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AUTOMORPHISMS OF A FINITE p-GROUP WITH CYCLIC FRATTINI SUBGROUP

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Abstract. Let G be a group and $\mathrm{Aut}^{\Phi}(G)$ denote the group of all automorphisms of G centralizing $G/\Phi(G)$ elementwise. In this paper, we characterize the finite p-groups G with cyclic Frattini subgroup for which $|\mathrm{Aut}^{\Phi}(G):\mathrm{Inn}(G)|=p$.

1. Introduction

Let G be a finite group and N a characteristic subgroup of G. We let $\mathrm{Aut}^N(G)$ denote the centralizer in $\mathrm{Aut}(G)$ of G/N. Clearly $\mathrm{Aut}^N(G)$ is a normal subgroup of $\mathrm{Aut}(G)$, the automorphism group of G, and $\alpha\in\mathrm{Aut}^N(G)$ if and only if $x^{-1}x^{\alpha}\in N$ for all $x\in G$. Now let M be a normal subgroup of G. We let $\mathrm{Aut}_M(G)$ denote the group of all automorphisms of G centralizing M. Moreover, $\mathrm{Aut}_M^N(G)=\mathrm{Aut}^N(G)\bigcap\mathrm{Aut}_M(G)$. It is well-known that if G is a finite p-group, then so is the group $\mathrm{Aut}^{\Phi}(G)$, where Φ denotes the Frattini subgroup of G. Clearly $\mathrm{Aut}^{\Phi}(G)$ is a normal subgroup of $\mathrm{Aut}(G)$ containing $\mathrm{Inn}(G)$, the group of inner automorphisms of G. Müller in [10] proved, using techniques from cohomology, that if G is a finite non-abelian p-group, then $\mathrm{Aut}_Z^{\Phi}(G)=\mathrm{Inn}(G)$ if and only if $\Phi\leq Z$ and Φ is cyclic, where $Z=Z(G)$. This turns out that $\mathrm{Aut}^{\Phi}(G)/\mathrm{Inn}(G)$ is non-trivial

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if and only if G is neither elementary abelian nor extraspecial. Curran and McCaughan in [4] proved that if G is a finite non-abelian p-group, with $\mathrm{Inn}(G)$ contained in $\mathrm{Aut}^Z(G)$, then

(i) $\mathrm{Aut}^Z(G) = \mathrm{Inn}(G)$ if and only if $G' = Z(G)$ and $Z(G)$ is cyclic.

(ii) $|\mathrm{Aut}^Z(G) : \mathrm{Inn}(G)| = p$ if and only if $Z(G)$ is cyclic and $|Z(G) : G'| = p$.

In this paper we characterize the finite p-groups G with cyclic Frattini subgroup for which $|\mathrm{Aut}^\Phi(G) : \mathrm{Inn}(G)| = p$. In §2 we give some basic results that are needed for the main results of the paper. Finally in §3 we prove the main results of the paper.

Throughout this paper all groups are assumed to be finite groups. Our notation is standard, and can be found in [7], for example. A group G of order p^m is said to be of maximal class if $m > 2$ and the nilpotency class of G is $m-1$. Recall that a group G is called a central product of its subgroups A and B if A and B commute elementwise and together generate G. In this situation, we write $G = A*B$. A non-abelian group that has no non-trivial abelian direct factor is said to be purely non-abelian. For a finite group G, $\exp(G)$, $\Omega_i(G)$, $o(x)$ and $|G|$ respectively denote the exponent of G, the subgroup of G generated by its elements of order dividing p^i, the order of $x \in G$ and the size of G. We use $\mathrm{Hom}(G, A)$ to denote the group of homomorphisms of G into an abelian group A and $\mathbb{Z}_n$ for the cyclic group of order n. If α is an automorphism of G and x is an element of G, we write x^α for the image of x under α. For $s \geq 1$, we use the notation G^{*s} for the iterated central product defined by $G^{*s} = G * G^{*(s-1)}$ with $G^{*1} = G$, where G is a finite p-group. We also make the convention $G^{*0} = 1$. We denote by D_{2^n}, Q_{2^n}, S_{2^n} and X_{p^3} for the dihedral, generalized quaternion, semidihedral group of order 2^n and non-abelian p-group of order p^3 and exponent p, where p is an odd prime; the group M_{p^n} is defined by

$$\langle a, b | a^{p^{n-1}} = b^p = 1, a^b = a^{1+p^{n-2}} \rangle,$$

when $p = 2$, assume that $n > 3$, while if p is odd, assume $n > 2$. Throughout the paper, we write Z and Φ for $Z(G)$ and $\Phi(G)$, respectively.

2. **Some basic results**

In this section we give some basic results which will be used in the rest of the paper.

In [1], Adney and Yen proved the following result.

Theorem 2.1. [1, Theorem 1] *For a finite purely non-abelian group G, there is a 1-1 correspondence between* $\mathrm{Hom}(G, Z(G))$ *and* $\mathrm{Aut}^Z(G)$, *whence*

$$|\mathrm{Hom}(G/G', Z(G))| = |\mathrm{Aut}^Z(G)|.$$

We now list two families of finite 2-groups of order 2^{n+3} introduced by Berger, Kovács and Newman in [2] which will be used in the rest of the paper.

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$$D^+_{2^{n+3}} = \langle a, b, c | a^{2^{n+1}} = b^2 = c^2 = 1, a^b = a^{-1+2^n}, a^c = a^{1+2^n}, [b, c] = 1\rangle,$$
$$Q^+_{2^{n+3}} = \langle a, b, c | a^{2^{n+1}} = b^2 = 1, a^b = a^{-1+2^n}, a^c = a^{1+2^n}, a^{2^n} = c^2, [b, c] = 1\rangle,$$

both with $n > 1$.

The following structure theorem plays an important role in our proofs.

Theorem 2.2. [2, Theorem 2] *If G is a finite p-group with $Z(\Phi(G))$ cyclic, then*

$$G = E \times (G_0 * G_1 * \cdots * G_s),$$

where E is an elementary abelian, $G_1, \ldots, G_s$ are non-abelian of order p^3, of exponent p for p odd and dihedral for $p = 2$, while $G_0 > 1$ if $E > 1$, $|G_0| > 2$ if $s > 0$, and G_0 has one of the following types: cyclic, non-abelian with a cyclic maximal subgroup, $D_{2^{n+2}} * \mathbb{Z}_4, S_{2^{n+2}} * \mathbb{Z}_4, D^+_{2^{n+3}}, Q^+_{2^{n+3}}, D^+_{2^{n+3}} * \mathbb{Z}_4$, all with $n > 1$. Conversely, every such group has cyclic Frattini subgroup.

In [5], Fouladi, Jamali and Orfi proved the following result giving some information on a finite non-abelian p-group with cyclic Frattini subgroup. We now give an alternative proof for their result [5, Theorem 2.4].

Theorem 2.3. *Let G be a finite non-abelian p-group with cyclic Frattini subgroup $\Phi(G)$.*

(i) *If $p > 2$, or $p = 2$ and $cl(G) = 2$, then $\Phi(G) \leq Z(G)$.*
(ii) *If $cl(G) > 2$, then $G' = \Phi(G)$.*

Proof. We will make use of the notation of Theorem 2.2 It is straightforward to observe that $\Phi(G) = \Phi(G_0)$. If G_0 is cyclic, then (i) is obvious. Next, by [7, Theorems 5.4.3 and 5.4.4], G_0 is one of the groups M_{p^n} $(n > 2, p > 2)$, D_8, Q_8 or M_{2^n} $(n > 3)$. So $|G'| = p$. By [9, Lemma 0.4], $\exp(G/Z(G)) = \exp(G') = p$, which implies that $\Phi(G) \leq Z(G)$. To prove (ii), we observe that $\mathrm{cl}(G_0) > 2$ and G_0 has one of the following types: D_{2^n}, Q_{2^n} or S_{2^n} all with $n \geq 4$; and $D_{2^{n+2}} * \mathbb{Z}_4, S_{2^{n+2}} * \mathbb{Z}_4, D^+_{2^{n+3}}, Q^+_{2^{n+3}}, D^+_{2^{n+3}} * \mathbb{Z}_4$, all with $n > 1$. It is easy to see that in each cases $\Phi(G) = \Phi(G_0) = G'_0 = G'$, as required. □

Lemma 2.4. *Let G be a finite group with $\Phi(G) \leq Z(G)$. Then there is a bijection from $\mathrm{Hom}(G/G', \Phi(G))$ onto $\mathrm{Aut}^{\Phi}(G)$ associating to every homomorphism $f : G \to \Phi(G)$ the automorphism $x \mapsto xf(x)$ of G. In particular, if G is a p-group and $\exp(\Phi(G)) = p$, then $\mathrm{Aut}^{\Phi}(G) \cong \mathrm{Hom}(G/G', \Phi(G))$.*

Proof. For any $\alpha \in \mathrm{Aut}^{\Phi}(G)$, define $f_\alpha : G \to \Phi(G)$ by $f_\alpha(x) = x^{-1}x^\alpha$. Clearly f_α is a homomorphism, and $\alpha \mapsto f_\alpha$ is an injective map from $\mathrm{Aut}^{\Phi}(G)$ to $\mathrm{Hom}(G, \Phi(G))$. Conversely, if $f \in \mathrm{Hom}(G, \Phi(G))$, then define $\alpha = \alpha_f : G \to G$ by $x^\alpha = xf(x)$. Since $x^{-1}x^\alpha \in \Phi(G)$, for all $x \in G$, we may write G as the product of the image of α and the Frattini subgroup of G and so the image of α must be G itself. Thus α is an automorphism of G. We have $\alpha = \alpha_f \in \mathrm{Aut}^{\Phi}(G)$ and $f_{\alpha_f} = f$. Finally suppose that

$\exp(\Phi(G)) = p$ and $\alpha \in \mathrm{Aut}^{\Phi}(G)$. We observe that α fixes any element of $\Phi(G)$. Consequently the map $\alpha \mapsto f_\alpha$ is an isomorphism, which completes the proof. □

Lemma 2.5. *Let $G = E * F$ be a central product of subgroups E and F. If $\alpha \in \mathrm{Aut}_{Z(E)}^{\Phi(E)}(E)$, then the map $\hat{\alpha} : xy \mapsto x^\alpha y$, where $x \in E$ and $y \in F$, defines an automorphism of G lying in $\mathrm{Aut}_{Z(G)}^{\Phi(G)}(G)$.*

Proof. It is straightforward. □

3. p-groups with a cyclic Frattini subgroup for which $|\mathrm{Aut}^{\Phi}(G) : \mathrm{Inn}(G)| = p$

It is well-known [8, Satz III. 3.17] that for a p-group G of order p^n, the order of $\mathrm{Aut}^{\Phi}(G)$ divides $p^{r(n-r)}$, where $|G/\Phi(G)| = p^r$. In this section we study the finite p-groups G with cyclic Frattini subgroup for which $|\mathrm{Aut}^{\Phi}(G) : \mathrm{Inn}(G)| = p$. Let G be an abelian p-group. It is easy to see that $|\mathrm{Aut}^{\Phi}(G)| = p$ if and only if $G \cong \mathbb{Z}_{p^2}$. Thus we assume that G is a non-abelian p-group.

Lemma 3.1. *Let G be a non-abelian p-group with cyclic Frattini subgroup. Assume that either $p > 2$, or $p = 2$ and $cl(G) = 2$. Then $|\mathrm{Aut}^{\Phi}(G)| = |G|/p$ and $|\mathrm{Aut}^{\Phi}(G) : \mathrm{Inn}(G)| = |Z(G)|/p$.*

Proof. According to Theorem 2.3, $\Phi(G) \leq Z(G)$. So G is of class 2 and $|G'| = p$. Assume that $|\Phi(G) : G'| = p^e$. Then $\Phi(G) \cong \mathbb{Z}_{p^{e+1}}$. Since $\exp(G/G') \leq p^{e+1}$, $|\mathrm{Aut}^{\Phi}(G)| = |G|/p$, by Lemma 2.4 and so $|\mathrm{Aut}^{\Phi}(G) : \mathrm{Inn}(G)| = |Z(G)|/p$. □

In the following theorem we will characterize the finite non-abelian p-groups G with cyclic Frattini subgroup when either $p > 2$, or $p = 2$ and $\mathrm{cl}(G) = 2$.

Theorem 3.2. *Let G be a finite non-abelian p-group with cyclic Frattini subgroup. Assume that either $p > 2$, or $p = 2$ and $cl(G) = 2$. Then $|\mathrm{Aut}^{\Phi}(G) : \mathrm{Inn}(G)| = p$ if and only if G has one of the following types: M_{p^4}, $\mathbb{Z}_4 * D_8^{*s}$, $M_{16} * D_8^{*s}$, $\mathbb{Z}_{p^2} * X_{p^3}^{*s}$, $M_{p^4} * X_{p^3}^{*s}$, $\mathbb{Z}_p \times M_{p^3}$, $\mathbb{Z}_p \times (M_{p^3} * X_{p^3}^{*s})$, $\mathbb{Z}_2 \times D_8$, $\mathbb{Z}_2 \times Q_8$, $\mathbb{Z}_2 \times D_8^{*(s+1)}$ or $\mathbb{Z}_2 \times (Q_8 * D_8^{*s})$, for some $s \geq 1$.*

Proof. By Lemma 3.1, $|Z(G)| = p^2$. We use Theorem 2.2, and consider two cases:
CASE I. $E = 1$.
If $s = 0$, then $G = G_0$ where G_0 is a non-abelian with a cyclic maximal subgroup. Thus by [7, Theorems 5.4.3 and 5.4.4], $G \cong M_{p^4}$. Let $s > 0$ and $G = G_0 * K$, where $|G_0| > 2$ and $K = G_1 * \cdots * G_s$. Since $G_0 \bigcap K \neq 1$ then $Z(K) \leq Z(G_0)$. Thus $Z(G) = Z(G_0)$, because $|Z(K)| = p$, and so G be one of the groups: $\mathbb{Z}_4 * D_8^{*s}$, $M_{16} * D_8^{*s}$, $\mathbb{Z}_{p^2} * X_{p^3}^{*s}$ or $M_{p^4} * X_{p^3}^{*s}$.
CASE II. $E \neq 1$.
In this case $G_0 > 1$. If $s = 0$ then $G = E \times G_0$, where $Z(G_0) \cong \mathbb{Z}_p$ and so G is one of the groups: $\mathbb{Z}_p \times M_{p^3}$, $\mathbb{Z}_2 \times D_8$ or $\mathbb{Z}_2 \times Q_8$. Next we assume that $s > 0$ and $G = E \times (G_0 * K)$, where $|G_0| > 2$ and $K = G_1 * \cdots * G_s$. We have $1 \neq G_0 \bigcap K = Z(K) \leq Z(G_0)$ and $Z(G) = E \times Z(G_0)$. It follows

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that $Z(G_0) \cong \mathbb{Z}_p$ and so G is isomorphic either to: $\mathbb{Z}_p \times (M_{p^3} * X_{p^3}^{*s})$, $\mathbb{Z}_2 \times D_8^{*(s+1)}$ or $\mathbb{Z}_2 \times (Q_8 * D_8^{*s})$. The converse follows at once from Lemma 3.1. □

Lemma 3.3. *Let G be one of the groups D_{2^n}, Q_{2^n}, both with $n \geq 3$ or S_{2^n}, where $n \geq 4$. Then $|\mathrm{Aut}^{\Phi}(G) : \mathrm{Inn}(G)| = 2^{n-3}$. In particular, if $n \geq 5$ then $|\mathrm{Aut}_Z^{\Phi}(G) : \mathrm{Inn}(G)| > 2$.*

Proof. We set $G = \langle a, b\rangle$. Since $|G| = 2^n$, $(n \geq 3)$, we can assume that $o(a) = 2^{n-1}$, then $Z(G) = \langle a^{2^{n-2}}\rangle$ and G is of maximal class. Therefore $G' = \Phi(G)$ and $|G'| = 2^{n-2}$. Now by [3, Theorem 3.2], for any $u, v \in G'$ the map sending $a \mapsto au$ and $b \mapsto bv$ is an automorphism lying in $\mathrm{Aut}^{G'}(G)$. Hence $|\mathrm{Aut}^{\Phi}(G)| = 2^{2n-4}$ and so $|\mathrm{Aut}^{\Phi}(G) : \mathrm{Inn}(G)| = 2^{n-3}$. Now, if G is either D_{2^n} or Q_{2^n}, then there are automorphisms α and β defined by $a^\alpha = a$, $b^\alpha = a^2 b$ and $a^\beta = a^{-1}$, $b^\beta = b$. Also if $G = S_{2^n}$, then there are automorphisms α and β defined by $a^\alpha = a$, $b^\alpha = a^{-2+2^{n-2}} b$ and $a^\beta = a^{-1+2^{n-2}}$, $b^\beta = b$. In both cases, it is then easy to check that $\mathrm{Inn}(G) = \langle \alpha, \beta\rangle \cong D_{2^{n-1}}$. Next we defined the non-inner automorphism γ by $a^\gamma = a^5$, $b^\gamma = b$. This shows that $(a^{2^{n-2}})^\gamma = a^{2^{n-2}}$ and so $\gamma \in \mathrm{Aut}_Z^{\Phi}(G)$. Also $o(\gamma) = 2^{n-3}$, where $n \geq 5$. We claim that γ^2 is not in $\mathrm{Inn}(G)$. To see this, suppose to the contrary $\gamma^2 = \alpha^i \beta^j$, where $0 \leq i < 2^{n-2}$ and $0 \leq j \leq 1$. Now $b = b^{\gamma^2} = a^{(-1)^j 2i} b$ or $a^{(-1)^j i(2^{n-2}-2)} b$ where $G = D_{2^n}, Q_{2^n}$ or S_{2^n} respectively. So $2^{n-1} \mid 2i$ or $2^{n-1} \mid i(2^{n-2} - 2)$. Then $i = 0$ and $\gamma^2 = \beta$, a contradiction. This implies that $|\mathrm{Aut}_Z^{\Phi}(G) : \mathrm{Inn}(G)| > 2$. □

Lemma 3.4. *Let G be one of the groups $D^+_{2^{n+3}}$, $Q^+_{2^{n+3}}$, $D_{2^{n+2}} * \mathbb{Z}_4, S_{2^{n+2}} * \mathbb{Z}_4$ or $D^+_{2^{n+3}} * \mathbb{Z}_4$, all with $n \geq 3$. Then $|\mathrm{Aut}_Z^{\Phi}(G) : \mathrm{Inn}(G)| > 2$.*

Proof. By using GAP [6], we have $|\mathrm{Aut}_Z^{\Phi}(G) : \mathrm{Inn}(G)| > 2$, where G stands for either D^+_{64}, Q^+_{64} or $D^+_{64} * \mathbb{Z}_4$. First assume that G is either $D^+_{2^{n+3}}$ or $Q^+_{2^{n+3}}$, where $n \geq 4$. Then $G = \langle a, b, c\rangle$, $Z(G) = \langle a^{2^n}\rangle$ and there are automorphisms α, β and γ defined by $a^\alpha = a$, $b^\alpha = a^{2^n-2} b$, $c^\alpha = a^{2^n} c$, $a^\beta = a^{2^n-1}$, $b^\beta = b$, $c^\beta = c$ and $a^\gamma = a^{2^n+1}$, $b^\gamma = b$, $c^\gamma = c$. It is then easy to check that $\mathrm{Inn}(G) = \langle \alpha, \beta\rangle \times \langle \gamma \rangle \cong D_{2^{n+1}} \times \mathbb{Z}_2$. By defining the automorphism $a^\delta = a^5$, $b^\delta = b$ and $c^\delta = c$, it follows that $\delta \in \mathrm{Aut}_Z^{\Phi}(G)$. We show that $\delta^2 \notin \mathrm{Inn}(G)$. Let $\delta^2 = \alpha^i \beta^j \gamma^k$, where $0 \leq i < 2^n, 0 \leq j \leq 1$ and $0 \leq k \leq 1$. Now $b = b^{\delta^2} = a^{(-1)^j i(2^n-2)} b$ and so $2^{n+1} \mid i(2^n - 2)$. Then $i = 0$, $\delta^2 = \beta^j \gamma^k$ and $o(\delta) = 2^{n-1} \leq 4$. Thus $n \leq 3$, a contradiction. If $G = D_{2^{n+2}} * \mathbb{Z}_4$, then

$$G \cong \langle a, b, c | a^{2^{n+1}} = b^2 = [a, c] = [b, c] = 1, a^{2^n} = c^2, a^b = a^{-1}\rangle,$$

and $Z(G) = \langle c\rangle$. We define the automorphisms α, β by $a^\alpha = a$, $b^\alpha = a^{-2} b$, $c^\alpha = c$, $a^\beta = a^{-1}$, $b^\beta = b$, $c^\beta = c$. Now $\mathrm{Inn}(G) = \langle \alpha, \beta\rangle \cong D_{2^{n+1}}$ and by considering the automorphism δ mentioned for the previous case, it follows that $\delta \in \mathrm{Aut}_Z^{\Phi}(G)$ which implies that $|\mathrm{Aut}_Z^{\Phi}(G) : \mathrm{Inn}(G)| > 2$. If $G = S_{2^{n+2}} * \mathbb{Z}_4$, then

$$G \cong \langle a, b, c | a^{2^{n+1}} = b^2 = [a, c] = [b, c] = 1, a^{2^n} = c^2, a^b = a^{-1+2^n}\rangle,$$

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and $Z(G) = \langle c \rangle$. Using the automorphisms α, β defined by $a^\alpha = a$, $b^\alpha = a^{2^n-2}b$, $c^\alpha = c$, $a^\beta = a^{2^n-1}$, $b^\beta = b$, $c^\beta = c$, we have $\mathrm{Inn}(G) = \langle \alpha, \beta \rangle \cong D_{2^{n+1}}$. Next by defining the automorphism δ mentioned earlier, it follows that $\delta \in \mathrm{Aut}^\Phi_Z(G)$ and $\delta^2 \notin \mathrm{Inn}(G)$, by a similar argument. Also if $G = D^+_{2^{n+3}} * \mathbb{Z}_4$, where $n \geq 4$, then

$$G \cong \langle a, b, c, d | b^2 = c^2 = d^4 = [a, d] = [b, d] = [c, d] = [b, c] = 1, a^b = a^{2^n-1}, a^c = a^{2^n+1}, a^{2^n} = d^2 \rangle.$$

We define the automorphisms α, β, γ and δ by $a^\alpha = a$, $b^\alpha = a^{2^n-2}b$, $c^\alpha = a^{2^n}c$, $d^\alpha = d$, $a^\beta = a^{2^n-1}$, $b^\beta = b$, $c^\beta = c$, $d^\beta = d$, $a^\gamma = a^{2^n+1}$, $b^\gamma = b$, $c^\gamma = c$, $d^\gamma = d$ and $a^\delta = a^5$, $b^\delta = b$, $c^\delta = c$, $d^\delta = d$. We observe that $\mathrm{Inn}(G) = \langle \alpha, \beta \rangle \times \langle \gamma \rangle \cong D_{2^{n+1}} \times \mathbb{Z}_2$ and $|\delta| = 2^{n-1}$, where $n \geq 4$. Finally since $Z(G) = \langle d \rangle$, it follows that $\delta \in \mathrm{Aut}^\Phi_Z(G)$ which implies that $|\mathrm{Aut}^\Phi_Z(G) : \mathrm{Inn}(G)| > 2$. □

From now on we shall consider the case that G is a finite non-abelian 2-group whose Frattini subgroup is cyclic and $\mathrm{cl}(G) > 2$.

Lemma 3.5. *Let G be a non-abelian 2-group with cyclic Frattini subgroup and $cl(G) > 2$ such that $|\mathrm{Aut}^\Phi(G) : \mathrm{Inn}(G)| = 2$. Then $Z(G) \leq \Phi(G)$ and so G is purely non-abelian group.*

Proof. Since $cl(G) > 2$, by [10, Proposition 3.1], $\mathrm{Inn}(G) \lneqq \mathrm{Aut}^\Phi_Z(G)$, so $\mathrm{Aut}^\Phi(G) = \mathrm{Aut}^\Phi_Z(G)$. Assume that $Z(G) \nleq \Phi(G)$. Then $G = M\langle z \rangle$ for some maximal subgroup M of G and for some z in $Z(G)\backslash M$. We choose an element u in $\Omega_1(\Phi(G) \bigcap Z(G))$. The map $\alpha : hz^i \mapsto h(zu)^i$, where $h \in M$ and $0 \leq i < 2$, is in $\mathrm{Aut}^\Phi(G)$ from which we conclude that $u = 1$, a contradiction. So that $Z(G) \leq \Phi(G)$ and G is purely non-abelian group. □

For the rest of the paper, we will make use of the notation of Theorem 2.2 without further mention.

Lemma 3.6. *If G has one of the following types: $D_{16} * \mathbb{Z}_4 * D_8^{*s}$, $S_{16} * \mathbb{Z}_4 * D_8^{*s}$ or $D^+_{32} * \mathbb{Z}_4 * D_8^{*s}$, for some $s \geq 0$, then $|\mathrm{Aut}^\Phi(G) : \mathrm{Inn}(G)| > 2$.*

Proof. Assume that $|\mathrm{Aut}^\Phi(G) : \mathrm{Inn}(G)| = 2$. Then by Lemma 3.5, $Z(G) \leq \Phi(G)$. We observe that $G' = \Phi(G) = Z(G) \cong \mathbb{Z}_4$. Therefore $\mathrm{Aut}^\Phi(G) = \mathrm{Inn}(G)$, a contradiction. □

Theorem 3.7. *Let G be a finite non-abelian 2-group with cyclic Frattini subgroup. If $|\mathrm{Aut}^\Phi(G) : \mathrm{Inn}(G)| = 2$ and $cl(G) > 2$, then G has one of the following types: $D_{16} * D_8^{*s}, Q_{16} * D_8^{*s}, S_{16} * D_8^{*s}, D^+_{32} * D_8^{*s}$ or $Q^+_{32} * D_8^{*s}$.*

Proof. If $s = 0$, this is straightforward by using GAP [6], Theorem 2.2 and Lemmas 3.3, 3.4, 3.5, 3.6. Let $s > 0$. By our assumption, $G = H * D_8^{*s}$, where H has one of the following types quoted in Theorem 2.2. Now by Lemma 3.3 and lemma 3.4, for $H = D_{2^n}, Q_{2^n}, S_{2^n} (n \geq 5)$ or $D^+_{2^{n+3}}, Q^+_{2^{n+3}}, D_{2^{n+2}} *$

$\mathbb{Z}_4, S_{2^{n+2}} * \mathbb{Z}_4, D^+_{2^{n+3}} * \mathbb{Z}_4 (n \geq 3)$, we let $\sigma \in \mathrm{Aut}^{\Phi(H)}_{Z(H)}(H)$ such that $\sigma^2 \notin \mathrm{Inn}(H)$. By Lemma 2.5, σ can be extended to an automorphism σ^* of G defined by $(hx)^{\sigma^*} = h^\sigma x$ for $h \in H$ and $x \in D_8^{*s}$. Now since $\sigma^* \in \mathrm{Aut}^\Phi(G)$, it follows that $\sigma^{*2} = i_g$, where i_g is the inner automorphism of G induced by $g \in G$. Writing $g = h_1 x_1$, where $h_1 \in H$ and $x_1 \in D_8^{*s}$, gives

$$h^{\sigma^2} = h^{\sigma^{*2}} = g^{-1}hg = x_1^{-1}h_1^{-1}hh_1x_1 = h_1^{-1}hh_1,$$

for all $h \in H$. We conclude that $\sigma^2 \in \mathrm{Inn}(H)$, a contradiction. So by Lemma 3.6, $H = D_{16}, Q_{16}, S_{16}, D^+_{32}$ or Q^+_{32}, completing the proof. □

The following theorem completes the proof of our main result when G is a finite non-abelian 2-group whose Frattini subgroup is cyclic and $\mathrm{cl}(G) > 2$.

Theorem 3.8. *If G is one of the groups $D_{16}, Q_{16}, S_{16}, D^+_{32}, Q^+_{32}, D_{16}*D_8^{*s}, Q_{16}*D_8^{*s}, S_{16}*D_8^{*s}, D^+_{32}*D_8^{*s}$ or $Q^+_{32} * D_8^{*s}$, where $s \geq 0$, then $|\mathrm{Aut}^\Phi(G) : \mathrm{Inn}(G)| = 2$.*

Proof. By Lemma 3.3 and using GAP [6], we have $|\mathrm{Aut}^\Phi(G) : \mathrm{Inn}(G)| = 2$, where G stands for either $D_{16}, Q_{16}, S_{16}, D^+_{32}$ or Q^+_{32}. So we give a proof for the group $G = D_{16} * D_8^{*s}$, where $s > 0$; the other groups are treated similarly. We have

$$G \cong \langle a, b, c_i, d_i | a^8 = b^2 = d_1^2 = \cdots = d_s^2 = [a, c_i] = [b, c_i] = [a, d_i] = [b, d_i] = (ab)^2 = (c_id_i)^2 = a^4c_i^2 = 1\rangle,$$

where $1 \leq i \leq s$. It is easily seen that $Z(G) \cong \mathbb{Z}_2$ and therefore $Z(G) \leq G'$. So by Theorem 2.1, $|\mathrm{Aut}^Z(G)| = 2^{2s+2}$. Now we observe that $\mathrm{Aut}^Z(G)\mathrm{Inn}(G)$ is a subgroup of $\mathrm{Aut}^\Phi(G)$ and so the order of $\mathrm{Aut}^\Phi(G)$ is greater than 2^{2s+4}. We claim that $|\mathrm{Aut}^\Phi(G)| \leq 2^{2s+4}$. To see this, since $\Phi(G) = \langle a^2\rangle$, it follows that for $\sigma \in \mathrm{Aut}^\Phi(G)$, $a^\sigma \in \{a, a^3, a^5, a^7\}$ and $b^\sigma \in \{b, ba^2, ba^4, ba^6\}$. If $c_i^\sigma = c_ia^2$ or $c_i^\sigma = c_ia^6$ $(1 \leq i \leq s)$, we find that $a^4 = 1$, which is impossible. Hence, $c_i^\sigma = c_i$ or $c_i^\sigma = c_ia^4$ $(1 \leq i \leq s)$. By the above argument, $d_i^\sigma = d_i$ or $d_i^\sigma = d_ia^4$ $(1 \leq i \leq s)$. Therefore, $|\mathrm{Aut}^\Phi(G)| = 2^{2s+4}$ and $|\mathrm{Aut}^\Phi(G) : \mathrm{Inn}(G)| = 2$. □

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