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ON NONINNER AUTOMORPHISMS OF FINITE p-GROUPS THAT FIX THE CENTER ELEMENTWISE

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Abstract. In this paper we show that every finite nonabelian p-group G in which the Frattini subgroup $\Phi(G)$ has order $\leq p^5$ admits a noninner automorphism of order p leaving the center $Z(G)$ elementwise fixed. As a consequence it follows that the order of a possible counterexample to the conjecture of Berkovich is at least p^8.

1. Introduction

One of the important conjectures in studying p-groups is the conjecture of Berkovich that states every finite nonabelian p-group G has at least one noninner automorphism of order p (See [16, Problem 4.13]). The conjecture was established for various classes of finite p-groups. Some of them are gathered in the following.

Remark 1.1. *Let G be a finite nonabelian p-group. If G satisfies one of the following conditions, then it admits a noninner automorphism of order p.*

(1) *The nilpotency class of G is ≤ 3* [2, 6, 15],
(2) *The coclass of G is* 2 [4], *or $p \neq 3$ and the coclass of G is* 3 [19],
(3) $C_G(Z(\Phi(G))) \neq \Phi(G)$ [10],
(4) *G is regular* [10, 20],
(5) *$G/Z(G)$ is powerful* [1],

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(6) *G is 2-generator p-group with abelian Frattini subgroup* [7],
(7) *The commutator subgroup of G is cyclic* [14].

For the other results see [13] and the references therein. In most of the above mentioned results, the noninner automorphism of order p can be chosen so that it fixes pointwise either the center $Z(G)$ or the Frattini subgroup $\Phi(G)$ of G. Schmid proved that every finite nonabelian p-group G admits a noninner automorphism of p-power order leaving the center $Z(G)$ elementwise fixed [21]. Thus it is reasonable to look for noninner automorphisms of order p that fix the the center $Z(G)$ elementwise.

Remark 1.2. *It was observed in* [4, 5] *that if a finite nonabelian p-group G satisfies one of the following conditions, then it admits a noninner automorphism of order p that fixes pointwise the center $Z(G)$ of G.*

(1) *G is regular,*
(2) *G is nilpotent of class 2,*
(3) *The commutator subgroup of G is cyclic,*
(4) *$G/Z(G)$ is powerful,*
(5) *The coclass of G is 2.*

The main result of this paper is the following.

Theorem 1.3. *Let p be any prime and let G be a finite nonabelian p-group such that the Frattini subgroup of G has order $\leq p^5$. Then G admits a noninner automorphism of order p, leaving the center $Z(G)$ elementwise fixed.*

Theorem 1.3, has the following consequence.

Corollary 1.4. *Let p be any prime and let G be a finite nonabelian p-group of order $\leq p^7$. Then G has a noninner automorphism of order p leaving the center $Z(G)$ elementwise fixed.*

Corollary 1.4 improves the result of Bodnarchuk and O. S. Pylyavska. They proved the conjecture of Berkovich in the case when G is a finite p-group of order p^n, $n \leq 6$, and $p \geq 5$ [8].

2. Proof of the main result

Throughout the section, G is a finite nonabelian p-group. By $Z(G)$, $Z_i(G)$, $\gamma_i(G)$, and $C_G(X)$ we denote the center of G, the ith term of the upper central series of G, the ith term of lower central series of G, and the centralizer of a subset X of G, respectively. If G is of order p^n and of nilpotency class $cl(G)$, then G is called to be of coclass $cc(G) = n - cl(G)$. For positive integer n, G^n denotes the subgroup $\langle x^n \mid x \in G\rangle$ and $\Omega_1(G)$ stands for subgroup $\langle x \in G \mid \ x^p = 1\rangle$. If $x \in G$ and α is an automorphism of G, then $[x, \alpha]$, denotes $x^{-1}\alpha(x)$. We use the following results.

Theorem 2.1. [9, Theorem 3.2] *Let $G = \langle a, b\rangle$ be a metabelian, two-generated group. Then the following are equivalent:*

(1) *For all* $u, v \in \gamma_2(G)$*, there is an automorphism of* G *that maps* a *to* au *and* b *to* bv*;*
(2) G *is nilpotent.*

Remark 2.2. *Let* G *be a finite nonabelian* p*-group. If* G *has no noninner automorphism of order* p *leaving the* $\Phi(G)$ *elementwise fixed, then by* [12]*, we have*

$$Z_2^\star(G) \leq C_G(Z_2^\star(G)) = \Phi(G), \tag{2.1}$$

where $Z_2^\star(G) = \{a \in Z_2(G) \mid a^p \in Z(G)\}$*. Condition* (2.1) *implies that* $C_G(Z(\Phi(G))) = \Phi(G)$ *and*

$$Z_2^\star(G) \leq Z(\Phi(G)). \tag{2.2}$$

Moreover, it follows from [1, Lemma 2.2] *that*

$$d(\frac{Z_2(G)}{Z(G)}) = d(G)d(Z(G)). \tag{2.3}$$

Remark 2.3. *Let* G *be a finite nonabelian* p*-group. Then it is easy to see that* $G = AH$ *for some subgroups* A *and* H *such that* $A \leq Z(G)$ *and* $Z(H) \leq \Phi(H)$*. If* H *has a noninner automorphism of order* p *leaving the center* $Z(H)$ *elementwise, then* G *has a noninner automorphism of order* p *which fixes both* $Z(H)$ *and* A *elementwise* (*See* [5, Lemma 2.1] *and* [10, Remark 4]).

Lemma 2.4. *Let* G *be a finite* p*-group with cyclic center.*

(1) *If* $cl(G) \geq 4$ *and* $\gamma_2(G) \leq G^{2p}\gamma_3(G)Z_2^\star(G)$*, then* G *is powerful.*
(2) *If* $\gamma_2(G)$ *is abelian noncyclic, then there is a noncentral element* $u \in \gamma_2(G) \cap Z_2(G)$ *of order* p*.*

Proof. (1) Note that $G^{2p} = \begin{cases} G^p & p > 2, \\ G^4 & p = 2 \end{cases}$. If $\gamma_2(G) \leq G^{2p}\gamma_3(G)Z_2^\star(G)$, then

$$\gamma_3(G) \leq [G^{2p}\gamma_3(G)Z_2^\star(G), G] \leq G^{2p}\gamma_4(G)\Omega_1(Z(G)).$$

Since $\Omega_1(Z(G)) \leq \gamma_4(G)$, we get $\gamma_3(G) \leq G^{2p}\gamma_4(G)$. By reiterating this argument we get $\gamma_2(G) \leq G^{2p}$, and therefore G is powerful.
(2) Suppose that $\gamma_2(G)$ is abelian noncyclic. Then $\frac{\Omega_1(\gamma_2(G))Z(G)}{Z(G)}$ is a nontrivial normal subgroup of $\frac{G}{Z(G)}$. Thus there exists an element $u \in (\Omega_1(\gamma_2(G)) \cap Z_2(G)) \setminus Z(G)$. □

Remark 2.5. *Let* G *be a finite nonabelian* p*-group such that* $Z(G)$ *is cyclic and* $C_G(\Phi(G)) = Z(\Phi(G))$*. Then* $Z_2^\star(G)$ *is abelian, since* $Z_2^\star(G) \leq C_G(\Phi(G))$*. To prove the main result we use the following arguments frequently.*

(1) *If* $d(Z_2^\star(G)) \geq 2$ *then* $Z_2^\star(G)$ *has a noncentral element* u *of order* p (Indeed, let t be an element of maximal order in $Z_2^\star(G)$. Then $Z_2^\star(G) = \langle t \rangle \times B$, for some nontrivial subgroup B of $Z_2^\star(G)$. If $\exp(Z_2^\star(G)) > \exp(Z(G))$, then $Z(G) = \langle t^p \rangle$ and if $\exp(Z_2^\star(G)) = \exp(Z(G))$, then we may assume that $\langle t \rangle = Z(G)$. In both cases B is a nontrivial elementary abelian subgroup, since $B^p \leq Z(G)$). Now consider the mapping $\varphi_u : G \to \Omega_1(Z(G))$, given by $x \mapsto [x, u]$, for all $x \in G$. Then φ_u is a homomorphism and $ker(\varphi_u) = C_G(u)$ is a maximal subgroup of G. Let

$M = C_G(u)$ and $g \in G \setminus M$. If p is odd, then $(gu)^p = g^p$. It is well-known that the mapping $g \mapsto gu$, $m \mapsto m$, for all $m \in M$, extends to an automorphism σ of G of order p leaving M elementwise fixed. Suppose that for some $h \in G$, $\sigma = \theta_h$, the inner automorphism induced by h. Since $g^{-1}g^{\sigma} \in Z_2(G)$, for all $g \in G$, and σ fixes the $\Phi(G)$ elementwise, we get that $h \in Z_3(G) \setminus Z_2(G)$ and $h \in Z(\Phi(G))$. Therefore, $h \in (Z_3(G) \cap Z(\Phi(G))) \setminus Z_2(G)$.

(2) If $d(Z_2^{\star}(G)) \geq 3$ then $Z_2^{\star}(G)$ has an elementary abelian subgroup $\langle u, v\rangle$of order p^2 such that $\langle u, v\rangle \cap Z(G) = 1$. Set $M = C_G(u)$ and $N = C_G(v)$. Then M and N are distinct maximal subgroups of G. Let $a \in M \setminus N$ and $b \in N \setminus M$. If $p = 2$, then let α be the mapping given by $a \mapsto au$, $b \mapsto bv$ and $m \mapsto m$, for all $m \in M \cap N$, and if $p > 2$, then let β, and γ be the mappings given by $a \mapsto av$, $m \mapsto m$, for all $m \in M$, and $b \mapsto bu$, $n \mapsto n$, for all $n \in N$. Then α, β and γ extend to automorphisms of G of order p that fix $M \cap N$, M, and N elementwise, respectively (See [3, Lemma 2.1]). If α, β, and γ are inner, then $\alpha = \theta_w$, $\beta = \theta_x$ and $\gamma = \theta_y$, for some $w, x, y \in G$. Similar to argument (1) we must have $w, x, y \in (Z_3(G) \cap Z(\Phi(G))) \setminus Z_2(G)$.

The following well-known fact can be easily proved by induction.

Lemma 2.6. *Let G be a group of nilpotency class 3. If $x, y \in G$, then for each positive integer n,*

(1) $[y, x^n] = [y, x]^n [y, x, x]^{\binom{n}{2}}$.

(2) *Moreover, if $[y, x, y] = 1$, then* $(xy)^n = x^n y^n [y, x]^{\binom{n}{2}} [y, x, x]^{\binom{n}{3}}$.

Lemma 2.7. *Let G be a finite p-group with abelian Frattini subgroup and nilpotency class 3. Then,*

(1) *$\gamma_3(G)$ is elementary abelian.*

(2) *$\Phi(G) \leq Z_2^{\star}(G)$.*

(3) *if $G = \langle a, b\rangle$ is two generated and $\gamma_2(G)$ is noncyclic, then $\gamma_2(G)$ is elementary abelian.*

Proof. Let $x, y, z \in G$. Then it follows form Lemma 2.6 that

$$[x, y, z]^p = [x^p, y, z] = [x, y, z^p] = [[x, y]^p, z] = 1.$$

It implies that $\gamma_3(G)$ has exponent p, $G^p \leq Z_2(G)$, and $\gamma_2(G) \leq Z_2^{\star}(G)$, since $\gamma_2(G) \leq Z_2(G)$. Moreover, for each $x, y \in G$,

$$[x^{p^2}, y] = [x, y]^{p^2} [x, y, x]^{\binom{p^2}{2}} = 1.$$

This shows that $G^p \leq Z_2^{\star}(G)$. As $\gamma_3(G)$ is abelian and $\Phi(G) = \gamma_2(G)G^p$, items (a) and (b) follow. Now suppose that $G = \langle a, b\rangle$ is two generated and $\gamma_2(G)$ is noncyclic. Then G has a noncentral element $u \in \gamma_2(G) \cap Z_2(G)$ of order p, by Lemma 2.4. But $\gamma_2(G) = \langle [a, b], \gamma_3(G)\rangle$. Hence $u = [a, b]^i c$ for some integer i and $c \in \gamma_3(G)$. Since $[a, b]^p \in Z(G)$ we must have $\gcd(i, p) = 1$. So $[a, b]$ is of order p. Therefore, $\gamma_2(G)$ is elementary abelian and item (c) follows. □

Remark 2.8. *Let G be a finite p-group. If α is an automorphism of G of p-power order, then the group $\langle\alpha\rangle$ acts on $Z(G)$. Therefore, $|C_{Z(G)}(\alpha)| \geq p$ (See for instance* [17, Section 5.4.1]*). In particular, if $Z(G)$ has order p, then α fixes $Z(G)$ pointwise.*

Lemma 2.9. *Let G be a finite p-group with cyclic center. If $Z(G) \leq \Phi(G) \leq Z_2(G)$, then G has a noninner automorphism of order p leaving the center $Z(G)$ elementwise fixed.*

Proof. Let G ba a counterexample to the lemma. Then G has no noninner automorphism of order p leaving $\Phi(G)$ elementwise fixed. Therefore G fulfils conditions (2.1) - (2.3) of Remark 2.2. If either $p \geq 3$, or $p = 2$ and $d(Z_2^\star(G)) \geq 3$, then Remark 2.5 gives a contradiction. Thus we may assume that $p = 2$ and $d(Z_2^\star(G)) = 2$. Hence, $d(G) = 2$ and $Z(G) = \Phi(Z_2^\star(G)) = (Z_2^\star(G))^2$, since by (2.3), $d(Z_2^\star(G)/Z(G)) = 2$. Clearly $cl(G) \leq 3$. But by Remark 1.2, we may assume that $cl(G) = 3$. Now it follows form Lemma 2.7 that $Z_2^\star(G) = \Phi(G)$.

Let $G = \langle a, b\rangle$. Then $\Phi(G) = \langle a^2, b^2, \gamma_2(G)\rangle$. By Lemma 2.7, $\gamma_2(G)$ is elementary abelian. Hence, $Z(G) = \langle a^4, b^4\rangle$. Without loss we may assume that $Z(G) = \langle a^4\rangle$. Thus $\Phi(G) = \langle a^2\rangle \times \langle [a, b]\rangle$. Clearly $a^2 \notin Z(G)$. Then $[a^2, b] = [a, b]^2[a, b, a] = [a, b, a] \neq 1$. We may suppose that $[a, b, b] = 1$. For if $[a, b, b] \neq 1$, then $[a, ab, ab] = [a, b, a][a, b, b] = 1$. Replacing b, by ab, we have $[a, b, b] = 1$. Next, we have $[a, b^2] = [a, b]^2[a, b, b] = 1$. Thus $b^2 \in Z(G)$, and hence $b^2 = a^{4i}$, for some integer i. Replacing b by ba^{-2i}, we may assume that $b^2 \in \Omega_1(Z(G))$. Let $2^m = |Z(G)|$. By Remark 2.8, we may suppose that $m \geq 2$. Hence $b^2 = a^{2^{m+1}j}$, for some integer j. Now, replacing b by $ba^{-2^m j}$ we have $b^2 = 1$. Therefore, $G = \langle a, b\rangle$ and $Z_2(G) = \Phi(G) = \langle a^2\rangle \times \langle [a, b]\rangle$, $a^{2^{m+2}} = 1 = b^2$, $[a^4, b] = 1$, and $[a, b]^2 = 1$. Let $x_1 = a$ and $x_2 = b$. Then G has the following power-commutator presentation.

$$G = \langle x_1,\ x_2,\ x_3,\ x_4,\ x_5 \mid x_3 = [x_2, x_1],\ x_4 = x_1^2,\ x_5 = x_4^2,\ x_3^2 = 1,\\ x_5^{2^m} = 1,\ x_2^2 = 1,\ [x_5, x_2] = 1,\ [x_1, x_2, x_1] = x_5^{2^{m-1}}\rangle.$$

It is straightforward to check that this presentation is consistent (See [22, page 424]). Now, let $y_1 = x_1x_2,\ y_2 = x_2,\ , y_3 = x_3,\ y_4 = x_3x_4,\ y_5 = x_5$. Then

$$\begin{aligned}
[y_2, y_1] &= [x_1x_2, x_1] = [x_2, x_1] = x_3 = y_3,\\
y_1^2 &= (x_1x_1)^2 = x_1^2x_2^2[x_2, x_1] = x_4x_3 = y_4,\\
y_4^2 &= (x_3x_4)^2 = x_3^2x_4^2 = x_5 = y_5,\\
[y_5, y_2] &= [x_5, x_2] = 1,\\
[y_1, y_2, y_1] &= [x_1x_2, x_2, x_1x_2] = [x_1, x_2, x_1] = x_5^{2^{m-1}} = y_5^{2^{m-1}}.
\end{aligned}$$

Thus by von Dyck's Theorem, the mapping $\alpha :\ x_i \mapsto y_i$, $1 \leq i \leq 5$ is an automorphism of G. Clearly α is noninner of order 2, and $\alpha(x_1^4) = (x_1x_2)^4 = x_1^4$. Therefore α fixes the center elementwise. □

Proof of Theorem 1.3. Let G be a counterexample of the theorem. By Remark 2.3 we may assume that $Z(G) \leq \Phi(G)$. Thus G has no noninner automorphism of order p leaving $\Phi(G)$ elementwise fixed. Hence G satisfies conditions (2.1) - (2.3) of Remark 2.2. Then

$$p^5 \geq |Z(\Phi(G))| \geq |Z_2^\star(G)| \geq p^{d(G)d(Z(G))}|Z(G)| \geq p^{d(G)d(Z(G))+d(Z(G))}. \tag{2.4}$$

Therefore $Z(G)$ is cyclic of order $\leq p^3$. Moreover, by Remark 1.2, we may assume that $\gamma_2(G)$ is not cyclic. Now, we consider three cases: (a) $Z(G) \cong \mathbb{Z}_{p^3}$, (b) $Z(G) \cong \mathbb{Z}_{p^2}$, and (c) $Z(G) \cong \mathbb{Z}_p$.

Case (a): Let $Z(G) \cong \mathbb{Z}_{p^3}$. Then $Z_2^\star(G) = Z(\Phi(G)) = \Phi(G)$ is of order p^5, by (2.4). Now Lemma 2.9 gives a contradiction in this case.

Case (b): Let $Z(G) \cong \mathbb{Z}_{p^2}$. By Lemma 2.9 we may assume that $Z_2^\star(G) \lneqq \Phi(G)$. Then it follows from (2.4) that $|Z_2^\star(G)| = p^4$, $d(G) = 2$, and $|G| = p^7$. Then $cl(G) \leq 4$, as $|Z_2^\star(G)| = p^4$. Thus we may assume by Remark 1.2 that $cl(G) = 3$ or $cl(G) = 4$. If $cl(G) = 3$, then it follows from Lemma 2.7, that $Z_2^\star(G) = \Phi(G)$, a contradiction. Hence $cl(G) = 4$ and G has the following upper central series;

$$1 < Z_1(G) < Z_2(G) < Z_3(G) < Z_4(G) = G,$$

where $|Z_1(G)| = p^2$, $|Z_2(G)/Z_1(G)| = p^2$, and $|Z_3(G)/Z_2(G)| = p$.

By Remark 1.2, we may assume that $\gamma_2(G)$ is not cyclic. Thus it follows from Lemma 2.4 that there exists a noncentral element u of order p in $\gamma_2(G) \cap Z_2(G)$. If $u \notin G^{2p}\gamma_3(G)$, then $u = [a,b]^i c$ for some $c \in G^{2p}\gamma_3(G)$ and $1 \leq i \leq p-1$, since $\gamma_2(G) = \langle [a,b],\ \gamma_3(G)\rangle$. This implies that $\gamma_2(G) \leq G^{2p}\gamma_3(G)Z_2^\star(G)$. Hence G is powerful, by Lemma 2.4, and then Remark 1.2 yields a contradiction. Therefore $u \in G^{2p}\gamma_3(G)$. Now, it follows from Theorem 2.1 that there are automorphisms α and α' of G such that $\alpha:\ a \mapsto au,\ b \mapsto b$ and $\alpha':\ a \mapsto a,\ b \mapsto bu$. Clearly α and α' have order p, as they fix $G^{2p}\gamma_3(G)$ elementwise. Then $\alpha = \theta_s$ and $\alpha' = \theta_{s'}$, for some $s, s' \in Z_3(G) \setminus Z_2(G)$. As $s' = s^i t$ for some integer $1 \leq i \leq p-1$ and some $t \in Z_2(G)$ we get

$$1 = [a, \alpha'] = [a, s'] = [a, s^i t] = [a, s]^i [a, t] = u^i [a, t].$$

Thus $u \in Z(G)$, a contradiction.

Case (c): Let $|Z(G)| = p$, then it suffices to show that G admits a noninner automorphism of order p, by Remark 2.8. Suppose that $\Phi(G)$ is nonabelian. It follows from (2.4) that $Z_2^\star(G) = Z(\Phi(G))$ is of order p^3, and $d(G) = 2$. Thus G has order p^7. Hence we have either $cl(G) \leq 3$ or $cc(G) \leq 3$. By [1, Corollary 2.4], G is not of maximal class, and by parts (1) and (2) of Remark 1.1 we have $p = 3$ and coclass G is 3. Now $d(Z_2^\star(G)) \geq 2$ and hence argument (1) of Remark 2.5 gives a contradiction.Therefore, $\Phi(G)$ is abelian. By part (6) of Remark 1.1 we may assume that $d(G) \geq 3$, and by Lemma 2.9, we have $\Phi(G) \nleq Z_2(G)$. Thus (2.4), implies that

$$d(G) = 3,\ |Z_2^\star(G)| = p^4,\ |\Phi(G)| = p^5,\ \text{and } |G| = p^8.$$

Then the argument (2) of Remark 2.5 holds.

Now, let $g \in Z_2(G)$. For every $h \in G$ we have $[g^p, h] = [g, h]^p = 1$, as $Z(G)$ is of order p. Hence

$$Z_2(G) = Z_2^\star(G). \tag{2.5}$$

Since $Z(\frac{G}{Z_2(G)}) \cap \frac{\Phi(G)}{Z_2(G)}$ is nontrivial, we get $\frac{Z_3(G)}{Z_2(G)} \cap \frac{\Phi(G)}{Z_2(G)}$ is cyclic of order p. Suppose that p is odd. Use the same notations and terminology as in the argument (2) of Remark 2.5. Then $y = x^i t$, for some integer $0 < i < p$ and $t \in Z_2(G)$. Thus

$$u = [b, \gamma] = [b, y] = [b, x^i t] = [b, x]^i [b, t] = [b, \beta]^i [b, t] = [b, t] \in Z(G),$$

that contradicts the choice of u. Hence we have $p = 2$. If $G^4\gamma_3(G) \not\leq Z_2^\star(G)$, then $\gamma_2(G) \leq \Phi(G) = G^4\gamma_3(G)Z_2^\star(G)$. Thus by Lemma 2.4, G is powerful and part (5) of Remark 1.1 gives a contradiction. Therefore,

$$G^4\gamma_3(G) \leq Z_2^\star(G). \tag{2.6}$$

Since $G/Z_2^\star(G)$ is nonabelian, we get $\gamma_2(G) \not\leq Z_2^\star(G)$. Hence $\Phi(G) = \gamma_2(G)Z_2^\star(G)$ and $|\gamma_2(G/Z_2^\star(G)| = 2$. Therefore, $G/Z_2^\star(G)$ is of nilpotency class 2.

It is easy to see that if H is finite 2-group such that H' is of order 2 and $d(H) = 3$, then $Z(H) \nleq \Phi(H)$. Therefore there is some $c \in G \setminus \Phi(G)$ such that $cZ_2^\star(G) \in Z(G/Z_2^\star(G))$. Thus $[c, G] \subseteq Z_2^\star(G)$.

Recall w from the argument (2) of Remark 2.5. We have $w = ts$, for some $t \in \gamma_2(G) \setminus Z_2^\star(G)$ and $s \in Z_2^\star(G)$. Thus $u = [a, ts] = [a, t][a, s]$ and $v = [b, ts] = [b, t][b, s]$. Then $u, v \in \gamma_3(G)$. This shows that $\Omega_1(Z_2(G)) = \gamma_3(G)$.

We know that $[\gamma_i(G), Z_j(G)] \leq Z_{j-i}$, for each $j \geq i$ (See for instance [23, Corollary 2, page 20]). Hence $[\gamma_3(G), c] = 1$. This shows that $c \in M \cap N$. Thus $G = \langle a, b, c\rangle$. Since $\gamma_2(G) = \langle [a, b], [a, c], [b, c], \gamma_3(G)\rangle$ and $[a, c], [b, c] \in Z_2^\star(G)$, it follows that $\Phi(G)/Z_2^\star(G) = \langle [a, b]Z_2^\star(G)\rangle$.

Note that $[c^2, a] = [c, a]^2[c, a, c] \in Z(G)$, since $[c, a] \in Z_2^\star(G)$. Similarly $[c^2, b] \in Z(G)$. Hence $c^2 \in Z_2(G)$. Set $N = \langle c, Z_2(G)\rangle$. Then $|N| = 2^5$, $N \trianglelefteq G$ and G/N is nonabelian of order 8. Let $\overline{G} = G/N$ and $\overline{g} = Ng$, for $g \in G$. We have $[N, \gamma_3(G)] = 1$. Thus $\overline{G}$ acts by conjugation on $\gamma_3(G)$. Therefore $\gamma_3(G)$ can be considered as a $\overline{G}$-module. If f is a derivation from $\overline{G} \to \gamma_3(G)$, then the mapping $\alpha_f : G \to G$, given by $\alpha_f(x) = xf(\overline(x)$, is an automorphism of G of order 2. We consider the following two cases.

- If $\overline{a}^2 = \overline{b}^2 = \overline{[a, b]}$, then

$$\overline{G} = \langle \overline{a},\ \overline{b} \mid\ \overline{a}^4 = \overline{1}, \overline{a}^2 = \overline{b}^2,\ [\overline{a}, \overline{b}] = \overline{a}^2\rangle \cong Q_8.$$

So every element of $\overline{G}$ can be written in the form $\overline{a}^i\overline{b}^j$ for some $i \in \{0, 1, 2, 3\}$ and $j \in \{0, 1\}$. Define the mapping $f : \overline{G} \to \gamma_3(G)$, by $f(\overline{a}^i\overline{b}^j) = v^{j+1+a+\cdots+a^{i-1}}u^{1+b+\ldots b^{j-1}}$. Then

$$f(\overline{a}^i\overline{b}^j\overline{a}^k\overline{b}^l) = f(\overline{a}^{i+k+2jk}\overline{b}^{j+l}) = v^{j+l+1+a+\cdots+a^{i+k+2jk-1}}u^{1+b+\cdots b^{j+l-1}}.$$

On the other hand,

$$\begin{aligned} f(\overline{a}^i\overline{b}^j)^{a^kb^l} f(\overline{a}^k\overline{b}^l) &= (v^{j+1+a+\cdots+a^{i-1}}u^{1+b+\cdots b^{j-1}})^{a^kb^l} v^{l+1+a+\cdots+a^{k-1}}u^{1+b+\cdots b^{l-1}} \\ &= v^{ja^k+a^k+\cdots+a^{i+k-1}}u^{b^l+b^{l+1}+\cdots b^{j+l-1}} v^{l+1+a+\cdots+a^{k-1}}u^{1+b+\cdots b^{l-1}} \\ &= v^{j+l+1+a+\cdots+a^{i+k-1}}u^{1+b+\cdots b^{j+l-1}}[v, a]^{jk}. \end{aligned}$$

Since $v^{a^{i+k}+\cdots+a^{i+k+2jk-1}} = [v,a]^{jk}$, f is a derivation. By assumption $\alpha_f = \theta_{w_0}$, for some $w_0 \in G$. Since $[G,\alpha_f] \leq Z_2(G)$ and α_f is identity on N, we have $w_0 \in Z_3(G) \setminus Z_2(G)$ and $w_0 \in C_G(N) \leq C_G(Z_2^{\star}(G)) = \Phi(G)$. Hence $w_0 = wt$, for some $t \in Z_2(G)$. Now,

$$uv = [b,\alpha_f] = [b,w_0] = [b,wt] = [b,w][b,t] = v[b,t]$$

This implies that $u \in Z(G)$, a contradiction.

- If $\overline{a}^2 = \overline{1}$ or $\overline{a}^2 = \overline{1}$, then by replacement may assume that $\overline{a}^2 = \overline{[a,b]}$ and $\overline{b}^2 = \overline{1}$. Indeed, if $\overline{b}^2 = \overline{[a,b]}$, and $\overline{a}^2 = \overline{1}$, then we replace a and b by each other, and u and v by each other, and if $\overline{a}^2 = \overline{b}^2 = \overline{1}$, then we replace a by ab and u by uv. Therefore we have

$$\overline{G} = \langle \overline{a},\ \overline{b} \mid\ \overline{a}^4 = \overline{b}^2 = \overline{1}, [\overline{a},\overline{b}] = \overline{a}^2 \rangle \cong D_8,$$

and each element of $\overline{G}$ is written in the form $\overline{a}^i\overline{b}^j$ for some $i \in \{0,1,2,3\}$ and $j \in \{0,1\}$. In this case the mapping $f : \overline{G} \to \gamma_3(G)$, defined by $f(\overline{a}^i\overline{b}^j) = v^{j+1+a+\cdots+a^{i-1}}$ is a derivation. Then α_f is an automorphism of G of order 2. Thus $\alpha_f = \theta_{w_0}$ is inner. Hence $w_0 = wt$, for some $t \in Z_2(G)$.

$$v = [a,\alpha_f] = [a,w_0] = [a,wt] = [a,w][a,t] = u[a,t]$$

It implies that $uv \in Z(G)$, a contradiction. This completes the proof. □

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