International Journal of Group Theory
ISSN (print): 2251-7650, ISSN (on-line): 2251-7669
Vol. 6 No. 4 (2017), pp. 41-51.

www.theoryofgroups.ir
www.ui.ac.ir

BIPARTITE DIVISOR GRAPH FOR THE SET OF IRREDUCIBLE CHARACTER DEGREES

ROGHAYEH HAFEZIEH

Communicated by Gunnar Traustason

Abstract. Let G be a finite group. We consider the set of the irreducible complex characters of G, namely $Irr(G)$, and the related degree set $cd(G) = \{\chi(1) : \chi \in Irr(G)\}$. Let $\rho(G)$ be the set of all primes which divide some character degree of G. In this paper we introduce the bipartite divisor graph for $cd(G)$ as an undirected bipartite graph with vertex set $\rho(G) \cup (cd(G) \setminus \{1\})$, such that an element p of $\rho(G)$ is adjacent to an element m of $cd(G) \setminus \{1\}$ if and only if p divides m. We denote this graph simply by $B(G)$. Then by means of combinatorial properties of this graph, we discuss the structure of the group G. In particular, we consider the cases where $B(G)$ is a path or a cycle.

1. Introduction

Let G be a finite group. It is well known that the set of irreducible characters of G, denoted by $Irr(G)$, can be used to obtain information about the structure of the group G. The value of each character at the identity is the degree of the character and by $cd(G)$ we mean the set of all irreducible character degrees of G. Let $\rho(G)$ be the set of all primes which divide some character degree of G. When studying problems on character degrees, it is useful to attach the following graphs which have been widely studied to the sets $\rho(G)$ and $cd(G) \setminus \{1\}$, respectively.

(i) Prime degree graph, namely $\Delta(G)$, which is an undirected graph whose set of vertices is $\rho(G)$; there is an edge between two different vertices p and q if pq divides some degree in $cd(G)$.

(ii) Common divisor degree graph, namely $\Gamma(G)$, which is an undirected graph whose set of vertices is $cd(G) \setminus \{1\}$; there is an edge between two different vertices m and k if $(m, k) \neq 1$.

MSC(2010): Primary 05C25; secondary 05C75.

Keywords: bipartite divisor graph, irreducible character degree, path, cycle.

Received: 01 July 2016, Accepted: 04 February 2017.

The notion of the bipartite divisor graph was first introduced by Iranmanesh and Praeger in [4] for a finite set of positive integers. As an application of this graph in group theory, in [2], the writers considered this graph for the set of conjugacy class sizes of a finite group and studied various properties of it. In particular they proved that the diameter of this graph is at most six, and classified those groups for which the bipartite divisor graphs of conjugacy class sizes have diameter exactly 6. Moreover, they showed that if the graph is acyclic, then the diameter is at most five and they classified the groups for which the graph is a path of length five. Similarly, Taeri in [20] considered the case that the bipartite divisor graph of the set of conjugacy class sizes is a cycle and (by using the structure of F-groups and the classification of finite simple groups) proved that for a finite nonabelian group G, the bipartite divisor graph of the conjugacy class sizes is a cycle if and only if it is a cycle of length 6, and for an abelian group A and $q \in \{4, 8\}$, $G \simeq A \times SL_2(q)$. Inspired by these papers, in this work we consider the bipartite divisor graph for the set of irreducible character degrees of a finite group and define it as follows:

Definition 1. *Let G be a finite group. The bipartite divisor graph for the set of irreducible character degrees of G, is an undirected bipartite graph with vertex set $\rho(G) \cup (cd(G) \setminus \{1\})$, such that an element p of $\rho(G)$ is adjacent to an element m of $cd(G) \setminus \{1\}$ if and only if p divides m.*

Since classifying groups whose associated graphs have prescribed isomorphism types is an important topic in this area, in this paper, we will discuss the cases where $B(G)$ is a path or a cycle for a group G.

In the second section, after finding the best upper bound for the diameter of $B(G)$, we consider the case where $B(G)$ is a path of length n. We prove G is solvable, $n \leq 6$ and in Theorem 3, which is the main theorem of this section, we give some group theoretical properties of such a group.

In the third section, we consider the case that G is nonsolvable and $B(G)$ is a union of paths where by union of paths we mean that each connected component of $B(G)$ is a path. Theorem 6 is the main theorem of this section.

Finally in section four, we consider the case where $B(G)$ is a cycle. We prove that $B(G)$ is a cycle if and only if G is solvable and $B(G)$ is a cycle of length four or six. By using these properties, we prove if $B(G)$ is a cycle of length four, then there exists a normal abelian Hall subgroup of G which explains the structure of the irreducible character degrees of G. Theorem 11 is the main theorem of this section.

Notation 1. *For positive integers m and n, we denote the greatest common divisor of m and n by (m, n); if p is a prime, then by n_p and $n_{p'}$ we mean the p-part and the p'-part (the part which is relatively prime to p) of n, respectively; the number of connected components of a graph $\mathcal{G}$ by $n(\mathcal{G})$; the diameter of a graph $\mathcal{G}$ by $diam(\mathcal{G})$ (where by the diameter we mean the maximum distance between vertices in the same connected component of the graph). If α is a vertex of the graph $\mathcal{G}$, then $deg_{\mathcal{G}}(\alpha)$ is the number of vertices adjacent to α in $\mathcal{G}$. If the graph is well-understood, then we denote it by $deg(\alpha)$. By length of a path or a cycle, we mean the number of edges in the path or in the cycle. Also, by P_n and C_n we mean a path of length n and a cycle of length n, respectively. Let G be a finite solvable* *group. We denote the first and second Fitting subgroups of G by $F(G)$ and $F_2(G)$, respectively. As usual, we write $dl(G)$ and $h(G)$ to denote the derived length and Fitting height of G, respectively. Also in a finite group G, $cd(G)^*$ denotes $cd(G) \setminus \{1\}$. Other notation throughout the paper is standard.*

2. Groups whose bipartite divisor graphs are paths

We begin by giving the best upper bound for $diam(B(G))$ which is also mentioned in [16, Theorem 3.1].

Theorem 1. *For a finite group G, $diam(B(G)) \leq 7$ and this bound is best possible.*

Proof. By [10, Corollary 4.2, Theorem 6.5, Theorem 7.2] we know that $diam(\Delta(G))$ and $diam(\Gamma(G))$ are both less than or equal to three. Now by [4, Lemma 1] we have one of the following cases:

(i) $diam(B(G)) = 2max\{diam(\Delta(G)), diam(\Gamma(G))\} \leq 2 \times 3 = 6$ or

(ii) $diam(B(G)) = 2diam(\Delta(G)) + 1 = 2diam(\Gamma(G)) + 1 \leq (2 \times 3) + 1 = 7$.

So in general, $diam(B(G)) \leq 7$. Now let G be the group as in [7], then

$$cd(G) = \{1, 3, 5, 3 \times 5, 7 \times 31 \times 151, 2^7 \times 7 \times 31 \times 151, 2^{12} \times 31 \times 151, 2^{12} \times 3 \times 31 \times 151,$$
$$2^{12} \times 7 \times 31 \times 151, 2^{13} \times 7 \times 31 \times 151, 2^{15} \times 3 \times 31 \times 151\}$$

It is easy to see that for $7 \in \rho(G)$ and $5 \in cd(G)^*$, we have $d_{B(G)}(5, 7) = 7$, so $diam(B(G)) = 7$. □

Proposition 2. *Let G be a finite group. Assume that $B(G)$ is a path of length n. Then $n \leq 6$, G is solvable and $dl(G) \leq 5$. In particular, if $B(G)$ is isomorphic to P_5 or P_6, then $h(G)$ is less than or equal 3 or 4, respectively.*

Proof. Since $B(G) \simeq P_n$, we can see that both $\Delta(G)$ and $\Gamma(G)$ are paths, (see [4, Theorem 3]). By [10, Corollary 4.2, Theorem 6.5], we conclude that $\Delta(G) \simeq P_m$ where $m \leq 3$. Furthermore by [10, Theorem 4.5] and [13, Theorem B] we deduce that a path of length three cannot be the prime degree graph of a finite group, so $m \leq 2$. On the other hand we know that $|diam(\Delta(G)) - diam(\Gamma(G))| \leq 1$, thus $n \leq 6$.

We claim G is solvable: Since a finite group with at most three distinct irreducible character degrees is solvable, we may assume that $n > 3$ and if $n = 4$, then $\Gamma(G) \simeq P_2$. Now [14, Theorem A, Theorem 3.1] implies that G is solvable. As $n \leq 6$, we have $|cd(G)| \leq 5$. Now by [9], we have $dl(G) \leq 5$. In the case that $B(G)$ is isomorphic to P_5 or P_6, since $|cd(G)| \geq 4$, [19, Theorem 1.2] verifies that $h(G) \leq |cd(G)| - 1$. Thus $h(G) \leq 3$ if $B(G) \simeq P_5$ and $h(G) \leq 4$ if $B(G) \simeq P_6$. □

Example 1. *Let G be a finite group with $cd(G) = \{1, p^a, q^b, p^a q^b\}$, where p and q are distinct primes and $a, b \geq 1$ are positive integers. It is clear that $B(G)$ is a path of length 4. It should be mentioned that such a group exists. We can see that for $G = S_3 \times A_4$ which is a solvable group we have $cd(G) = \{1, 2, 3, 6\}$.*

Theorem 3. *Let G be a finite group. Assume that $B(G)$ is a path of length n. Then we have one of the following cases:*

(i) *G has an abelian normal subgroup N such that $cd(G) = \{1, [G : N]\}$ and $\frac{G}{N}$ is abelian. Furthermore, $n \in \{1, 2\}$.*

(ii) *There exist normal subgroups N and K of G and a prime number p with the following properties:*

(a) *$\frac{G}{N}$ is abelian.*

(b) *$\pi(G/K) \subseteq \rho(G)$.*

(c) *Either p divides all the nontrivial irreducible character degrees of N which implies that N has a normal p-complement, or $cd(N) = \{1, l, k, \frac{h}{m}\}$, where $cd(G) = \{1, m, h, l, k\}$.*

Furthermore, $n \in \{4, 5, 6\}$.

(iii) *$cd(G) = \{1, p^\alpha, q^\beta, p^\alpha q^\beta\}$, where p and q are distinct primes. Thus $n = 4$.*

(iv) *There exists a prime s such that G has a normal s-complement H. Either H is abelian and $n \in \{1, 2\}$ or it is nonabelian and we have one of the following cases:*

(i) *$cd(G) = \{1, h, hl\}$, for some positive integers h and l and we have $n = 3$.*

(ii) *$n = 4$ and $\frac{G}{H}$ is abelian. In the sense of [8], either H is a group of type one and $\{[H : F(H)]\} \cup cd(F(H)) = cd(H)$ or H is a group of type four and $cd(H) = \{1, [F_2(H) : F(H)], [H : F(H)]\}$. Also $[G : F(G)] \in cd(G)$ and $cd(F(G)) = \{1, h_{s'}\}$, where $[G : F(G)] \neq h \in cd(G)$.*

(iii) *$n = 3$, $\frac{G}{H}$ is abelian, $h := [G : F(G)] \in cd(G)$, $F(G) = P \times A$, where P is a p-group for a prime p, $A \leq Z(G)$, $cd(G) = cd(\frac{G}{A})$, and $cd(P) = \{1, m_{s'}\}$ for $h \neq m \in cd(G)$.*

Proof. Since $B(G)$ is a path of length n, Proposition 2 implies that $n \leq 6$ and G is solvable.

First suppose that $n \geq 4$. This implies that nonlinear irreducible character degrees of the solvable group G are not all equal. We claim that there exists a normal subgroup $K > 1$ of G such that $\frac{G}{K}$ is nonabelian. If G' is not a minimal normal subgroup of G, then G has a nontrivial normal subgroup N such that $\frac{G}{N}$ is nonabelian. Otherwise, if G' is a minimal normal subgroup of G, then it cannot be unique since nonlinear irreducible character degrees of the solvable group G are not all equal, [5, Lemma 12.3]. So we can see that G has a nontrivial normal subgroup N such that $\frac{G}{N}$ is nonabelian. Let K be maximal with respect to the property that $\frac{G}{K}$ is nonabelian. It is clear that $(\frac{G}{K})'$ is the unique minimal normal subgroup of $\frac{G}{K}$. Thus $\frac{G}{K}$ satisfies the hypothesis of [5, Lemma 12.3]. So all nonlinear irreducible characters of $\frac{G}{K}$ have equal degree f and we have the following cases:

Case 1. $B(G) \simeq P_6$.

By [10, Theorem 4.5], $\Delta(G)$ cannot be a path of length 3, so we may assume that $B(G) : m - p - h - q - l - r - k$, where p, q, and r are distinct prime numbers. If $\frac{G}{K}$ is an s-group for a prime s, then by symmetry we may assume that $s = p$ and $f = m$. Let $\chi \in Irr(G)$ with $\chi(1) = k$. Since p does not divide $\chi(1)$, we have $\chi_K \in Irr(K)$. Now by Gallagher's theorem [5, Corollary 6.17], we conclude that $mk \in cd(G)^*$, which is impossible since $p - r$ is not an edge in $\Delta(G)$. Now [5, Lemma 12.3] implies that $\frac{G}{K}$ is a Frobenius group with abelian Frobenius complement of order f and Frobenius kernel $\frac{N}{K} = (\frac{G}{K})'$ which is an elementary abelian s-group for some prime s. Thus $\frac{G}{N}$ is abelian and by Ito's theorem [5, Theorem 6.15] it is clear that N is not abelian.

First suppose that $f = m$. It is obvious that $l, k \in cd(N)$. Let χ be a nonlinear irreducible character of G with $\chi(1) = k$. If $s \notin \rho(G)$, then $(\chi(1), [G : K]) = 1$. This verifies that $\chi_K \in Irr(K)$ and by Gallagher's theorem [5] we have $km \in cd(G)$ which is impossible. Therefore $\pi(G/K) \subseteq \rho(G)$. Let θ be a nonlinear irreducible character of N such that s does not divide $\theta(1)$. By [5, Theorem 12.4] we conclude that $[G : N]\theta(1) \in cd(G)$, so q divides $\theta(1)$, $s = r$, $[G : N]\theta(1) = h$ which implies that $\theta(1) = \frac{h}{m}$. Since $B(G)$ is a path, it is obvious that $\theta(1) \in cd(N)$ is unique with this property that $s \nmid \theta(1)$. Let ψ be a nonlinear irreducible character of N such that $\psi(1) \neq \theta(1)$. Let μ be an irreducible constituent of ψ^G, so s divides $\mu(1)$ and we have $\mu(1) \in \{l, k\}$. Thus $(\mu(1), [G : N]) = 1$. This implies that $\mu(1) = \psi(1)$. All together N has one of the following properties:

(a) s divides every nonlinear irreducible character of N, which implies that N has a normal s-complement; or

(b) $cd(N) = \{1, l, k, \frac{h}{m}\}$.

Hence case (ii) occurs. The case $f = k$ is similar. Now assume $f = h$. Suppose that $\theta \in Irr(N)$ with $\theta(1) \neq 1$. It is easy to see that $[G : N]\theta(1) \notin cd(G)$. Now by [5, Theorem 12.4], we conclude that $s|\theta(1)$. This implies that $\pi(\frac{G}{K}) \subseteq \rho(G)$. The case $f = l$ is similar.

Case 2. $B(G) \simeq P_5$.

We may assume that $B(G) : p - m - q - l - r - h$, where p, q, and r are distinct prime numbers. As $mh \notin cd(G)$, similar to the previous case, we can see that $\frac{G}{K}$ is a Frobenius group with abelian Frobenius complement of order f and Frobenius kernel $\frac{N}{K} = (\frac{G}{K})'$ which is an elementary abelian s-group for some prime s. Thus $\frac{G}{N}$ is abelian.

First suppose $f = h$. As $mh \notin cd(G)$, we conclude that $s \in \rho(G)$. If there exists $\theta \in Irr(N)$ such that $s \nmid \theta(1)$, then by [5, Theorem 12.4] we have $[G : N]\theta(1) \in cd(G)$, so q divides $\theta(1)$ and $s = p$. Let $\psi \in Irr(N)$ be nonlinear and $\psi(1) \neq \theta(1)$. Then s divides $\psi(1)$. Let χ be an irreducible constituent of ψ^G, then s divides $\chi(1)$. Therefore $\chi(1) = m$. Since $(\chi(1), [G : N]) = 1$, we have $\chi(1) = \psi(1) = m$. Thus q divides $\psi(1)$. Hence q divides each nonlinear character degree of N. This implies that N has a normal q-complement. Therefore, case (ii) occurs. Now suppose that $f = m$. As $hm \notin cd(G)$, we have $s \in \rho(G)$, so $s = r$. Let $\theta \in Irr(N)$ be nonlinear. As $[G : N]\theta(1) \notin cd(G)$, [5, Theorem 12.4] implies that $s|\theta(1)$. Thus N has a normal s-complement, so case (ii) occurs. Finally suppose $f = l$. Let $\theta \in Irr(N)$ be nonlinear. As $[G : N]\theta(1) \notin cd(G)$, [5, Theorem 12.4] implies that $s|\theta(1)$, so N has a normal s-complement and case (ii) occurs.

Case 3. $B(G) \simeq P_4$.

First suppose that $B(G) : p - m - q - h - r$, where p, q, and r are distinct prime numbers. Since q divides every nonlinear character degree, G has a normal q-complement, ([5, Corollary 12.2]). Hence $G = HQ$, where Q is a sylow q-subgroup of G and H is its normal complement in G. As $cd(G)$ contains no powers of q, we conclude that $\frac{G}{H} \simeq Q$ is abelian. By the structure of $B(G)$ we can see that $cd(H) = \{1, a := p^\alpha, b := r^\beta\}$, for some positive integers α and β. As $ab \notin cd(H)$, H is not nilpotent, therefore $2 \leq h(H) \leq 3$. Now [6, Lemma 3.1] and [17, Theorem 3.5] imply that $dl(H) = 3$, either $h(H) = 2$, $[H : F(H)] = a$ and $cd(F(H)) = \{1, b\}$ or $h(H) = 3$ and $cd(H) = \{1, [F_2(H) : F(H)], [H : F(H)]\}$. As $\Delta(H)$ has two connected components, we can see that H is either a group of type one or four in the sense of [8]. Also by [17, Lemma 5.1] we deduce that $m = [G : F(G)]$ and $cd(F(G)) = \{1, h_{q'}\}$. Thus case (iv) occurs with $s = q$. Now suppose $B(G) : m - q - l - p - h$, where p and q are distinct prime numbers. Suppose $\frac{G}{K}$ is a Frobenius group with abelian Frobenius complement of order f and Frobenius kernel $\frac{N}{K} = (\frac{G}{K})'$ which is an elementary abelian s-group for some prime s. Thus $\frac{G}{N}$ is abelian. If $f = m$ and $s \notin \rho(G)$, then by Gallagher's theorem [5] we conclude that $cd(G) = \{1, p^\alpha, q^\beta, p^\alpha q^\beta\}$, so case (iii) occurs. If $f = m$ and $s \in \rho(G)$, then $s = p$. Let $\theta \in Irr(N)$ be nonlinear. If s does not divide $\theta(1)$, then $[G : N]\theta(1) \in cd(G)$. Thus $l = m\theta(1)$. As $(s, f) = 1$, s divides $\theta(1)$ which is a contradiction. Thus s divides each nonlinear irreducible character degree of N, so N has a normal s-complement. Hence we have case (ii). Let $f = l$. As $(s, f) = 1$, $s \notin \rho(G)$. On the other hand, for $\psi \in Irr(N)$ with $\psi(1) \neq 1$, $[G : N]\psi(1) \notin cd(G)$. [5, Theorem 12.4] implies that $s|\psi(1)$ and $s \in \rho(G)$ which is a contradiction. So $f \neq l$. Finally, suppose $\frac{G}{K}$ is an s-group for a prime s. By symmetry we may assume that $s = p$ and $f = p^\alpha = h$. Let $\chi \in Irr(G)$ with $\chi(1) = m$. By Gallagher's theorem [5], we conclude that $mh \in cd(G)^*$, so $l = mh$, $cd(G) = \{1, p^\alpha, q^\beta, p^\alpha q^\beta\}$ and case (iii) holds.

Now suppose that $B(G)$ is a path of length n where $n \leq 3$. We have the following cases:

Case 4. $B(G) \simeq P_3$.

Assume $B(G) : p - m - q - h$, where p and q are distinct prime numbers. Since q divides every nonlinear character degree of G, we conclude that $G = HQ$, where Q is a sylow q-subgroup of G and H is its normal complement in G, ([5, Corollary 12.2]). If $h|m$ then $cd(G) = \{1, h, hl\}$, where l is a positive integer. Suppose $h \nmid m$. We claim that Q is abelian. It is clear that $cd(H) = \{1, p^\alpha\}$ for a positive integer α. If Q is not abelian, then $cd(\frac{G}{H}) = cd(Q) = \{1, h\}$. Let $\theta \in Irr(H)$ such that $\theta(1) = p^\alpha$. As $(|H|, [G : H]) = 1$, θ is extendible to $\vartheta \in Irr(I_\theta(G))$ so that $[G : I_\theta(G)] = m_q$. Since $cd(\frac{G}{H}) = \{1, h\}$ and $h \nmid m$, $\frac{I_\theta(G)}{H}$ is not abelian by Ito's theorem, so there exists $\psi \in Irr(\frac{I_\theta(G)}{H})$ with $\psi(1) > 1$. By Gallagher's theorem [5], $\psi\vartheta \in Irr(I_\theta(G))$, which implies that $(\psi\vartheta)^G \in Irr(G)$. Hence $m < \vartheta(1)\psi(1)[G : I_\theta(G)] \in cd(G)$, a contradiction. Hence Q is abelian. As $h \nmid m$ and $\pi(h) \subseteq \pi(m) \neq \pi(h)$, [17, Lemma 5.2] implies that $[G : F(G)] = h$, $F(G) = P \times A$, where P is a p-group, $A \leq Z(G)$, $cd(G) = cd(\frac{G}{A})$, and $cd(P) = \{1, m_{q'}\}$. Thus case (iv) occurs with $s = q$.

Case 5. $B(G) \simeq P_2$.

Assume first that $cd(G) = \{1, m\}$ where m is not a prime power. Then $m = p^\alpha q^\beta$ for some primes $p \neq q$ and integers $\alpha, \beta \geq 1$. Now by [5, Theorem 12.5], G has an abelian normal subgroup N such that $[G : N] = m$. By the structure of G we can see that $\frac{G}{N}$ is abelian, so case (i) holds. If $cd(G) = \{1, m, h\}$, then both $1 < m < h$ are powers of a prime s and so G has a normal s-complement, say H. As $\rho(G) = \{s\}$, we conclude that H is abelian and case (iv) holds.

Case 6. $B(G) \simeq P_1$.

Thus we have $cd(G) = \{1, p^a\}$ for some prime p and integer $a \geq 1$. Now [5, Theorem 12.5] implies that either $G \simeq P \times A$, where P is a p-group, A is abelian and case (iv) holds, or G has an abelian normal subgroup of index p^a and case (i) occurs. $\square$

It should be mentioned that we were unable to construct groups G with $B(G) \simeq P_n$ where $n \in \{5, 6\}$, so we may ask the following question:

Question 1. *Is there any finite group G whose $B(G) \simeq P_n$ for $n \in \{5, 6\}$?*

3. Nonsolvable groups whose bipartite divisor graphs are union of paths

Let G be a finite nonsolvable group. By [10, Theorem 6.4], we know that $\Delta(G)$ has at most three connected components. Since [4] implies that $n(B(G)) = n(\Gamma(G)) = n(\Delta(G))$, we conclude that $n(B(G)) \leq 3$. In the rest of this section, we consider the case where each connected component of $B(G)$ is a path and we start by looking at simple groups.

Lemma 4. *For a nonabelian simple group S, $B(S)$ is disconnected and all the connected components are paths if and only if S is isomorphic to one of the following groups:*

(i) $PSL(2, 2^n)$ *where* $|\pi(2^n \pm 1)| \leq 2$*;*
(ii) $PSL(2, p^n)$ *where p is an odd prime and* $|\pi(p^n \pm 1)| \leq 2$.

Proof. By [4] we know that $n(B(S)) = n(\Gamma(S)) = n(\Delta(S))$. Since all connected components of $B(S)$ are paths, so are the connected components of $\Delta(S)$. This implies that $\Delta(S)$ has no triangles. Thus by [21, Lemma 3.1], one of the following cases holds:

(i) $S \simeq PSL(2, 2^n)$ where $|\pi(2^n \pm 1)| \leq 2$ and so $|\pi(S)| \leq 5$;
(ii) $S \simeq PSL(2, p^n)$ where p is an odd prime and $|\pi(p^n \pm 1)| \leq 2$ and so $|\pi(S)| \leq 4$.

Since $cd(PSL(2, 2^n)) = \{1, 2^n, 2^n + 1, 2^n - 1\}$, by [10, Theorem 6.4], we conclude that $B(S)$ has three connected components in case (i) while $B(S)$ has two connected components in the case (ii). Also it is clear that in both cases all the connected components are paths. $\square$

Lemma 5. *Let G be a finite group. Assume that $B(G)$ is a union of paths. If $|\rho(G)| = 5$, then $G \simeq PSL(2, 2^n) \times A$, where A is an abelian group and $|\pi(2^n \pm 1)| = 2$.*

Proof. Since each connected component of $B(G)$ is a path, we can see that $\Delta(G)$ is triangle-free. We claim that $B(G)$ is disconnected. If $B(G)$ is connected, then Proposition 2 implies that G is solvable and $n \leq 6$. This contradicts our hypothesis that $|\rho(G)| = 5$. Thus $n(\Delta(G)) = n(B(G)) > 1$. By [21, Theorem B], we have $G \simeq PSL(2, 2^n) \times A$, where A is an abelian group and $|\pi(2^n \pm 1)| = 2$. So $B(G)$ is a graph with three connected components, one of them is a path of length one and the other two components are paths of length two. $\square$

Theorem 6. *Let G be a finite nonsolvable group and let N be the solvable radical of G. Assume that $B(G)$ is a union of paths. Then $B(G)$ is disconnected and there exists a normal subgroup M of*

G such that G/N is an almost simple group with socle M/N. Furthermore $\rho(M) = \rho(G)$ and we have one of the following cases:

(1) *If $n(B(G)) = 2$, then $|cd(G)| = 5$ or $|cd(G)| = 4$, $G/N \in \{PGL(2,q), M_{10}\}$, for $q > 3$ odd, and either $cd(G) = \{1, q-1, q, q+1\}$ or $cd(G) = \{1, 9, 10, 16\}$. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be the connected components of $B(G)$. Then $\mathcal{C}_1$ is a path of length one and $\mathcal{C}_2$ is isomorphic to P_n, where $n \in \{|\rho(G)|, |\rho(G)|+1\}$.*

(2) *If $n(B(G)) = 3$, then $G \simeq PSL(2, 2^n) \times A$, where A is an abelian group and $n \geq 2$.*

Proof. As G is nonsolvable, Proposition 2 implies that $B(G)$ is disconnected. Since $B(G)$ is a union of paths, we deduce that $\Delta(G)$ is triangle-free. Now [21, Theorem A] implies that $|\rho(G)| \leq 5$. Also [21, Lemma 4.1] verifies that there exists a normal subgroup M of G such that G/N is an almost simple group with socle M/N. Furthermore $\rho(M) = \rho(G)$. Now it follows from Ito-Michler and Burnside's $p^a q^b$ theorems that $|\rho(M/N)| \geq 3$, so $|\rho(G)| \geq |\rho(M/N)| \geq 3$. Hence $3 \leq |\rho(G)| \leq 5$. In Lemma 5 we observe that if $|\rho(G)| = 5$, then $G \simeq PSL(2, 2^n) \times A$, where A is an abelian group and $|\pi(2^n \pm 1)| = 2$. This implies that $n(B(G)) = 3$. On the other hand, since G is a nonsolvable group, by [10, Theorem 6.4] we conclude that $n(B(G)) = 3$ if and only if $G \simeq PSL(2, 2^n) \times A$, where A is an abelian group and $n \geq 2$. So we may assume that $|\rho(G)| \leq 4$ and $n(B(G)) = 2$. As $n(\Gamma(G)) = n(B(G)) = 2$, [10, Theorem 7.1] implies that one of the connected components of $\Gamma(G)$ is an isolated vertex and the other one has diameter at most two. Therefore $|cd(G)| \in \{4, 5\}$.

Suppose that $|cd(G)| = 4$. It is clear that $|cd(G/N)| \geq 4$. As $cd(G/N) \subseteq cd(G)$, we have $cd(G/N) = cd(G)$. Since $n(B(G)) = 2$ and G/N is an almost simple group with $|cd(G)| = |cd(G/N)| = 4$, by [15, Theorem 1, Corollary B] we conclude that $G/N \in \{PGL(2,q), M_{10}\}$, and either $cd(G) = \{1, q-1, q, q+1\}$ for $q > 3$ odd or $cd(G) = \{1, 9, 10, 16\}$. Suppose $cd(G) = \{1, q-1, q, q+1\}$ for $q > 3$ odd. Since $B(G)$ is a union of paths and 2 divides $q-1$ and $q+1$, it is clear that $2 \leq |\pi(q^2-1)| \leq 3$. If one of $q-1$ or $q+1$ is a power of 2, then $|\pi(q^2-1)| = 2$, otherwise $|\pi(q^2-1)| = 3$ and by [13, Lemma 2.5] we conclude that one of the following cases occurs for q:

(i) $q \in \{3^4, 5^2, 7^2\}$;

(ii) $q = 3^f$, where f is an odd prime;

(iii) $q = p$, where $p \geq 11$ is a prime.

On the other hand, it is clear that the isolated vertex of $\Gamma(G)$ generates a connected component in $B(G)$ which is a path of length one. We denote this component by $\mathcal{C}_1$. Let $\mathcal{C}_2$ be the other component of $B(G)$. Since G is nonsolvable, we can easily see that $\mathcal{C}_2$ is either a path of length 3 or 4 if $|\rho(G)| = 3$ and $\mathcal{C}_2$ is either a path of length 4 or 5 if $|\rho(G)| = 4$.

□

Example 2. *Let $G_1 = M_{10}$ and $G_2 = PSL(2, 25)$. Since $cd(M_{10}) = \{1, 9, 10, 16\}$ and $cd(PSL(2, 25)) = \{1, 13, 24, 25, 26\}$, it is easy to see that $B(G_i)$ has two connected components $\mathcal{C}_{i,1}$ and $\mathcal{C}_{i,2}$, for $i \in \{1, 2\}$. $\mathcal{C}_{i,1}$ is a path of length one for each i, $\mathcal{C}_{1,2}$ is a path of length 3 and $\mathcal{C}_{2,2}$ is a path of length 5.*

4. Groups whose bipartite divisor graphs are cycles

Lemma 7. *Let G be a finite group whose $B(G)$ is a cycle of length $n \geq 6$. Then both $\Delta(G)$ and $\Gamma(G)$ are cycles.*

Proof. Suppose that $B(G) \simeq C_n$ and $n \geq 6$. Let $\Phi \in \{\Delta(G), \Gamma(G)\}$. By graph theory we know that Φ is a cycle if and only if it is a connected graph such that every vertex in Φ has degree two. Since $n(B(G)) = 1$, [4, Lemma 3.1] implies that Φ is a connected graph. Let α be a vertex of Φ. It is clear that $deg_{B(G)}(\alpha) = 2$. Since $n \geq 6$, one can see that $deg_{\Phi}(\alpha) = 2$. Thus Φ is a cycle. □

Theorem 8. *Let G be a finite group whose $B(G)$ is a cycle of length n. Then $n \in \{4, 6\}$.*

Proof. Since $B(G)$ is a cycle of length n, it is clear that $n \geq 4$. Furthermore, by [4, Theorem 3] both $\Delta(G)$ and $\Gamma(G)$ are acyclic if and only if $B(G) \simeq C_4$. In this case $\Delta(G) \simeq \Gamma(G) \simeq P_2$. On the other hand, if $n \geq 6$, then Lemma 7 implies that both $\Delta(G)$ and $\Gamma(G)$ are cycles. So $\Delta(G)$ is either a cycle or a path (which is a tree). Now [21, Theorem C] implies that $\Delta(G)$ has at most four vertices. Thus $B(G)$ can be isomorphic to C_4, C_6 or C_8. We claim that $B(G)$ cannot be a cycle of length eight. Otherwise, if $B(G) \simeq C_8$, then $\Delta(G) \simeq \Gamma(G) \simeq C_4$. First suppose that G is solvable. By the main theorem of [11], we have $G \simeq H \times K$, where $\rho(H) = \{p, q\}$, $\rho(K) = \{r, s\}$ and both $\Delta(H)$ and $\Delta(K)$ are disconnected graphs. This implies that there exists $m, n \in cd(H)^*$ and $l, k \in cd(K)^*$ such that $m = p^\alpha$, $n = q^\beta$, $l = r^\gamma$ and $k = s^\delta$, for some positive integers α, β, γ and δ. By the structure of G it is clear that $\{1, m, n, l, k, ml, mk, nl, nk\} \subseteq cd(G)$, which contradicts the form of $B(G)$. So in the solvable case, $B(G)$ is not a cycle of length eight. On the other hand, [13, Theorem B] implies that a square (i.e. C_4) cannot be the prime degree graph of a nonsolvable group. Thus also when G is nonsolvable $B(G)$ is not isomorphic to C_8. □

Corollary 9. *Let G be a finite group and $B(G)$ be a cycle. Then G is solvable and $dl(G) \leq |cd(G)| \leq 4$.*

Proof. Let $B(G) \simeq C_n$. By Theorem 8, we deduce that $n \in \{4, 6\}$ and $\Gamma(G)$ is a complete graph. Since $\Gamma(G)$ is complete, [1] implies that G is solvable. As $|cd(G)| \leq 4$ in the solvable group G, we conclude that $dl(G) \leq |cd(G)| \leq 4$ by [3]. □

Example 3. *From* [12, Section 6] *we know that for every pair of odd primes p and q such that p is congruent to 1 modula 3 and q is a prime divisor of $p + 1$, there exists a solvable group G such that $cd(G) = \{1, 3q, p^2q, 3p^3\}$. This gives an example of a solvable group G whose bipartite divisor graph related to the set of character degrees, is a cycle of length 6.*

Example 4. *There are exactly 66 groups of order 588. Among these groups, there are exactly two nonabelian groups whose bipartite divisor graphs are cycles of length four. These groups have $\{1, 6, 12\}$ as their irreducible character degrees set.*

Example 5. *Let G be a nonabelian finite group, $P \in Syl_p(G)$ such that $|P| = p^2$, where $p \geq 7$, $p \neq 11$ and $[G : P] = 12$. Since P is abelian and normal and $[G : P] = 12$, by Ito's theorem, every degree in $cd(G)$ divides 12. Since $B(G)$ is a cycle, no prime power can occur in $B(G)$, so the only possible* *degrees in $cd(G)$ are 6 and 12, and both must occur for $B(G)$ to be a cycle. Thus $B(G)$ is a cycle if and only if $cd(G) = \{1, 6, 12\}$.*

Remark 10. *Let G be a finite group with $B(G) \simeq C_4$. Let $\rho(G) = \{p, q\}$. We claim that $\pi(G) \neq \rho(G) = \{p, q\}$. Since p divides every nonlinear irreducible character degree of G,* [5, Corollary 12.2] *implies that G has a normal p-complement Q. If $\pi(G) = \rho(G) = \{p, q\}$, then Q is the normal Sylow q-subgroup of G. Let P be a Sylow p-subgroup of G. Similarly, we can see that P is normal in G. Thus all the Sylow subgroups of G are normal which implies that G is nilpotent. Therefore G is the direct product of its Sylow subgroups which contradicts the structure of $B(G)$. So if $B(G) \simeq C_n$ and $n = 4$, then we have $\rho(G) \subset \pi(G)$. But this is not always the case if $B(G) \simeq C_6$, because as we can see in Example 3, G is a group generated by P, σ and τ, where P is a Camina p-group of nilpotent class three and σ, τ are two commuting automorphisms of P with orders q and 3, respectively. As it is explained in* [12]*, we have $|G| = p^7 3q$. So in this case $\pi(G) = \rho(G)$.*

Suppose that G is a finite group. The following theorem shows that if $B(G)$ is a cycle of length four, then G has a normal abelian Hall subgroup which explains the structure of $cd(G)$.

Theorem 11. *Let G be a finite group. Assume that $B(G)$ is a cycle of length 4. There exists a normal abelian Hall subgroup N of G such that $cd(G) = \{[G : I_G(\lambda)] : \lambda \in Irr(N)\}$.*

Proof. Suppose that G is a finite group of order $p_1^{a_1} p_2^{a_2} \cdots p_l^{a_l}$. Without loss of generality, we may assume that $p_1 = p$, $p_2 = q$ and $B(G) : p - m - q - n - p$. Thus for each $p_i \in \pi(G) \setminus \{p, q\}$, the Sylow p_i-subgroup of G is a normal abelian subgroup of G. Let $N = P_3 \times ... \times P_l$. Since $B(G)$ is a cycle, we deduce from Remark 10 that G is not a $\{p, q\}$-group. Thus N is a nontrivial normal abelian Hall subgroup of G. Now $\frac{G}{N}$ is a $\{p, q\}$-group, so its bipartite divisor graph is not a cycle of length four. As $cd(\frac{G}{N}) \subseteq cd(G)$, there exists no element of $cd(\frac{G}{N})$ which is a prime power. So for each nonlinear $\chi \in Irr(\frac{G}{N})$, $\chi(1) = p^\alpha q^\beta$, for some positive integers α and β. This implies that $\frac{G}{N}$ is the direct product of its Sylow subgroups which are nonabelian. But this contradicts the form of $cd(\frac{G}{N})$. Thus $\frac{G}{N}$ is abelian and $G = N \rtimes H$, where H is the hall $\{p, q\}$-subgroup of G. Now by [18, Lemma 2.3], we have $cd(G) = \{\beta(1)[G : I_G(\lambda)] : \lambda \in Irr(N), \beta \in Irr(\frac{I_G(\lambda)}{N})\}$. Since $\frac{G}{N}$ is abelian, we conclude that $\frac{I_G(\lambda)}{N}$ is abelian and $cd(G) = \{[G : I_G(\lambda)] : \lambda \in Irr(N)\}$. □

Example 6. *Consider $G = S_3 \times N$, where N is a cyclic group of order 5. It is clear that $B(G)$ is not a cycle and N is a normal abelian Hall subgroup of G. Let x be the generator of N and let ϵ be a primitive fifth root of unity. This implies that $Irr(N) = \{\lambda_1, \ldots, \lambda_5\}$ such that for each $1 \leq i \leq 5$ and each $0 \leq a \leq 4$, we have*

$$\lambda_i(x^a) = \epsilon^{a(i-1)}$$

Now we have:

$$I_G(\lambda_i) = \{g \in G : \lambda_i^g = \lambda_i\} = \{g \in G : \lambda_i(g^{-1}xg) = \lambda_i(x)\} = G$$

Hence $[G : I_G(\lambda_i)] = 1$ for each $1 \leq i \leq 5$ which implies that

$$\{1, 2\} = cd(G) \neq \{[G : I_G(\lambda_i)] : 1 \leq i \leq 5\} = \{1\}.$$

Acknowledgments

The author would like to thank Prof. Mark L. Lewis for his helpful conversations while she was preparing this paper and the referees for their constructive comments.

References

[1] M. Bianchi, D. Chillag, M. L. Lewis and E. Pacifici, Character degree graphs that are complete graphs, *Proc. Amer. Math. Soc.*, **135** (2007) 671–676.

[2] D. Bubboloni, S. Dolfi, M. A. Iranmanesh and C. E. Praeger, On bipartite divisor graphs for group conjugacy class sizes, *J. Pure Appl. Algebra*, **213** (2009) 1722–1734.

[3] S. C. Garrison, *On groups with a small number of character degrees*, Ph.D. Thesis, University of Wisconsin, Madison, 1973.

[4] M. A. Iranmanesh and C. E. Praeger, Bipartite divisor graphs for integer subsets, *Graphs Combin.*, **26** (2010) 95–105.

[5] I. M. Isaacs, *Character theory of finite groups*, Academic Press, New York, 1976.

[6] M. L. Lewis, Irreducible character degree sets of solvable groups, *J. Algebra*, **206** (1998) 208–234.

[7] M. L. Lewis, A Solvable group whose character degree graph has diameter 3, *Proc. Amer. Math. Soc.*, **130** (2001) 625–630.

[8] M. L. Lewis, Solvable groups whose degree graphs have two connected components, *J. Group Theory*, **4** (2001) 255–275.

[9] M. L. Lewis, Derived lengths of solvable groups having five irreducible character degrees I, *Algeb. Represent. Theory*, **4** (2001) 469–489.

[10] M. L. Lewis, An overview of graphs associated with character degrees and conjugacy class sizes in finite groups, *Rocky Mountain J. Math.*, **38** (2008) 175–211.

[11] M. L. Lewis and Q. Meng, Square character degree graphs yield direct products, *J. Algebra*, **49** (2012) 185–200.

[12] M. L. Lewis, A. Moretó and T. R. Wolf, Nondivisibility among character degrees, *J. Group Theory*, **8** (2005) 561–588.

[13] M. Lewis and D. White, Four-vertex degree graphs of nonsolvable groups, *J. Algebra*, **378** (2013) 1–11.

[14] H. LiGuo and Q. GuoHua, Graphs of nonsolvable groups with four degree-vertices, *Sci. China Math.*, **58** (2015) 1305–1310.

[15] G. Malle and A. Moretó, Nonsolvable groups with few character degrees, *J. Algebra*, **294** (2005) 117–126.

[16] S. A. Moosavi, On bipartite divisor graph for character degrees, *Int. J. Group Theory*, **6** (2017) 1–7.

[17] T. Noritzsch, Groups having three complex irreducible character degrees, *J. Algebra*, **175** (1995) 767–798.

[18] A. Previtali, Orbit lengths and character degrees in a p-Sylow subgroup of some classical Lie groups, *J. Algebra*, **177** (1995) 658–675.

[19] J. M. Riedl, Fitting heights of odd-order groups with few character degrees, *J. Algebra*, **267** (2003) 421–442.

[20] B. Taeri, Cycles and bipartite graph on conjugacy class of groups, *Rend. Sem. Mat. Univ. Padova.*, **123** (2010) 233–247.

[21] H. P. Tong-Viet, Groups whose prime graph has no triangles, *J. Algebra, **378** (2013) 196–206.*

Roghayeh Hafezieh

Department of Mathematics, Gebze Technical University, P. O. B. 41400, Gebze, Turkey

Email: roghayeh@gtu.edu.tr