www.theoryofgroups.ir

International Journal of Group Theory
ISSN (print): 2251-7650, ISSN (on-line): 2251-7669
Vol. 8 No. 1 (2019), pp. 11-22.

www.ui.ac.ir

ON THE RANKS OF FISCHER GROUP Fi'_{24} AND THE BABY MONSTER GROUP $\mathbb{B}$

MOHAMMED ALI FAYA IBRAHIM, FARYAD ALI*, MOHAMMED A. AL-KADHI AND ABDULLAH ALJOUIEE

Communicated by Mohammad Reza Darafsheh

Abstract. If G is a finite group and X a conjugacy class of elements of G, then we define rank$(G{:}X)$ to be the minimum number of elements of X generating G. In the present article, we determine the ranks for the Fischer's simple group Fi'_{24} and the baby monster group $\mathbb{B}$.

1. Introduction

Group generations have played a significant role in solving outstanding problems in representation theory, topology, geometry and number theory. Generation of a group by suitable subsets has been studied since the origins of group theory. This paper is intended as a sequel to the author's earlier work on the determination of ranks for the sporadic simple groups. In a series of articles [1, 2, 3, 4, 5, 6, 7], the authors determined the ranks for the sporadic simple groups J_1, J_2, J_3, J_4, HS, McL, He, Co_1, Co_2, Co_3, $O'N$, Ly and Fi_{22}. We encourage the reader to consult [10] for summary of results on generation of sporadic groups by minimal sets of conjugate elements. The motivation for this study is outlined in these papers and we encourage the reader to consult [1, 2] and [7] for backgroup material as well as basic computational techniques.

MSC(2010): Primary: 20C15; Secondary: 20D08.
Keywords: Fischer group Fi'_{24} , rank, generating triple, Baby Monster group $\mathbb{B}$.
Received: 01 March 2018, Accepted: 23 March 2018.
*Corresponding author.
DOI: http://dx.doi.org/10.22108/ijgt.2018.109973.1471

.

Suppose that G is a finite group and $X \subseteq G$. We denote the rank of X in G by $rank(G{:}X)$, the minimum number of elements of X generating G. This paper focuses on the determination of rank$(G{:}X)$ where X is a conjugacy class of G and G is a sporadic simple group. In the present article, we completely determine the ranks for Fischer's largest sporadic simple group Fi_{24}' and the Fischer's Baby Monster group $\mathbb{B}$. In particular, we prove the following two theorems.

Theorem 3.5 . *Let Fi_{24}' be the Fischer's largest sporadic simple group. Then*

(i) $\text{rank}(Fi_{24}' : nX) = 3$, *if* $nX \in \{2A, 2B, 3A, 3B\}$,
(ii) $\text{rank}(Fi_{24}' : nX) = 2$, *if* $nX \notin \{1A, 2A, 2B, 3A, 3B\}$.

Theorem 4.5 . *Let $\mathbb{B}$ be the Fischer's Baby Monster group. Then*

(i) $\text{rank}(\mathbb{B} : 2A) = 4$,
(ii) $\text{rank}(\mathbb{B} : nX) = 3$, *if* $nX \in \{2B, 2C, 2D\}$,
(iii) $\text{rank}(\mathbb{B} : nX) = 2$, *if* $nX \notin \{1A, 2A, 2B, 2C, 2D\}$.

2. Preliminaries

Throughout this paper we use the same notion as in [1, 2, 7]. If G is a finite group, C_1, C_2, C_3 conjugacy classes of the elements of G, and g_3 is a fixed representative of C_3, then $\Delta_G(C_1, C_2, C_3)$ denotes the number of distinct ordered pairs $(g_1, g_2) \in (C_1 \times C_2)$ such that $g_1g_2 = g_3$. It is well known that $\Delta_G(C_1, C_2, C_3)$ is *structure constant* for the conjugacy classes C_1, C_2, C_3 and can easily be computed from the character table of G by the following formula

$$\Delta_G(C_1, C_2, C_3) = \frac{|C_1||C_2|}{|G|} \sum_{i=1}^{m} \frac{\chi_i(g_1)\chi_i(g_2)\overline{\chi_i(g_3)}}{\chi_i(1_G)}$$

where $\chi_1, \chi_2, \ldots, \chi_m$ are the irreducible complex characters of G. Also, let $\Delta_G^*(C_1, C_2, C_3)$ denotes the number of distinct ordered pairs $(g_1, g_2) \in (C_1 \times C_2)$ such that $g_1g_2 = g_3$ and $G = \langle g_1, g_2 \rangle$. If $\Delta_G^*(C_1, C_2, C_3) > 0$, then we say that G is (C_1, C_2, C_3)-generated. If H is any subgroup of G containing the fixed element $g_3 \in C_3$, then $\Sigma_H(C_1, C_2, C_3)$ denotes the number of distinct ordered pairs $(g_1, g_2) \in (C_1 \times C_2)$ such that $g_1g_2 = g_3$ and $\langle g_1, g_2 \rangle \leq H$ where $\Sigma_H(C_1, C_2, C_3)$ is obtained by summing the structure constants $\Delta_H(c_1, c_2, c_3)$ of H over all H-conjugacy classes c_1, c_2 satisfying $c_i \subseteq H \cap C_i$ for $1 \leq i \leq 2$.

Similarly, if C_1, C_2, C_3, C_4 are conjugacy classes of G and g_4 a fixed representative of C_4 then $\Delta_G(C_1, C_2, C_3, C_4)$ denote the number of distinct triples (g_1, g_2, g_3) with $g_i \in C_i$ $1 \leq i \leq 3$ such that $g_1g_2g_3 = g_4$. This number can be computed with formula

$$\Delta_G(C_1, C_2, C_3, C_4) = \frac{|C_1||C_2||C_3|}{|G|} \sum_{i=1}^{m} \frac{\chi_i(g_1)\chi_i(g_2)\chi_i(g_3)\overline{\chi_i(g_4)}}{(\chi_i(1_G))^2}$$

where $\chi_1, \chi_2, \ldots, \chi_m$ are the irreducible complex characters of G.

The $\mathbb{ATLAS}$ [8] serves as a valuable source of information and we use its notation for conjugacy classes, maximal subgroups, character tables, permutation characters, etc. A general conjugacy class of elements of order n in G is denoted by nX. For example, $2A$ represents the first conjugacy class of involutions in a group G.

Lemma 2.1. [7] *Let G be a finite simple group such that G is (lX, mY, nZ)-generated. Then G is $(\underbrace{lX, lX, \ldots, lX}_{m-times}, (nZ)^m)$-generated.*

Corollary 2.2. *Let G be a finite simple group such that G is (lX, mY, nZ)-generated, then* $\mathrm{rank}(G : lX) \leq m$.

Proof. Immediately follows from Lemma 2.1. □

Lemma 2.3. *Conder et al.* [9] *Let G be a simple $(2X, mY, nZ)$-generated group. Then G is $(mY, mY, (nZ)^2)$-generated.*

Theorem 2.4. [14] *Let G be a finite group and H a subgroup of G containing a fixed element x such that $gcd(o(x), [N_G(H){:}H]) = 1$. Then the number h of conjugates of H containing x is $\chi_H(x)$, where χ_H is the permutation character of G with action on the conjugates of H. In particular,*

$$h = \sum_{i=1}^{m} \frac{|C_G(x)|}{|C_{N_G(H)}(x_i)|} \ ,$$

where $x_1, \ldots, x_m$ are representatives of the $N_G(H)$-conjugacy classes that fuse to the G-class $[x]_G$.

The following result in certain situations is very effective at establishing non-generations.

Lemma 2.5. [9] *Let G be a finite centerless group and suppose lX, mY, nZ are G-conjugacy classes for which $\Delta^*(G) = \Delta^*_G(lX, mY, nZ) < |C_G(z)|, z \in nZ$. Then $\Delta^*(G) = 0$ and therefore G is not (lX, mY, nZ)-generated.*

3. Ranks of Fi'_{24}

A group G is called a 3-*transposition group* if it is generated by a conjugacy class D of involutions in G such that $o(de) \leq 3$ for all $d, e \in D$. The conjugacy class D is called a class of conjugate 3-transpositions. *B. Fischer* [11] introduced and investigated 3-transposition groups. He classified all finite 3-transposition groups with no non-trivial normal soluble subgroups. In the process of classifying the 3-transposition groups, Fischer discovered three new groups Fi_{22}, Fi_{23} and Fi_{24} with 3510, 31671 and 306936 transpositions respectively. Of these, the first two groups are simple, while the third contains a simple normal subgroup Fi'_{24} of index 2 (consisting of the products of evenly many transpositions). The group Fi'_{24} has order

$$1255205709190661721292800 = 2^{21} \cdot 3^{16} \cdot 5^2 \cdot 7^3 \cdot 11 \cdot 13 \cdot 17 \cdot 23 \cdot 29\cdot$$

DOI: http://dx.doi.org/10.22108/ijgt.2018.109973.1471

The group Fi_{24}' has 108 conjugacy classes of its elements in total including two classes of involutions and five classes of elements of order 3, namely $2A, 2B, 3A, 3B, 3C, 3D$ and $3E$ as represented in $\mathbb{ATLAS}$ [8]. Linton and Wilson [16] classified all the maximal subgroups of Fi_{24}' and its automorphism group Fi_{24} which are available in the $\mathbb{ATLAS}$[8]. We will use these maximal subgroups and the permutation characters of Fi_{24}' on the conjugates (right cosets) of the maximal subgroups listed in the above theorem extensively.

Next, we compute rank for each conjugacy class of Fi_{24}'. It is well known that each sporadic simple group can be generated by three involutions. In the following lemma we show that the group Fi_{24}' can be generated by three conjugate involutions $a, b, c \in 2X$, where $X \in \{A, B\}$ such that $abc \in 23A$.

Lemma 3.1. *Let $2A$ and $2B$ be two conjugacy classes of involutions in the Fischer group Fi_{24}'. Then* $\text{rank}(Fi_{24}' : 2X) = 3$, *where $X \in \{A, B\}$.*

Proof. We consider the triples $(2A, 3E, 23A)$ and $(2B, 3E, 23A)$. Simple computation shows that the structure constants $\Delta_{Fi_{24}'}(2A, 3E, 23A)$=138 and $\Delta_{Fi_{24}'}(2B, 3E, 23A)$=199962. From the maximal subgroups of Fi_{24}' (see [8]), we observe that, up to isomorphism, Fi_{23} and $2^{11}{:}M_{24}$ are the only maximal subgroup of Fi_{24}' with order divisible by 23. However, Fi_{24}'-class $3E$ fails to meet Fi_{23}. Further, a fixed $z \in 23A$ is contained in precisely a unique conjugate copy of $2^{11}{:}M_{24}$. Since $\Sigma_{2^{11}:M_{24}}(2A, 3E, 23A) = 46$ and $\Sigma_{2^{11}:M_{24}}(2B, 3A, 23A) = 506$, we have

$$\begin{aligned}
\Delta^*_{Fi_{24}'}(2A, 3E, 23A) &\geq \Delta_{Fi_{24}'}(2A, 3E, 23A) - \Sigma_{2^{11}:M_{24}}(2A, 3E, 23A) \\
&= 138 - 46 > 0, \\
\Delta^*_{Fi_{24}'}(2B, 3E, 23A) &\geq \Delta_{Fi_{24}'}(2B, 3E, 23A) - \Sigma_{2^{11}:M_{24}}(2B, 3E, 23A) \\
&= 199962 - 506 > 0.
\end{aligned}$$

Hence, Fi_{24}' is $(2X, 3E, 23A)$-generated, where $X \in \{A, B\}$. By applying Corollary 2.2, we have $\text{rank}(Fi_{24}' : 2X) \leq 3$, $X \in \{A, B\}$. But $\text{rank}(Fi_{24}' : 2X) = 2$ is not possible since if there are $x, y \in 2X$ such that $Fi_{24}' = \langle x, y \rangle$, then $Fi_{24}' = D_{2n}$ where $o(xy) = n$. This concluded that $\text{rank}(Fi_{24}' : 2X) = 3$, where $X \in \{A, B\}$. □

Lemma 3.2. *The Fischer group Fi_{24}' is not $(3X, 3X, nY)$-generated for any integer n, where $X \in \{A, B\}$.*

Proof. It is well known that if G is (l, m, n)-generated simple group then $\frac{1}{l} + \frac{1}{m} + \frac{1}{n} < 1$. It follows that in $(3X, 3X, nY)$- generations of Fi_{24}', we have $n > 3$.

Let $P = \{4A, 5A, 6A, 6B, 6D, 6E, 6F, 6G, 6H, 6I, 6J, 7A, 8B, 9A, 9B, 9C, 9E, 9F, 10A, 12A, 12B, 12E, 12H, 13A, 15A, 15C, 18B\}$. For each $nY \notin P$, non-generation of the triple $(3X, 3X, nY)$ follows immediately since the structure constant $\Delta_{Fi_{24}'}(3X, 3X, nY) = 0$. Further, computation reveals that for

DOI: http://dx.doi.org/10.22108/ijgt.2018.109973.1471

each $nY \in P$, we have

$$\Delta_{Fi'_{24}}(3X, 3X, nY) < |C_{Fi'_{24}}(nY)|$$

and by Lemma 2.5, we obtain $\Delta^{\star}_{Fi'_{24}}(3X, 3X, nY) = 0$. Hence, the Fischer group Fi'_{24} is not $(3X, 3X, nY)$-generated for any n. This completes the proof. □

Lemma 3.3. *Let $3X$ denote a conjugacy class of element of order 3 in the Fischer's simple group Fi'_{24}. Then*

(i) $\mathrm{rank}(Fi'_{24} : 3X) = 3$ *if* $X \in \{A, B\}$

(ii) $\mathrm{rank}(Fi'_{24} : 3X) = 2$ *if* $X \in \{C, D, E\}$.

Proof. We treat both cases separately.

(i) From the Lemma 3.2, we know that Fi'_{24} is not $(3X, 3X, tZ)$-generated, where $X \in \{A, B\}$ and $tZ \in Fi'_{24}$. Thus, $\mathrm{rank}(Fi'_{24}{:}3X) > 2$ for $X \in \{A, B\}$. Next, we show that $\mathrm{rank}(Fi'_{24}{:}3X) = 3$ where $X \in \{A, B\}$. Consider the quadruple $(3X, 3X, 3X, 23A)$. As mentioned earlier, up to isomorphisms, Fi_{23} and $2^{11}{:}M_{24}$ are the only maximal subgroups of Fi'_{24} that may admit $(2X, 2X, 2X, 23A)$-generation. However, from the fusion map of $2^{11}{:}M_{24}$ into the group Fi'_{24}, we observe that $2^{11}{:}M_{24}$ does not meet classes $3A$ and $3B$ of Fi'_{24}. In particular,

$$2^{11}{:}M_{24} \cap 3A = \emptyset = 3A \cap 2^{11}{:}M_{24}.$$

So, clearly any proper $(3X, 3X, 3X, 23A)$-generated subgroup of Fi'_{24} must lie in Fi_{23}. By looking at the fusion map of Fi_{23} into Fi'_{24}, we compute $\Delta_{Fi'_{24}}(3A, 3A, 3A, 23A) = 16148783$, $\Delta_{Fi'_{24}}(3B, 3B, 3B, 23A) = 110561539051$, $\Sigma_{Fi_{23}}(3A, 3A, 3A, 23A) = 5484067$, $\Sigma_{Fi_{23}}(3B, 3B, 3B, 23A) = 3766015009$. Since a fixed element of order 23 is contained in a unique conjugate subgroup of Fi_{23}, we have

$$\begin{aligned}
\Delta^{*}_{Fi'_{24}}(3A, 3A, 3A, 23A) &\geq \Delta_{Fi'_{24}}(3A, 3A, 3A, 23A) - \Sigma_{Fi_{23}}(3A, 3A, 3A, 23A) \\
&= 16148783 - 5884067 > 0, \\
\Delta^{*}_{Fi'_{24}}(3B, 3B, 3B, 23A) &\geq \Delta_{Fi'_{24}}(3B, 3B, 3B, 23A) - \Sigma_{Fi_{23}}(3B, 3B, 3B, 23A) \\
&= 110561539051 - 3766015009 > 0,
\end{aligned}$$

proving the generation of Fi'_{24} by the $(3X, 3X, 3X, 23A)$. Thus, $\mathrm{rank}(Fi'_{24} : 3X) = 3$ for $X \in \{A, B\}$ since $\mathrm{rank}(Fi'_{24} : 3X) > 2$.

(ii) Consider the triple $(2B, 3Y, 29A)$, where $Y \in \{C, D, E\}$. We calculate the structure constants

$$\Delta_{Fi'_{24}}(2B, 3C, 29A) = 261, \Delta_{Fi'_{24}}(2B, 3D, 29A) = 47096, \Delta_{Fi'_{24}}(2B, 3E, 29A) = 205001.$$

DOI: http://dx.doi.org/10.22108/ijgt.2018.109973.1471

From the list of maximal subgroups (see ATLAS [8]), we observe that, up to isomorphisms, 29:14 is the only maximal subgroup of Fi_{24}' with order divisible by 29. However, $29{:}14 \cap 3Y = \emptyset$ for $Y \in \{C, D, E\}$. Thus no proper subgroup of Fi_{24}' is $(2B, 3Y, 29A)$-generated and hence

$$\Delta^*_{Fi_{24}'}(2B, 3Y, 29A) = \Delta_{Fi_{24}'}(2B, 3Y, 29A) > 0.$$

It follows that Fi_{24}' is $(2B, 3Y, 29A)$-generated for $Y \in \{C, D, E\}$ and by Lemma 2.3, Fi_{24}' is $(3Y, 3Y, (29A)^2)$-generated where $Y \in \{C, D, E\}$. This concludes that $\text{rank}(Fi_{24}'{:}3Y) = 2$ for $Y \in \{C, D, E\}$. □

Lemma 3.4. *Let nX be a conjugacy class of the Fischer's sporadic simple group Fi_{24}' such that $n \geq 4$, then* $\text{rank}(Fi_{24}' : nX) = 2$.

Proof. Set $T = \{7B, 14B, 29A, 29B\}$. First we consider the triple $(nX, nX, 29A)$ with $nX \notin T$ and $n \geq 4$. By looking at the maximal subgroups of the group Fi_{24}', we observe that the only maximal subgroup that may contain $(nX, nX, 29A)$-generated proper subgroups, up to isomorphisms, is 29:14. However, each Fi_{24}'-class nX in the triple doest not meet the maximal subgroup 29:14. That is, $19{:}14 \cap nX = \emptyset$. Since, the structure constant $\Delta_{Fi_{24}'}(nX, nX, 29A) > 0$ we obtain

$$\Delta^*_{Fi_{24}'}(nX, nX, 29A) = \Delta_{Fi_{24}'}(nX, nX, 29A) > 0,$$

proving generation of Fi_{24}' by the triple $(nX, nX, 29A)$ with $n \geq 4$ and $nX \notin T$.

Next, we consider the triple $(nX, nX, 29A)$ for $nX \in T$. From the fusion map of the maximal subgroup 29:14 into the group Fi_{24}', we obtain that all the classes of order 7 in 29:14 fuses to $7B$-class of Fi_{24}', all the classes of order 14 in 29:14 fuses to $14B$-class of Fi_{24}', and the classes $29A$ and $29B$ in 29:14 fuses to $29A$ and $29B$ classes of Fi_{24}', respectively. Further, we compute $\Delta_{Fi_{24}'}(7B, 7B, 29A) = 296363124652705632$, $\Delta_{Fi_{24}'}(14B, 14B, 29A) = 7115678621348544239968$, $\Delta_{Fi_{24}'}(29A, 29A, 29A) = 1492487887225908548178$, $\Delta_{Fi_{24}'}(7B, 7B, 29A) = 1492487886962906026578$, $\Sigma_{29:14}(7B, 7B, 29A) = 174 = \Sigma_{29:14}(14B, 14B, 29A)$, $\Sigma_{29:14}(29A, 29A, 29A) = 6$ and $\Sigma_{29:14}(29B, 29B, 29A) = 7$. Since a fixed element of order 29 in Fi_{24}' is contained in a unique conjugate class of 29:14, we calculate

$$\begin{aligned}
\Delta^*_{Fi'_{24}}(7B,7B,29A) &\geq \Delta_{Fi'_{24}}(7B,7B,29A) - \Sigma_{29:14}(7B,7B,29A)\\
&= 296363124652705632 - 174 > 0,\\
\Delta^*_{Fi'_{24}}(14B,14B,29A) &\geq \Delta_{Fi'_{24}}(14B,14B,29A) - \Sigma_{29:14}(14B,14B,29A)\\
&= 7115678621348544239968 - 174 > 0,\\
\Delta^*_{Fi'_{24}}(29A,29A,29A) &\geq \Delta_{Fi'_{24}}(29A,29A,29A) - \Sigma_{29:14}(29A,29A,29A)\\
&= 1492487887225908548178 - 6 > 0,\\
\Delta^*_{Fi'_{24}}(29B,29B,29A) &\geq \Delta_{Fi'_{24}}(29B,29B,29A) - \Sigma_{29:14}(29B,29B,29A)\\
&= 1492487886962906026578 - 7 > 0.
\end{aligned}$$

Thus, Fi'_{24} is $(nX, nX, 29A)-$generated for $nX \in T$ and hence $\text{rank}(Fi'_{24}{:}nX) = 2$ for each conjugacy class $nX \in Fi'_{24}$ with $n \geq 4$. This completes the lemma. □

We summarize our results of this section in the following theorem:

Theorem 3.5. *Let Fi'_{24} be the Fischer's largest sproadic simple group and nX be a conjugacy class of Fi'_{24}. Then*

(i) $\text{rank}(Fi'_{24} : nX) = 3$, *if* $nX \in \{2A, 2B, 3A, 3B\}$

(ii) $\text{rank}(Fi'_{24} : nX) = 2$, *if* $nX \notin \{1A, 2A, 2B, 3A, 3B\}$

Proof. The proof follows from the lemmas proved in this section. □

4. **Ranks of the Baby Monster group $\mathbb{B}$**

The baby monster group, also known as Fischer's baby monster group, is the second-largest sporadic group and is denoted by $\mathbb{B}$. The Fischer's baby monster group is a $\{3,4\}$-transposition simple group of order

$$4,154,781,481,226,191,177,580,544,000,000 = 2^{41} \cdot 3^{13} \cdot 5^6 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 31 \cdot 47\cdot$$

It was first discovered by B. Fischer [12], who himself determined much of its properties including $\{3,4\}$-transposition properties. J.S. Leon and C.C. Sims [15] proved its existence and uniqueness by constructed the group $\mathbb{B}$ as a permutation group on 13571955 points. D.C. Hunt [13] was the first to compute its character table. Later, R.A. Wilson constructed it in a new way, studied its subgroup structure, classified much of its maximal subgroups along with its application (see [18, 19]). The character table of the group $\mathbb{B}$ and its all known properties are now available in [8] and [17]. The baby monster group $\mathbb{B}$ has exactly 30 conjugacy classes of maximal subgroups and 184 conjugacy classes of its elements including four classes of involutions $2A$, $2B$, $2C$ and $2D$ as represented in $\mathbb{ATLAS}$ ([8]).

DOI: http://dx.doi.org/10.22108/ijgt.2018.109973.1471

Next, we compute the rank for each conjugacy class of the group $\mathbb{B}$. In the following lemma we show that the baby monster group $\mathbb{B}$ can be generated by three conjugate involutions from its conjugacy class $2A$.

Lemma 4.1. *The Baby Monster sporadic group $\mathbb{B}$ is not $(2A, 2A, 2A, nX)$-generated for any integer n.*

Proof. Let $T_1 = \{2A, 2D, 4A, 4B, 4C, 4D, 4F, 4G, 4H, 4I, 6A, 6B, 6D, 6E, 6H, 8A, 8B, 8C, 8F, 12A, 12B, 12E, 14A\}$. For each $nX \notin T_1$, the non-generation of the group $\mathbb{B}$ by the quadruples $(2A, 2A, 2A, nX)$ follows immediately since the structure constant $\Delta_{\mathbb{B}}(2A, 2A, 2A, nX) = 0$ for each $nX \notin T_1$. Next, suppose $nX \in T_1$ and consider the triple $(2A, 2A, 2A, nX)$. Our computation reveals that for each $nX \in T_1$, we obtain

$$\Delta_{\mathbb{B}}(2A, 2A, 2A, nX) < |C_{\mathbb{B}}(nX)|.$$

Now by an application of lemma 2.5, we have $\Delta^*_{\mathbb{B}}(2A, 2A, 2A, nX) = 0$. Hence the group $\mathbb{B}$ is not $(2A, 2A, 2A, nX)$-generated for any n. This completes the proof. □

Lemma 4.2. $\text{rank}(\mathbb{B} : 2A) = 4$.

Proof. Direct computation shows that $\Delta_{\mathbb{B}}(2A, 2A, 2A, 2A, 47A) = 4049097$. The only maximal subgroups of group $\mathbb{B}$ with order divisible by 47, up to isomorphism, is 47:23. Since the group 47:23 has odd order, it contains no involutions. This proves that $\mathbb{B}$ has no proper subgroup of type $(2A, 2A, 2A, 2A, 47A)$ and so in this case, we have

$$\Delta^*_{\mathbb{B}}(2A, 2A, 2A, 2A, 47A) = \Delta_{\mathbb{B}}(2A, 2A, 2A, 2A, 47A) = 4049097.$$

Thus, the group $\mathbb{B}$ is $(2A, 2A, 2A, 2A, 47A)$-generated and therefore by Corollary 2.2, we have $\text{rank}(\mathbb{B} : 2A) \leq 4$. Since $\text{rank}(\mathbb{B} : 2A) > 2$ (as we argue in Lemma 3.1, that two involutions can not generate a simple group), the result follows by considering the previous lemma. □

Lemma 4.3. *If $nX \in \{2B, 2C, 2D\}$ then $\text{rank}(\mathbb{B} : nX) = 3$.*

Proof. First we prove that the Baby Monster group $\mathbb{B}$ is $(2X, 3Y, 23A)$-generated, for $(X, Y) \in \{(B, B), (C, B), (D, A)\}$. We treat each case separately.

Case $(2B, 3B, 23A)$: The maximal subgroups of the group $\mathbb{B}$ with orders divisible by 23 and having non-empty intersection with the classes in this triple are isomorphic to $2^{1+22}{\cdot}Co_2$ and Fi_{23}. We calculate that $\Delta_{\mathbb{B}}(2B, 3B, 23A) = 92$, $\Sigma_{2^{1+22}\cdot Co_2}(2B, 3B, 23A) = 0$, $\Sigma_{Fi_{23}}(2B, 3B, 23A) = 23$. Further, a fixed element of order 23 in $\mathbb{B}$ is contained in precisely unique conjugate of $2^{1+22}{\cdot}Co_2$ and two conjugate copies of Fi_{23}. Hence

$$\begin{aligned} \Delta^*_{\mathbb{B}}(2B, 3B, 23A) &\geq \Delta_{\mathbb{B}}(2B, 3B, 23A) - 2\Sigma_{Fi_{23}}(2B, 3B, 23A) \\ &= 92 - 2(23) > 0, \end{aligned}$$

proving the generation of $\mathbb{B}$ by the triple $(2B, 3B, 23A)$.

Case $(2C, 3B, 23A)$*:* Amongst the maximal subgroups of $\mathbb{B}$ with order divisible by $2 \times 3 \times 23$, the only maximal subgroup with non-empty intersection with all the classes in this triple is isomorphic to $2^{1+22}{\cdot}Co_2$. We compute

$$\begin{aligned}\Delta^*_{\mathbb{B}}(2C, 3B, 23A) &\geq \Delta_{\mathbb{B}}(2C, 3B, 23A) - \Sigma_{2^{1+22}\cdot Co_2}(2C, 3B, 23A)\\ &= 2503504 - 4416 > 0.\end{aligned}$$

This shows that the group $\mathbb{B}$ is $(2C, 3B, 23A)$-generated.

Case $(2D, 3A, 23A)$*:* We calculate the structure constant $\Delta_{\mathbb{B}}(2D, 3A, 23A) = 598$. The $(2D, 3A, 23A)$-generated proper subgroups of $\mathbb{B}$ are contained in the maximal subgroups isomorphic to $2^{1+22}{\cdot}Co_2$ and Fi_{23}. As mentioned above, a fixed element $z \in \mathbb{B}$ of order 23 is contained in precisely a unique conjugate of $2^{1+22}{\cdot}Co_2$ and two conjugate copies of Fi_{23}. Thus

$$\begin{aligned}\Delta^*_{\mathbb{B}}(2D, 3A, 23A) &\geq \Delta_{\mathbb{B}}(2D, 3A, 23A) - \Sigma_{2^{1+22}\cdot Co_2}(2D, 3A, 23A) - 2\Sigma_{Fi_{23}}(2D, 3A, 23A),\\ &= 598 - 138 - 2(161) > 0,\end{aligned}$$

and generation of $\mathbb{B}$ by the triple $(2B, 3B, 23A)$ follows.

Thus by applying Corollary 2.2 on the triples $(2B, 3B, 23A)$-, $(2C, 3B, 23A)$-, and $(2D, 3A, 23A)$-generations of $\mathbb{B}$, we obtain that $\operatorname{rank}(\mathbb{B} : 2X) = 2$ where $X \in \{B, C, D\}$. This completes the proof. □

Lemma 4.4. *Let nX be a conjugacy class in the Baby Monster group $\mathbb{B}$ such that $nX \notin \{1A, 2A, 2B, 2C, 2D\}$. Then* $\operatorname{rank}(\mathbb{B} : nX) = 2$*.*

Proof. Set $W = \{23A, 23B, 47A, 47B\}$. Consider the triple $(nX, nX, 47A)$ with $nX \notin W$. Amongst the maximal subgroups of the group $\mathbb{B}$ with order divisible by 47, the only maximal subgroup that may admit $(nX, nX, 47A)$-generated proper subgroups is isomorphic to 47:23. However, each non-involution $\mathbb{B}$-class $nX \notin W$ doest not meet the maximal subgroup 47:23. That is, $47{:}23 \cap nX = \emptyset$. Since, the structure constant $\Delta_{\mathbb{B}}(nX, nX, 47A) > 0$ we obtain

$$\Delta^*_{\mathbb{B}}(nX, nX, 47A) = \Delta_{\mathbb{B}}(nX, nX, 47A) > 0.$$

Hence, the group $\mathbb{B}$ is $(nX, nX, 47A)$-generated with $nX \notin W$

Now we consider the triple $(nX, nX, 47A)$ for $nX \in W$. By looking at the fusion map of the maximal subgroup 47:23 into the group $\mathbb{B}$, we observe that $23A, 23B, 23C, 23D, 23F, 23H, 23I, 23L, 23M, 23P, 23R$ classes in maximal subgroup 47:23 fuses to $23A$-class of $\mathbb{B}$, while the classes $23E, 23G, 23J, 23K, 23N,$

DOI: http://dx.doi.org/10.22108/ijgt.2018.109973.1471

$23O, 23Q, 23S, 23T, 23U, 23V$ in 47:23 fuses to $23B$-class of $\mathbb{B}$. The 47:23-classes $47A$ and $47B$ fuses to $47A$ and $47B$ classes of $\mathbb{B}$, respectively. We compute

$$\begin{aligned}\Delta_{\mathbb{B}}(23A, 23A, 47A) &= 1963507316354951733772778733568,\\ \Delta_{\mathbb{B}}(23B, 23B, 47A) &= 1963507316354951733772778733568,\\ \Delta_{\mathbb{B}}(47A, 47A, 47A) &= 1880823140432881168431474063924,\\ \Delta_{\mathbb{B}}(47B, 47B, 47A) &= 1880823141406367549686998153780.\end{aligned}$$

Further, we calculate

$$\Sigma_{47:23}(23A, 23A, 47A) = 0 = \Sigma_{47:23}(23B, 23B, 47A),$$

$\Sigma_{\mathbb{B}}(47A, 47A, 47A) = 11$ and $\Sigma_{47:23}(47B, 47B, 47A) = 12$. Since a fixed element of order 47 in $\mathbb{B}$ is contained in precisely a unique conjugates of the group 47:23, we have

$$\begin{aligned}\Delta^*_{\mathbb{B}}(23A, 23A, 47A) &= \Delta_{\mathbb{B}}(23A, 23A, 47A) - \Sigma_{47:23}(23A, 23A, 47A)\\ &= 1963507316354951733772778733568 - 0 > 0,\\ \Delta^*_{\mathbb{B}}(23B, 23B, 47A) &= \Delta_{\mathbb{B}}(23B, 23B, 47A) - \Sigma_{29:14}(23B, 23B, 47A)\\ &= 1963507316354951733772778733568 - 0 > 0,\\ \Delta^*_{\mathbb{B}}(47A, 47A, 47A) &\geq \Delta_{\mathbb{B}}(47A, 47A, 47A) - \Sigma_{29:14}(47A, 47A, 47A)\\ &= 1880823140432881168431474063924 - 11 > 0,\\ \Delta^*_{\mathbb{B}}(47B, 47B, 47A) &\geq \Delta_{\mathbb{B}}(47B, 47B, 47A) - \Sigma_{29:14}(47B, 47B, 47A)\\ &= 1880823141406367549686998153780 - 12 > 0,\end{aligned}$$

showing that the group $\mathbb{B}$ is $(nX, nX, 47A)$-generated for $nX \in T$ and hence together with the above case, we have $\text{rank}(\mathbb{B} : nX) = 2$ for each conjugacy class of $nX \in \mathbb{B}$ such that $nX \notin \{1A, 2A, 2B, 2C, 2D\}$. This completes the lemma. □

We now summarize the above results of this section in the following theorem.

Theorem 4.5 . *Let $\mathbb{B}$ be the Fischer's Baby Monster group and nX a conjugacy class of elements of $\mathbb{B}$. Then*

(i) $\text{rank}(\mathbb{B} : 2A) = 4$.
(ii) $\text{rank}(\mathbb{B} : nX) = 3$ *if* $nX \in \{2B, 2C, 2D\}$
(iii) $\text{rank}(\mathbb{B} : nX) = 2$ *if* $nX \notin \{1A, 2A, 2B, 2C, 2D\}$

Proof. Proof follows from Lemmas 4.1 to 4.4. □

DOI: http://dx.doi.org/10.22108/ijgt.2018.109973.1471

Acknowledgments

This work was funded by the National Plan for Science, Technology and Innovation (MARRIFAH) - King Abdulaziz City for Science and Technology - Kingdom of Saudi Arabia, award number (13-MAT264-08). The authors would also like to thank the referee for all his comments, corrections and valuable suggestions.

References

[1] F. Ali, On the ranks of Fi_{22}, *Quaest. Math.*, **37** (2014) 591–600.

[2] F. Ali, On the ranks of O'N and Ly, *Discrete Appl. Math.*, **155** (2007) 394–399.

[3] F. Ali and M. A. F. Ibrahim, On the ranks of the Conway groups Co_2 and Co_3, *J. Algebra Appl.*, **4** (2005) 557–565.

[4] F. Ali and M. A. F. Ibrahim, On the ranks of Conway group Co_1, *Proc. Japan Acad. Ser. A Math. Sci.*, **81** (2005) 95–98.

[5] F. Ali and M. A. F. Ibrahim, On the ranks of HS and McL, *Util. Math.*, **70** (2006) 187–195.

[6] F. Ali and M. A. F. Ibrahim, On the simple sporadic group He generated by the $(2, 3, t)$ generators, *Bull. Malays. Math. Sci. Soc. (2)*, **35** (2012) 745–753.

[7] F. Ali and J. Moori, On the ranks of Janko groups J_1, J_2, J_3 and J_4, *Quaestiones Mathematicae*, **31** (2008) 37–44.

[8] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker and R. A. Wilson, *An Atlas of Finite Groups*, Oxford University Press, 1985.

[9] M. D. E. Conder, R. A. Wilson and A. J. Woldar, The symmetric genus of sporadic groups, *Proc. Amer. Math. Soc.*, **116** (1992) 653–663.

[10] L. Di Martino, M. A. Pellegrini and A. E. Zalesski, On generators and representations of the sporadic simple groups, *Comm. Algebra*, **42** (2014) 880–908.

[11] B. Fischer, Finite groups generated by 3-transpositions. I, *Invent. Math.*, **13** (1971) 232–246.

[12] B. Fischer, *A note concerning* $\{3, 4\}$*-transposition groups*, unpublished.

[13] D. C. Hunt, The character table of the Baby Monster, unpublished.

[14] S. Ganief, 2-*Generations of the sporadic simple groups*, PhD Thesis, University of Natal 1997.

[15] J. S. Leon and C. C. Sims, The existence and uniqueness of a simple group generated by $\{3, 4\}$-transpositions, *Bull. Amer. Math. Soc.*, **87** (1977) 1039–1040.

[16] S. A. Linton and R. A. Wilson, The maximal subgroups of the Fischer groups Fi_{24} and Fi'_{24}, *Proc. London Math. Soc.*, **63** (1991) 113–164.

[17] The GAP Group, $\mathbb{GAP}$ - *Groups, Algorithms and Programming, Version 4.3*, Aachen, St Andrews, 2003, (http://www.gap-system.org).

[18] R. A. Wilson, Some subgroups of the Baby Monster, *Invent. Math.*, **89** (1987) 197–218.

[19] R. A. Wilson, A new construction of the Baby Monster, and its applications, *Bull. London Math. Soc.*, **25** (1993) 431–437.

Mohammed Ali Faya Ibrahim

Department of Mathematics, College of Arts ans Science, Najran University, Najran, Saudi Arabia

Email: mfaya2000@gmail.com

DOI: http://dx.doi.org/10.22108/ijgt.2018.109973.1471

Faryad Ali
Department of Mathematics and Statistics, College of Science, Al Imam Mohammad Ibn Saud Islamic University, Riyadh 11623, Saudi Arabia
Email: `faryada@hotmail.com`

Mohammed A. Al-Kadhi
Department of Mathematics and Statistics, College of Science, Al Imam Mohammad Ibn Saud Islamic University, Riyadh 11623, Saudi Arabia
Email: `malki2@gmail.com`

Abdullah Aljouiee
Department of Mathematics and Statistics, College of Science, Al Imam Mohammad Ibn Saud Islamic University, Riyadh 11623, Saudi Arabia
Email: `joa111@gmail.com`

DOI: http://dx.doi.org/10.22108/ijgt.2018.109973.1471