finite groups of Lie type, *Quart. J. Math. Oxford Ser. (2)*, **35** (1984) 49–71.

[13] K. Hickin, Universal locally finite central extensions of groups, *Proc. London Math. Soc. (3)*, **52** (1986) 53–72.

[14] O. H. Kegel, Über einfache, lokal endliche Gruppen, *Math. Z.*, **95** (1967) 169–195.

[15] O. H. Kegel and B. A. F. Wehrfritz, *Locally Finite Groups*, North-Holland Mathematical Library, **3**, North-Holland Publishing Co., Amsterdam-London, American Elsevier Publishing Co., Inc., New York, 1973.

[16] P. B. Kleidman and R. A. Wilson, A characterization of some locally finite simple groups of Lie type, *Arch. Math. (Basel)*, **48** (1987) 10–14.

[17] M. Kuzucuoğlu, Centralizers of abelian subgroups in locally finite simple groups, *Proc. Edinburgh Math. Soc. (2)*, **40** no. 2 (1997) 217-225.

[18] M. Kuzucuoğlu, Centralizers of semisimple subgroups in locally finite simple groups, *Rend. Sem. Mat. Univ. Padova*, **92** (1994) 79-90.

[19] M. Kuzucuoğlu and R. E. Phillips, Locally finite minimal non-FC groups, *Math. Proc. Cambridge Philos. Soc.*, **105** (1989) 417–420.

[20] V. D. Mazurov, E. I. Khukhro and editors, *The Kourovka Notebook (unsolved problems in group theory)*, Russian Academy of Sciences, Siberian Division, Institute of Mathematics, Novosibirsk, 5th issue, 1976.

[21] U. Meierfrankenfeld, A non-linear Locally Finite Simple Group with a p-group as centralizer, *Turkish J. Math.*, **31** (2007) 95–103.

[22] S. Thomas, The classification of the simple periodic linear groups, *Arch. Math. (Basel)*, **41** (1983) 103–116.

[23] D. J. S. Robinson, *A Course in the Theory of Groups*, Graduate Texts in Mathematics, **80**, Springer-Verlag, New York, 1996.

[24] A. E. Zalesskii and V. N. Serejkin, Linear Groups Generated by Transvections, *Math. USSR Izv.*, **10** (1976) 25–46.

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