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ON FREE SUBGROUPS OF FINITE EXPONENT IN CIRCLE GROUPS OF FREE NILPOTENT ALGEBRAS

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Abstract. Let K be a commutative ring with identity and N the free nilpotent K-algebra on a non-empty set X. Then N is a group with respect to the circle composition. We prove that the subgroup generated by X is relatively free in a suitable class of groups, depending on the choice of K. Moreover, we get unique representations of the elements in terms of basic commutators. In particular, if K is of characteristic 0 the subgroup generated by X is freely generated by X as a nilpotent group.

1. **Introduction**

In [4], Magnus investigated the groups of units in the $\mathbb{Z}$-algebra of power series on a non-empty set X. He showed that the subgroup generated by $\{1 + x \mid x \in X\}$ is a free group, freely generated by this set. In terms of the circle composition, this corresponds to the statement that the subgroup generated by X is a free group, freely generated by X.

In this article, we investigate the circle group in the free nilpotent algebra on a non-empty set X. We mainly consider the subgroup generated by X and show that this group is freely generated by X with regard to a suitable class of groups. Furthermore, in case of a finite set X, all the elements in this group have a unique representation in terms of basic commutators. We also analyse the occurring classes of groups and provide some applications.

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Throughout this article let K denote a commutative ring with identity, X a non-empty set, p a prime and let $k, l \in \mathbb{N}$.

2. Preliminaries

Let A be an associative K-algebra. Recall the definition of the circle composition, which we will denote by $*$:

$$a * b := a + b + ab \quad \text{for all } a, b \in A.$$

Note that $*$ is associative and 0 is the neutral element. If A is nil, then $(A, *)$ is a group and if A is a nilpotent algebra of nilpotency class k, then the group $(A, *)$ is nilpotent of class at most k. We will denote the inverse element of $a \in A$ with respect to $*$ by a^- if it exists. Set

$$\mathop{\bigstar}_{i=1}^{n} a_i := a_1 * \cdots * a_n, \quad a^{(n)} := \mathop{\bigstar}_{i=1}^{n} a \quad \text{and} \quad a^{(-n)} := \left(a^{(n)}\right)^-$$

for all $n \in \mathbb{N}$ and $a, a_1, \ldots, a_n \in A$.

The following lemma is easily proved by induction on n:

Lemma 2.1. *Let $n \in \mathbb{N}$ and $a, a_1, \ldots, a_n \in A$. Then*

$$\mathop{\bigstar}_{i=1}^{n} a_i = \sum_{j=1}^{n} \sum_{1 \leq i_1 < \cdots < i_j \leq n} a_{i_1} \cdots a_{i_j}, \quad a^{(n)} = \sum_{i=1}^{n} \binom{n}{i} a^i \quad \text{and}$$

$$a^{k+1} = 0 \quad \Rightarrow \quad a^{(-n)} = \sum_{i=1}^{k} (-1)^i \binom{n+i-1}{i} a^i.$$

Let $N_{k,X}(K)$ denote the free nilpotent K-algebra of class k on X. We write $X^{=i}$ for the set of words in X of length i and use the description $N_{k,X}(K) = KX^{\leq k}$, where $X^{\leq k} = \bigcup_{i=1}^{k} X^{=i}$. The multiplication on $X^{\leq k}$ is given by concatenation and yields 0 if the resulting word is longer than k. Note that

$$N_{k,X}(K) = \bigoplus_{i=1}^{k} KX^{=i} = \bigoplus_{f \in X^{\leq k}} Kf.$$

Let $\pi_i : N_{k,X}(K) \to KX^{=i}$ denote the i-th projection and $\pi_f : N_{k,X}(K) \to Kf$ the projection with respect to the second decomposition. If there is no doubt about k, X (and K), we write only $N(K)$ (or N) instead of $N_{k,X}(K)$. We obviously have

$$N^j = \bigoplus_{i=j}^{k} KX^{=i}$$

for all $j \in \{1, \ldots, k\}$.

The aim of this article is to analyse the subgroup of $(N, *)$ generated by X. Due to Lemma 2.1 this subgroup is contained in $N(K_0)$, where K_0 denotes the prime ring of K. Thus, we only need to consider the case where K is a factor ring of $\mathbb{Z}$.

Proposition 2.2. *Let $K = \mathbb{Z}/c\mathbb{Z}$ for some $c \in \mathbb{N} \setminus \{1\}$. Let $c = p_1^{r_1} \cdots p_q^{r_q}$ be the prime factorization of c. Then*

$$N(\mathbb{Z}/c\mathbb{Z}) \cong \bigoplus_{i=1}^{q} N(\mathbb{Z}/p_i^{r_i}\mathbb{Z}).$$

Proof. It is easy to check that $N(K \oplus L) \cong N(K) \oplus N(L)$ for any two commutative rings K and L with identity. The Chinese remainder theorem now implies the result. □

Note that Proposition 2.2 also gives us a direct decomposition into subgroups with respect to $*$. For finite X, this decomposition corresponds to the Sylow decomposition.
From now on, we will only consider $K = \mathbb{Z}$ or $K = \mathbb{Z}/p^l\mathbb{Z}$.

Our next goal is to determine the order of certain elements in $(N, *)$. In order to do so, we need some elementary number theory as well as a partition of $\{1, \ldots, k\}$:

Proposition 2.3. *Let $m, n \in \mathbb{N}$, $m \leq n$, and $i, j \in \mathbb{N}_0$ such that $i \leq j$ and $p^j \mid n$ and $p^i \mid m$, $p^{i+1} \nmid m$. Then we have $p^{j-i} \mid \binom{n}{m}$ and $n = p^j$ yields $p^{j-i+1} \nmid \binom{n}{m}$.*

Proof. Let $s \in \mathbb{N}$ and $m_0, \ldots, m_s, n_0, \ldots, n_s \in \{0, \ldots, p-1\}$ such that $m = \sum_{r=0}^{s} m_r p^r$ and $n = \sum_{r=0}^{s} n_r p^r$. We have $m_i \neq 0$ and $n_i = n_{i+1} = \cdots = n_{j-1} = 0$.
Observing that we have carries in the positions $i, i+1, \ldots, j-1$ when adding m and $n-m$ in base p, we obtain the result by Kummer's theorem[1] [1, Theorem 10.2.2].
If $n = p^j$, those positions are the only ones where we have carries. □

Definition 2.4. *For every $t \in \mathbb{N}_0$ let*

$$c_t := \left\lfloor \frac{k}{p^t} \right\rfloor, \quad I_t := (c_{t+1}, c_t] \cap \mathbb{N}, \quad \textit{and} \quad s := \max\{t \in \mathbb{N}_0 \mid c_t \neq 0\}.$$

This partition of $\{1, \ldots, k\}$ now allows us to determine the order of certain elements in $(N, *)$. For any $a \in N$ we denote by $o_+(a)$ its additive order and by $o_*(a)$ its order in the group $(N, *)$.

Lemma 2.5. (a) *All non-zero elements in $(N(\mathbb{Z}), *)$ are of infinite order.*

(b) *Let $a \in N(\mathbb{Z}/p^l\mathbb{Z})$, $i \in \{1, \ldots, k\}$ and $t_i \in \{1, \ldots, s\}$ such that $i \in I_{t_i}$. Then*

$$a \in N(\mathbb{Z}/p^l\mathbb{Z})^i \quad \Rightarrow \quad o_*(a) \mid p^{l+t_i} \quad \textit{and}$$

$$\left(a \in N(\mathbb{Z}/p^l\mathbb{Z})^i \wedge o_+(a\pi_i) = p^l\right) \quad \Rightarrow \quad o_*(a) = p^{l+t_i}.$$

Proof. (a) Let $K = \mathbb{Z}$. Let $a \in N \setminus \{0\}$ and i minimal with $a\pi_i \neq 0$. Then we have by Lemma 2.1 for all $n \in \mathbb{N}$:

$$a^{(n)}\pi_i = \sum_{j=1}^{n} \binom{n}{j} a^j \pi_i = \binom{n}{1} a\pi_i \neq 0 \quad \Rightarrow \quad a \neq 0$$

[1] Kummer's theorem states: There are l carries when adding m and $n-m$ in base p if and only if we have $p^l \mid \binom{n}{m}$.

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(b) Let $K = \mathbb{Z}/p^l\mathbb{Z}$. Let $a \in N$ and $i \in \{1, \dots, k\}$ with $a\pi_j = 0$ for all $j < i$. By Lemma 2.1 we have

$$a^{(p^{l+t_i})} = \sum_{j=1}^{p^{l+t_i}} \binom{p^{l+t_i}}{j} a^j.$$

For all $j \in \{1, \dots, p^{t_i}\}$ we have $\binom{p^{l+t_i}}{j} \equiv 0 \mod p^l$ by Proposition 2.3 and for all $j > p^{t_i}$ we have $a^j \in (N^i)^{p^{t_i}+1} = \{0\}$ by the choice of t_i. Hence $a^{(p^{l+t_i})} = 0$.
Now let $o_+(a\pi_i) = p^l$. Then there exists $f \in X^{=i}$ with $a\pi_f = \alpha f$ and α is a unit in $\mathbb{Z}/p^l\mathbb{Z}$. By the choice of t_i we know that $f^{p^{t_i}} \neq 0$. This implies

$$\begin{aligned} a^{(p^{l+t_i-1})}\pi_{f^{p^{t_i}}} &= \sum_{j=1}^{p^{l+t_i-1}} \binom{p^{l+t_i-1}}{j} a^j \pi_{f^{p^{t_i}}} = \sum_{j=1}^{p^{l-1}} \binom{p^{l+t_i-1}}{jp^{t_i}} a^{jp^{t_i}} \pi_{f^{p^{t_i}}} \\ &= \binom{p^{l+t_i-1}}{p^{t_i}} a^{p^{t_i}} \pi_{f^{p^{t_i}}} = \binom{p^{l+t_i-1}}{p^{t_i}} \alpha^{p^{t_i}} f^{p^{t_i}} \neq 0 \end{aligned}$$

by Proposition 2.3. □

The following estimate for the elements of the lower central series[2] of $(N, *)$ is easily shown by induction on i since $(N^i/N^{i+1}, *) = (N^i/N^{i+1}, +)$ is abelian:

Lemma 2.6. *For all $i \in \{1, \dots, k\}$ we have $\gamma_i(N) \subseteq N^i$.*

Now we fix an order on X and consider the basic commutators in the free group $\mathcal{F}_X$ with respect to the given order[3]. If X is finite, $|X| = n$, there exist only finitely many basic commutators of a given weight i, say n_i. Note that by [2, 11.2.4] we have:

Lemma 2.7. *Let X be finite. Then*

$$\mathcal{F}_X/\gamma_{k+1}(\mathcal{F}_X) = \left\{ \prod_{i=1}^{k} \prod_{j=1}^{n_i} e_{i,j}^{\alpha_{i,j}} \middle| \alpha_{i,j} \in \mathbb{Z} \right\}$$

where $e_{i,1}, \dots, e_{i,n_i}$ denote the basic commutators of weight i.

Similarly, we define the basic Lie commutators in the free algebra F_X. If $|X| = n$, we denote by $\mathfrak{B}_i = (b_{i,1}, \dots, b_{i,n_i})$ the n_i-tuple of all basic Lie commutators of weight i.
Note that $n_i = \frac{1}{i}\sum_{d|i} \mu(d) n^{\frac{i}{d}}$ by the Witt formula [2, Theorem 11.2.2], where μ denotes the Möbius function. Furthermore, the basic Lie commutators of a given weight i form a basis for the intersection of $KX^{=i}$ with the Lie algebra generated by X.
Recall that by the Poincaré-Birkhoff-Witt theorem [3, Theorem 5.3.7] the set

$$\left\{ b_{i_1,j_1} \cdots b_{i_m,j_m} \middle| m \in \mathbb{N}, (i_1, j_1) \underset{lex}{\leq} \cdots \underset{lex}{\leq} (i_m, j_m) \right\}$$

[2]We denote the lower central series of a group G with $\gamma_i(G)$, starting with $\gamma_1(G) = G$.
[3]For the definition of basic commutators see for instance [2, Chapter 11].

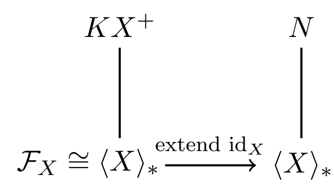
is a K-basis of F_X. In particular,

$$B = \left\{ b_{i_1,j_1} \cdots b_{i_m,j_m} \middle| m \in \mathbb{N}, (i_1, j_1) \underset{lex}{\leq} \cdots \underset{lex}{\leq} (i_m, j_m), \sum_{r=1}^{m} i_r \leq k \right\}$$

is a K-basis of N.

A well-known theorem by Magnus [4] states, that the multiplicative group generated by $\{1+x \mid x \in X\}$ in the $\mathbb{Z}$-algebra of power series in X is free, freely generated by $\{1 + x \mid x \in X\}$. Since $1 + a \mapsto a$ is a monomorphism from the multiplicative structure into the structure with respect to $*$, we know that $\langle X \rangle_*$ as a subgroup of the algebra of power series is a free group, freely generated by X. If we consider $\mathcal{F}_X$ to be the free group $\langle X \rangle_*$ as described above, for every basic commutator e of weight i we have that $e\pi_i$ is a basic Lie commutator by [2, Lemma 11.2.3].

We now define the basic commutators in $(N, *)$ by extending id_X to a homomorphism of $\mathcal{F}_X$ into N and taking images. If X is finite, we denote by $\mathfrak{E}_i = (e_{i,1}, \ldots, e_{i,n_i})$ the n_i-tuple of all basic commutators of weight i in N. Without loss of generality we can assume $e_{i,j}\pi_i = b_{i,j}$ for all $i \in \{1, \ldots, k\}$ and $j \in \{1, \ldots, n_i\}$.



Having the basis B, we now construct a K-basis of N using the basic commutators.

Proposition 2.8. *For all $i \in \{1, \ldots, k\}$ let $B_i \subseteq N^i$ such that $B_i\pi_i$ is a K-basis of $N\pi_i$. Then $\cup_{i=1}^{k} B_i$ is a K-basis of N.*
In particular,

$$E := \left\{ e_{i_1,j_1} \cdots e_{i_m,j_m} \middle| m \in \mathbb{N}, (i_1, j_1) \underset{lex}{\leq} \cdots \underset{lex}{\leq} (i_m, j_m), \sum_{r=1}^{m} i_r \leq k \right\}$$

is a K-basis of N.

Proof. Showing that $\cup_{i=k-j}^{k} B_i$ is a K-basis of N^{k-j} by induction on j yields the first claim for $j = k-1$. Set $B_i := E \cap N^i$. Then $B_i\pi_i = B \cap N^i$ since

$$(e_{i_1,j_1} \cdots e_{i_m,j_m})\pi_{\sum_{r=1}^{m} i_r} = b_{i_1,j_1} \cdots b_{i_m,j_m} \in B \cap N^{\sum_{r=1}^{m} i_r}$$

and the second claim is an application of the first. □

By taking images under a homomorphism $\varphi : \mathcal{F}_X \to G$ we can consider basic commutators in any group G (with respect to φ). In case $K = \mathbb{Z}/p^l\mathbb{Z}$ we can now compute the order of the basic commutators in $\langle X \rangle_* \leq N(\mathbb{Z}/p^l\mathbb{Z})$ (where φ is the homomorphic extension of id_X).

Lemma 2.9. *Let $K = \mathbb{Z}/p^l\mathbb{Z}$. Each basic commutator e of weight i in $(N, *)$ is of order p^{l+t_i} where $t_i \in \{1, \ldots, s\}$ with $i \in I_{t_i}$.*

Proof. This immediately follows from Lemma 2.5 (b) since $e\pi_i$ is an element of the $\mathbb{Z}/p^l\mathbb{Z}$-basis B. □

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3. Decompositions of groups

Throughout this section, let G be a group, $m \in \mathbb{N}_{>1}$ and $U_1, \ldots, U_m$ subgroups of G.

Definition 3.1. (a) *We call (U_1, U_2) a decomposition of G if $U_1U_2 = G$ and $U_1 \cap U_2 = \{1\}$.*

(b) *We call $(U_1, \ldots, U_m)$ a decomposition of G if $H_j := \prod_{i=1}^{j} U_i$ is a subgroup of G for all $j \in \{1, \ldots, m\}$, $H_m = G$ and (H_{j-1}, U_j) is a decomposition of H_j for all $j \in \{2, \ldots, m\}$.*

The following lemma is readily proved:

Lemma 3.2. *Let $H_j := \prod_{i=1}^{j} U_i$ be a subgroup of G for all $j \in \{1, \ldots, m\}$. The following statements are equivalent:*

(i) *$(U_1, \ldots, U_m)$ is a decomposition of G.*

(ii) *For each $g \in G$, there exist uniquely determined $u_i \in U_i$ for all $i \in \{1, \ldots, m\}$ such that $g = u_1 \cdots u_m$.*

(iii) *Let $u_i \in U_i$ for all $i \in \{1, \ldots, m\}$ such that $u_1 \cdots u_m = 1$. Then we have $u_i = 1$ for all $i \in \{1, \ldots, m\}$.*

The next proposition gives us a criterion on how to get a decomposition by using decompositions of the factor groups in a subnormal series of G.

Proposition 3.3. *Let $\{1\} = M_{k+1} \trianglelefteq M_k \trianglelefteq \cdots \trianglelefteq M_1 = G$ be a subnormal series of G. For all $i \in \{1, \ldots, k\}$ let $m_i \in \mathbb{N}$ and $U_{i,1}, \ldots, U_{i,m_i} \leq G$ such that*

$$(M_{i+1}U_{i,1}/M_{i+1}, \ldots, M_{i+1}U_{i,m_i}/M_{i+1})$$

is a decomposition of M_i/M_{i+1}.
If $U_{i,j} \cap M_{i+1} = \{1\}$ for all $i \in \{1, \ldots, k\}$ and $j \in \{1, \ldots, m_i\}$, then

$$(U_{k,1}, U_{k,2}, \ldots, U_{k,m_k}, U_{k-1,1}, \ldots, U_{1,m_1})$$

is a decomposition of G.

Proof. For all $i \in \{1, \ldots, k\}$ and $j \in \{1, \ldots, m_i\}$ let

$$H_{i,j} := U_{k,1} \cdots U_{k,m_k} U_{k-1,1} \cdots U_{i,j}.$$

It is easy to see that $H_{i,j}$ is a subgroup of G and that $H_{i,m_i} = M_i$ holds. In particular, we have $H_{1,m_1} = G$.
Let $i \in \{1, \ldots, k\}$ and $j \in \{1, \ldots, m_i\}$. If $j = 1$, we have $H_{i+1,m_{i+1}} \cap U_{i,1} = M_{i+1} \cap U_{i,1} = \{1\}$ by assumption. Let $j > 1$ and $u \in H_{i,j-1} \cap U_{i,j}$. By assumption and Lemma 3.2, there exist $u_s \in U_{i,s}$ for all $s \in \{1, \ldots, j-1\}$ satisfying $M_{i+1}u = M_{i+1}u_1 \cdots u_{j-1}$, which gives us $u \in M_{i+1} \cap U_{i,j} = \{1\}$ by using Lemma 3.2 again. □

We will use this proposition for finitely generated nilpotent groups by choosing the subnormal series to be the lower central series. Then each of the arising factor groups is a direct product of cyclic groups.

4. Free nilpotent groups of finite exponent

In this section let $n \in \mathbb{N}$ and $\varepsilon = (\varepsilon_1, \dots, \varepsilon_k) \in \mathbb{N}^k$.

Let $\mathfrak{N}_k$ denote the class of nilpotent groups of class at most k and $\mathfrak{N}_{k,n}$ the class of groups in $\mathfrak{N}_k$ that are generated by at most n elements. Furthermore, let $\mathfrak{N}_k(\varepsilon)$ be the class of groups $G \in \mathfrak{N}_k$ satisfying:

$$\text{The exponent of } \gamma_i(G) \text{ is a divisor of } \varepsilon_i \text{ for all } i \in \{1, \dots, k\},$$

where the $\gamma_i(G)$ denote the members of the lower central series starting with $\gamma_1(G) = G$. Let $\mathfrak{N}_{k,n}(\varepsilon)$ be defined respectively.

We denote by $\mathcal{N}_{k,n}$ the free object in $\mathfrak{N}_{k,n}$, by $\mathcal{F}_X$ the free group and by $\mathcal{N}_{k,X}$ the $\mathfrak{N}_k$-free group on X.

Note that for every set X, there exists a $\mathfrak{N}_k(\varepsilon)$-free group, freely generated by X. It is isomorphic to

$$\mathcal{F}_X \Big/ \left(\prod_{i=1}^{k} (\gamma_i(\mathcal{F}_X))^{\varepsilon_i} \gamma_{k+1}(\mathcal{F}_X) \right).$$

We denote this group by $\mathcal{N}_{k,X}(\varepsilon)$ or $\mathcal{N}_{k,n}(\varepsilon)$ if $|X| = n$. By applying Lemma 2.7, we have

$$|\mathcal{N}_{k,n}(\varepsilon)| \leq \prod_{i=1}^{k} \varepsilon_i^{n_i}.$$

In particular, $\mathcal{N}_{k,n}(\varepsilon)$ is finite and we have the following:

Lemma 4.1. *Let $G \in \mathfrak{N}_{k,n}(\varepsilon)$. Then*

$$|G| \leq \prod_{i=1}^{k} \varepsilon_i^{n_i} \quad \textit{where } n_i = \frac{1}{i} \sum_{d|i} \mu(d) n^{\frac{i}{d}}, \ \mu \textit{ the Möbius function.} \tag{4.1}$$

Since ε_1 is an upper bound for the exponent of G, we get the weaker estimate

$$|G| \leq \varepsilon_1^{\sum_{i=1}^{k} n_i}$$

for every $G \in \mathfrak{N}_{n,k}$ with exponent $\leq \varepsilon_1$.

Later in this section we will see that $|\mathcal{N}_{k,n}(\varepsilon)| < \prod_{i=1}^{k} \varepsilon_i^{n_i}$ for some ε and we will also investigate cases where equality holds. In section 5, we will see more examples for the equality. The estimate (4.1) for the order is interesting to compare with the estimate from R. Quintana in [6, Theorem 1]:

Theorem 4.2. *Let $n > 1$ and $G \in \mathfrak{N}_{k,n}$ of exponent ε_1. Then*

$$|G| \leq n^{\frac{1}{2}\varepsilon_1 k(1+k)}.$$

In the case $k = 2$, we often can find the free nilpotent group of finite exponent as a Heisenberg group $\mathcal{H}(K)$ for a suitable ring K.

Theorem 4.3. *We have $\mathcal{H}(\mathbb{Z}) \cong \mathcal{N}_{2,2}$ and $\mathcal{H}(\mathbb{Z}/c\mathbb{Z}) \cong \mathcal{N}_{2,2}(c, c)$ for all $c \in \mathbb{N}_{>1} \setminus 2\mathbb{N}$.*

Proof. The case $K = \mathbb{Z}$ is well-known. Let $c \in \mathbb{N}_{>1} \setminus 2\mathbb{N}$. It is easy to see that $\mathcal{H}(\mathbb{Z}/c\mathbb{Z})$ is generated by

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix},$$

is nilpotent of class 2, of exponent c, and $|\mathcal{H}(\mathbb{Z}/c\mathbb{Z})| = c^3$. Hence $\mathcal{H}(\mathbb{Z}/c\mathbb{Z}) \in \mathfrak{N}_{2,2}(c,c)$ and, since $|\mathcal{N}_{2,2}(c,c)| \le c^2 \cdot c = c^3$, we have $\mathcal{H}(\mathbb{Z}/c\mathbb{Z}) \cong \mathcal{N}_{2,2}(c,c)$. □

In this case, the equality in (4.1) holds. Note that $c \in \mathbb{N}_{>1} \setminus 2\mathbb{N}$ is equivalent to $I_0 = \{1,2\} = \{1,\dots,k\}$ for all primes p dividing c. The impact of those intervals (cf. Definition 2.4) on the structure of these groups will be further investigated later in this section.

The following reduction to finite sets X is readily shown:

Proposition 4.4. *Let $\mathcal{N} \in \mathfrak{N}_k$ and let X be a generating set for $\mathcal{N}$.*

(a) *If Y is $\mathfrak{N}_k$-free for all finite $Y \subseteq X$ then X is a $\mathfrak{N}_k$-free generating set for $\mathcal{N}$.*
(b) *Let $\mathcal{N} \in \mathfrak{N}_k(\varepsilon)$. If Y is $\mathfrak{N}_k(\varepsilon)$-free for all finite $Y \subseteq X$ then X is a $\mathfrak{N}_k(\varepsilon)$-free generating set for $\mathcal{N}$.*

We now reduce the choices for ε to powers of a prime:

Lemma 4.5. (a) *Let $j \in \{2,\dots,k\}$, $\tilde{\varepsilon}_i := \varepsilon_i$ for all $i \neq j$ and $\tilde{\varepsilon}_j := \gcd(\varepsilon_{j-1}, \varepsilon_j)$. Let $\tilde{\varepsilon} = (\tilde{\varepsilon}_1,\dots,\tilde{\varepsilon}_k)$. Then we have $\mathfrak{N}_k(\varepsilon) = \mathfrak{N}_k(\tilde{\varepsilon})$ and $\mathfrak{N}_{k,n}(\varepsilon) = \mathfrak{N}_{k,n}(\tilde{\varepsilon})$.*
(b) *Let $q \in \mathbb{N}$, $p_1,\dots,p_q$ be distinct primes and $r_{i,1},\dots,r_{i,q} \in \mathbb{N}_0$ satisfying $\varepsilon_i = p_1^{r_{i,1}} \cdots p_q^{r_{i,q}}$ for all $i \in \{1,\dots,k\}$. Then we have*

$$\mathcal{N}_{k,n}(\varepsilon) \cong \bigtimes_{j=1}^{q} \mathcal{N}_{k,n}\left(p_j^{r_{1,j}}, \cdots, p_j^{r_{k,j}}\right).$$

Proof. (a) is easy to verify.
(b) For all $j \in \{1,\dots,q\}$ let S_j be the p_j-Sylow subgroup of $\mathcal{N}_{k,n}(\varepsilon)$. Since $\mathcal{N}_{k,n}(\varepsilon)$ is nilpotent, we have $\mathcal{N}_{k,n}(\varepsilon) = \times_{j=1}^{q} S_j$. it is readily proved that

$$S_j \in \mathfrak{N}_{k,n}\left(p_j^{r_{1,j}}, \dots, p_j^{r_{k,j}}\right) \text{ and } \bigtimes_{j=1}^{q} \mathcal{N}_{k,n}\left(p_j^{r_{1,j}}, \dots, p_j^{r_{k,j}}\right) \in \mathfrak{N}_{k,n}(\varepsilon).$$

This gives us

$$\prod_{j=1}^{q} |S_j| \le \left| \bigtimes_{j=1}^{q} \mathcal{N}_{k,n}\left(p_j^{r_{1,j}}, \dots, p_j^{r_{k,j}}\right) \right| \le |\mathcal{N}_{k,n}(\varepsilon)| = \prod_{j=1}^{q} |S_j|.$$

This implies $S_j \cong \mathcal{N}_{k,n}\left(p_j^{r_{1,j}}, \dots, p_j^{r_{k,j}}\right)$ which completes the proof. □

Combining (a) and (b), we see that it suffices to analyse $\mathcal{N}_{k,n}(\varepsilon)$ where $\varepsilon = (p^{r_1},\dots,p^{r_k})$ for some prime p and $r_1,\dots,r_k \in \mathbb{N}_0$ satisfying $r_1 \ge \cdots \ge r_k$.

We are now trying to give a representation of $\mathcal{N}_{k,n}(\varepsilon)$ in terms of generators and relations. Let $Y = \{y_1, \ldots, y_n\}$ be a set with n elements and $\mathfrak{E}_i = (e_{i,1}, \ldots, e_{i,n_i})$ the n_i-tuple of basic commutators of weight i over Y. Furthermore, let

$$\mathcal{M}_{k,n}(\varepsilon) = \langle y_1, \ldots, y_n \mid \quad \{e_{i,j}^{\varepsilon_i} \mid i \in \{1, \ldots, k\}, j \in \{1, \ldots, n_i\}\} \\ \cup \{[y_{i_1}, \ldots, y_{i_{k+1}}] \mid i_1, \ldots, i_{k+1} \in \{1, \ldots, n\}\}\rangle.$$

Lemma 4.6. *We have $\mathcal{M}_{k,n}(\varepsilon) \in \mathfrak{N}_{k,n}$ and there is an epimorphism $\varphi : \mathcal{M}_{k,n}(\varepsilon) \to \mathcal{N}_{k,n}(\varepsilon)$. Furthermore, we have $|\mathcal{M}_{k,n}(\varepsilon)| \leq \prod_{i=1}^{k} \varepsilon_i^{n_i}$.*
In particular, we have $\mathcal{M}_{k,n}(\varepsilon) \cong \mathcal{N}_{n,k}(\varepsilon)$ if the equality in (4.1) holds.

Proof. $\mathcal{M}_{k,n}(\varepsilon) \in \mathfrak{N}_{k,n}$ and the existence of φ are clear by definition.
$|\mathcal{M}_{k,n}(\varepsilon)| \leq \prod_{i=1}^{k} \varepsilon_i^{n_i}$ follows from Lemma 2.7. □

Note that $\mathcal{M}_{k,n}(\varepsilon) \ncong \mathcal{N}_{k,n}(\varepsilon)$ in general: We consider $\mathcal{M}_{2,2}(2,2)$. It is easy to see that $\mathcal{M}_{2,2}(2,2) \cong D_8$, but the dihedral group with eight elements is of exponent four. This implies $|\mathcal{N}_{2,2}(2,2)| < 8$ so in this case, we have a strict inequality in (4.1).

Proposition 4.7. *We consider the subgroup of $(N(K), *)$ generated by X.*

(a) *Let $K = \mathbb{Z}$. Then we have $\langle X\rangle_* \in \mathfrak{N}_k$.*
If $|X| = n$, we have $\langle X\rangle_* \in \mathfrak{N}_{k,n}$.
(b) *Let $K = \mathbb{Z}/p^l\mathbb{Z}$. For all $j \in \{1, \ldots, k\}$ let $t_j \in \{0, \ldots, s\}$ with $j \in I_{t_j}$. Let $\varepsilon = \left(p^{l+t_1}, \ldots, p^{l+t_k}\right)$.*
Then we have $\langle X\rangle_* \in \mathfrak{N}_k(\varepsilon)$.
If $|X| = n$, we have $\langle X\rangle_* \in \mathfrak{N}_{k,n}(\varepsilon)$.

Proof. (a) is obvious and (b) follows from Lemma 2.5 since $\gamma_i(N) \subseteq N^i$ for all $i \in \{1, \ldots, k\}$ by Lemma 2.6. □

5. The main theorem

The aim of this section is to prove the following main theorem:

Theorem 5.1. *We consider the subgroup $\langle X\rangle_*$ of $(N(K), *)$.*

(a) *Let char $K = 0$. Then $\langle X\rangle_* \cong \mathcal{N}_{k,X}$.*
(b) *Let char $K = p^l$ and $\varepsilon = \left(p^{l+t_1}, \ldots, p^{l+t_k}\right)$ as in Proposition 4.7. Then $\langle X\rangle_* \cong \mathcal{N}_{k,X}(\varepsilon)$.*

Note that this theorem allows us to determine $\langle X\rangle_*$ in $N(K)$ for arbitrary K by applying Proposition 2.2 since $\langle X\rangle_* \subseteq N(K_0)$ where K_0 denotes the prime ring of K.

Proposition 5.2. *Let $n \in \mathbb{N}$ and $|X| = n$. Let $(e_{i,1}, \ldots, e_{i,n_i})$ be the n_i-tuple of basic commutators in $(N, *)$ for all $i \in \{1, \ldots, k\}$. For all $i \in \{1, \ldots, k\}$ and $j \in \{1, \ldots, n_i\}$ let*

$$O_{i,j} := \begin{cases} \{0, \ldots, o_*(e_{i,j}) - 1\} & o_*(e_{i,j}) < \infty \\ \mathbb{Z} & otherwise \end{cases}.$$

Then we have

$$\langle X \rangle_* = \left\{ \mathop{\bigstar}_{i=1}^{k} \mathop{\bigstar}_{j=1}^{n_i} e_{i,j}^{(\alpha_{i,j})} \middle| \alpha_{i,j} \in O_{i,j} \right\}$$

and the $\alpha_{i,j} \in O_{i,j}$ are uniquely determined for each element in $\langle X \rangle_*$.

Proof. The equality is a direct consequence of Lemma 2.7.
By Proposition 3.3 and Lemma 3.2, for the uniqueness we need to prove that

($\star$) $\gamma_i(N)/\gamma_{i+1}(N) = \bigtimes_{j=1}^{n_i} \langle e_{i,j}\gamma_{i+1}(N)\rangle_*$ for all $i \in \{1, \ldots, k\}$ and
($\star\star$) $\langle e_{i,j}\rangle_* \cap \gamma_{i+1}(N) = \{0\}$ for all $i \in \{1, \ldots, k\}$ and $j \in \{1, \ldots, n_i\}$.

Let E be the K-basis of N defined in Proposition 2.8. For all $i \in \{1, \ldots, k\}$ and $j \in \{1, \ldots, n_i\}$ we consider the following subsets of $E \cup \{0\}$:

$$\begin{aligned} M_{i,1} &:= \{e_{r_1,t_1} \cdots e_{r_m,t_m} \mid m \in \mathbb{N}, r_1, \ldots, r_m \in \{i+1, \ldots, k\}, \\ &\qquad \forall u \in \{1, \ldots, m\} : t_u \in \{1, \ldots, n_{r_u}\}, (r_1, t_1) \underset{lex}{\le} \cdots \underset{lex}{\le} (r_m, t_m)\}, \\ M_{i,2} &:= \{e_{i,t_1} \cdots e_{i,t_m} \mid m \in \mathbb{N}, t_1, \ldots, t_m \in \{1, \ldots, n_i\}, t_1 \le \cdots \le t_m\}, \\ M_{(i,j)} &:= \{e_{i,j}^m \mid m \in \mathbb{N}\}, \quad \text{and} \\ M_{(i,>j)} &:= \{e_{i,t_1} \cdots e_{i,t_m} \mid m \in \mathbb{N}, t_1, \ldots, t_m \in \{j+1, \ldots, n_i\}, t_1 \le \cdots \le t_m\}. \end{aligned}$$

For fixed i and j we have

$$\langle M_{i,1}\rangle_K \cap \langle M_{i,2}\rangle_K = \{0\} = \langle M_{i,1}\rangle_K \cap \langle M_{(i,j)}\rangle_K = \langle M_{(i,j)}\rangle_K \cap \langle M_{(i,>j)}\rangle_K.$$

Since $\langle e_{i,j}\rangle_* \subseteq \langle M_{(i,j)}\rangle_K$ by Lemma 2.1 and $\gamma_{i+i}(N) \subseteq N^{i+1} = \langle M_{i,1}\rangle_K$ by Lemma 2.6 this yields ($\star\star$).
By Lemma 2.7,

$$\gamma_i(N)/\gamma_{i+1}(N) = \prod_{j=1}^{n_i} \langle e_{i,j}\gamma_{i+1}(N)\rangle_*$$

obviously holds.

Let $\alpha_1, \dots, \alpha_{n_i} \in \mathbb{Z}$ with $\mathop{\bigstar}\limits_{j=1}^{n_i} e_{i,j}^{(\alpha_j)} \in \gamma_{i+1}(N)$. Then

$$\begin{aligned} & \mathop{\bigstar}_{j=1}^{n_i} e_{i,j}^{(\alpha_j)} \in \langle M_{i,2}\rangle_K \cap \langle M_{i,1}\rangle_K = \{0\} \\ \Rightarrow \quad & \mathop{\bigstar}_{j=1}^{n_i} e_{i,j}^{(\alpha_j)} = 0 \\ \Rightarrow \quad & e_{i,1}^{(-\alpha_1)} = \mathop{\bigstar}_{j=2}^{n_i} e_{i,j}^{(\alpha_j)} \in \langle M_{(i,1)}\rangle_K \cap \langle M_{(i,>1)}\rangle_K = \{0\}. \end{aligned}$$

This inductively implies $e_{i,j}^{(\alpha_j)} = 0$ for all $j \in \{1, \dots, n_i\}$ which proves $(\star)$. □

Now we are able to prove our main result:

Proof of Theorem 5.1. By Proposition 4.4, we only need to prove the theorem for finite X. Let X be finite and $n := |X|$.

If char $K = 0$, the result directly follows by using [2, Theorem 11.2.4] from the unique representation in Proposition 5.2.

Let char $K = p^l$. By Proposition 5.2 and Lemma 2.9, we have $|\langle X\rangle_*| = \prod_{i=1}^k \varepsilon_i^{n_i} \geq |\mathcal{N}_{k,n}(\varepsilon)|$. Since $\langle X\rangle_* \in \mathfrak{N}_{k,n}(\varepsilon)$ by Proposition 4.7, we conclude $\langle X\rangle_* \cong \mathcal{N}_{k,n}(\varepsilon)$. □

Note that we have not only proved that $\langle X\rangle_*$ is freely generated by X in a suitable class of groups, but we also have shown that $|\langle X\rangle_*| = |\mathcal{N}_{k,n}(\varepsilon)| = \prod_{i=1}^k p^{(l+t_i)n_i}$ and the existence of unique representations of the elements of $\langle X\rangle_*$ in terms of basic commutators if $|X| = n$. In particular, choosing ε as in Proposition 4.7 gives us an equality in (4.1) and we have $\mathcal{M}_{k,n}(\varepsilon) \cong \mathcal{N}_{k,n}(\varepsilon)$ by Lemma 4.6. So in the case that ε naturally arises from the order of the elements in $(N, *)$, we can describe the group $\mathcal{N}_{k,n}(\varepsilon)$ with generators and relations.

Now, we finally want to describe the impact of our result on the algebra of power series and its circle group. Let $P_X(K)$ denote the K-algebra of power series in the non-commuting set X without constant terms. Recall that by Magnus' theorem $\langle X\rangle_*$ is freely generated by X as a subgroup of $(P_X(\mathbb{Z}), *)$ (cf. page 33).

Lemma 5.3. *Let X be finite. In $P_X(\mathbb{Z})$ we have*

$$\langle X\rangle_* \cap (P_X(\mathbb{Z}))^{k+1} = \gamma_{k+1}(\langle X\rangle_*).$$

Proof. $\pi_{\leq k} := \sum_{i=1}^k \pi_i$ is a homomorphism of algebras mapping $P_X(\mathbb{Z})$ onto $N_{k,X}(\mathbb{Z})$. We have

$$\ker \pi_{\leq k|\langle X\rangle_*} = \langle X\rangle_* \cap (P_X(\mathbb{Z}))^{k+1} \quad \text{and} \quad \langle X\rangle_* / \ker \pi_{\leq k|\langle X\rangle_*} \cong \mathcal{N}_{k,X}$$

by Theorem 5.1. This yields $\gamma_{k+1}(\langle X\rangle_*) \leq \langle X\rangle_* \cap (P_X(\mathbb{Z}))^{k+1}$. Since finitely generated nilpotent groups are hopfian [5, Theorem 5.5], this implies

$$\gamma_{k+1}(\langle X\rangle_*) = \ker \pi_{\leq k|\langle X\rangle_*} = \langle X\rangle_* \cap (P_X(\mathbb{Z}))^{k+1}.$$

□

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