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ON GROUPS WITH A RESTRICTION ON NORMAL SUBGROUPS

ALESSIO RUSSO

In memory of Mario Curzio

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Abstract. The structure of infinite groups in which every (proper) normal subgroup is the only one of its cardinality is investigated in the universe of groups without infinite simple sections. The corrisponding problem for finite soluble groups was considered by M. Curzio (1958).

1. **Introduction**

A group G is said to be an N-*group* (or to have the N-*property*) if every normal subgroup of G is the only one of its cardinality, while G is called an N_0-*group* if for every *proper* normal subgroup H of G there is no other proper subgroup with the same cardinality of H. Clearly the properties N and N_0 are equivalent for finite groups, and it is a simple exercise to show that every finite nilpotent group satisfying the N-property is cyclic. The structure of finite soluble N-groups was investigated by M. Curzio [2] in 1958. He proved that a finite soluble group G is an N-group if and only if G satisfies the following conditions: (1) Its Sylow subgroups are cyclic or elementary abelian or quaternion; (2) if G contains a quaternion subgroup, it has no normal non-cyclic subgroups of order $4m$, where m is odd; (3) the order of every elementary abelian Sylow subgroup appears as the order of a principal factor of G. As a consequence of this result he obtained that a finite group is a supersoluble N-group if and only if its Sylow subgroups are cyclic.

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The aim of this short article is to describe the structure of infinite N_0-groups in the universe of groups without infinite simple sections. Clearly every locally soluble group has no infinite simple sections. In particular, it will be showed that an infinite non-abelian group G without infinite simple sections satisfies the N_0-property if and only if any two proper subgroups of G with the same cardinality are isomorphic. Recently [3], infinite groups satisfying this latter property have been investigated.

Most of our notation is standard, and can for instance be found in [5].

2. **Results**

Recall that an uncountable group whose proper subgroups have strictly smaller cardinality is said to be a *Jónsson group*. An example of Jónsson group with cardinality of continuum was contructed by S. Shelah [6] in 1980. As remarked by A. Macintyre about the Shelah's example, it is easy to show (see also [4, Corollary 2.6]) that the factor group $G/Z(G)$ is simple with the same cardinality of G, whenever G is a Jónsson group. It follows that a group without infinite simple sections cannot be a Jónsson group.

Theorem 2.1. *An infinite group G with no infinite simple sections has the N_0-property if and only if it satisfies the following conditions:*

(1) *G is a group of type p^∞ for some prime number p;*
(2) *$G = \langle x\rangle \ltimes P$, where P is a group of type p^∞ for some prime number p and x has prime order $q \neq p$.*

Proof. Clearly, all groups described in (1) and (2) have the N_0-property. Conversely, let G be an infinite N_0-group without infinite simple sections. Assume first that G has no proper infinite normal subgroups. It follows by hypothesis that G contains infinitely many finite normal subgroups, and hence it is generated by them. Let N be any finite normal subgroup of G. As $G/C_G(N)$ is finite, then G centralizes N. Therefore G is abelian and hence also a torsion group. Moreover G has no proper subgroup of finite index. Thus G is divisible and it is a group of type p^∞ for some prime number p. Now let P be a proper infinite normal subgroup of G. If x is an element in G which does not belong to P, then $G = \langle x\rangle P$ since the subgroups P and $\langle x\rangle P$ have the same cardinality. It follows that P is the unique proper infinite normal subgroup of G and by hypothesis every proper subgroup of P has cardinality smaller than the cardinality of P. If P is uncountable, then it is a Jónsson group, a contradiction by the quoted remark. Therefore G is countable. Moreover P is locally finite since otherwise it is finitely generated and hence it contains a maximal normal subgroup of finite index. As a consequence P is abelian by the well-known theorem of Hall-Kulatilaka and Kargapolov (see [5, Part 1, Theorem 3.43]). In addition, as above we have that P is divisible and so it is of type p^∞ for some prime number p. We have already seen that $G = \langle x\rangle P$ and it is clear that x has order a prime $q \neq p$. Thus $G = \langle x\rangle \ltimes P$ and the statement is proved. □

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Clearly the above result yields that any locally nilpotent N_0-group is abelian. Moreover it cannot be extended to locally graded groups as the consideration of infinite alternating group shows. Here, a group G is called *locally graded* if every finitely generated non-trivial subgroup of G contains a proper subgroup of finite index.

Recall that a group G is said to be a *C-group* (a *C_0-group*, respectively) if any two (proper) subgroups of G with the same cardinality are isomorphic. The properties C and C_0 are obviously equivalent within the universe of finite groups. The structure of finite soluble C-groups was investigated by R. Armstrong [1], while recently infinite groups in the classes C and C_0 have been studied, with the particular target locally graded groups (see [3]). Our next result shows that for infinite groups there is a strictly relation between the classes N_0 and C_0.

Corollary 2.2. *Let G be an infinite group without infinite simple sections. Then the properties N_0 and C_0 are equivalent provided G neither is infinite cyclic nor abelian of prime exponent.*

Proof. The statement follows immediately from Theorem 2.1 and [3, Corollary 2.4]. □

Recall that a finite soluble C-group has derived length at most 4 (see [1, Theorem 4.2]). On the other hand, it is possible to construct finite soluble N-groups of arbitrary derived length. For this purpose, let n be a positive integer. It is easy to show that there exist n prime numbers

$$5 = p_1 < p_2 < \cdots < p_n$$

such that 3 does not divide $p_i - 1$, $p_i \equiv -1 \ (mod p_1), \ldots, p_i \equiv -1 \ (mod p_{i-1})$, for every $i = 2, \ldots, n$. Denote by $P_i = \langle c_i \rangle \times \langle d_i \rangle$ an abelian group of order p_i^2 and exponent p_i. Put

$$S = \langle a, b \mid a^2 = 1 = b^3, b^{-1}ab = a^{-1} \rangle.$$

Clearly, S is isomorphic to a subgroup of $GL(2, p_i)$ which acts irreducibly over all P_i by the rules:

$$c_i^a = c_i, d_i^a = c_i^{-1} d_i^{-1}, c_i^b = d_i, d_i^b = c_i^{-1} d_i^{-1}.$$

Put $G_1 = S \ltimes P_1$. Then G_1 is a soluble N-group with derived length 3. Note that G does not satisfy the property C, since it contains the subgroups $\langle a, c_1 \rangle$ and $\langle a, c_1 d_1^2 \rangle$ of order 10 which are not isomorphic. Now consider the group $G_2 = G_1 \ltimes P_2$, where c_1 acts over P_2 as an automorphism of order p_1 while d_1 acts identically. Clearly G_2 is a soluble N-group with derived length 4. Iterating the construction, we may define by induction the group

$$G = G_{n-1} \ltimes P_n = (\cdots((S \ltimes P_1) \ltimes P_2) \ltimes \cdots \ltimes P_{n-1}) \ltimes P_n$$

which is a soluble N-group with derived length $n + 2$.

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Alessio Russo

Dipartimento di Matematica e Fisica, Seconda Università di Napoli, Viale Lincoln 5, I-81100 Caserta, Italy

Email: alessio.russo@unina2.it

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