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RECOGNITION OF JANKO GROUPS AND SOME SIMPLE K_4-GROUPS BY THE ORDER AND ONE IRREDUCIBLE CHARACTER DEGREE OR CHARACTER DEGREE GRAPH

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Abstract. In this paper we prove that some Janko groups are uniquely determined by their orders and one irreducible character degree. Also we prove that some finite simple K_4-groups are uniquely determined by their character degree graphs and their orders.

1. Introduction

Let G be a finite group, $\mathrm{Irr}(G)$ be the set of complex irreducible characters of G, and denote by $\mathrm{cd}(G) = \{\chi(1) : \chi \in \mathrm{Irr}(G)\}$ the set of irreducible character degrees of G. Classifying finite groups by the properties of their characters is an interesting problem in representation theory. In [3], Huppert conjectured that each finite non-abelian simple group G is characterized by the set $\mathrm{cd}(G)$. In [3, 4, 13, 15], it was shown that the conjecture holds for simple groups such as $L_2(q)$ and $\mathrm{Sz}(q)$. In this paper, we attempt to characterize the Janko groups J_1, J_3 and J_4 by their orders and one irreducible character degree. Also authors guess that the result is not correct for Janko group J_2, but they have not found any counterexample yet. Let G be a finite group; $L(G)$ denotes the largest irreducible character degree of G. The following result is our main theorem in the third section.

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Theorem 1.1. *Let G be a finite group, and S be one of the Janko groups, J_1, J_2 and J_3. Then $G \cong S$ if and only if the following conditions hold:*

(i) $|G| = |S|$;
(ii) $L(G) = L(S)$.

The character degree graph $\Gamma(G)$ associated to a finite group G is a graph whose vertex set is the prime divisors of irreducible character degrees of G and there is an edge between two distinct vertices p and q if and only if pq divides some irreducible character degree of G. This graph was introduced in [10] and studied by many authors (see [9, 14]).

A finite group G is called a K_n-group if $|G|$ has exactly n distinct prime divisors. Recently Xu et al. in [16] proved that K_3-groups are uniquely determined by their orders and one or both of their largest and second largest irreducible character degrees. Khosravi et al. [7] proved that the group $L_2(p^2)$, where p is a prime, is characterizable by its character degree graph and its order. Khosravi et al. [6] invested the influence of the character degree graph and order of the simple groups of order less than 6000, on the structure of group. Some simple K_4-groups are invested by Khosravi et al. in [8]. In this paper we invest some other simple K_4-groups which are dropped in their list. In the last section we prove these groups can be uniquely determined by their orders and character degree graph.

Theorem 1.2. *Let G be a finite group, and let S be one of the K_4-groups $L_2(23)$, $L_2(25)$, $L_2(47)$, $L_2(81)$, $U_3(7)$, $U_3(8)$, $\mathrm{Sz}(8)$, $\mathrm{Sz}(32)$ and $U_3(9)$. Then $G \cong S$ if and only if the following conditions hold:*

(i) $|G| = |S|$;
(ii) $\Gamma(G) = \Gamma(S)$.

2. Preliminaries

In this section, we present some results that will be needed for the proofs of our theorems.

Proposition 2.1. ((Ito's Theorem)[5, Corollary 6.15]) *Let $A \trianglelefteq G$ be abelian. Then $\chi(1)$ divides $|G : A|$ for all $\chi \in \mathrm{Irr}(G)$*

Let $N \trianglelefteq G$ and $\theta \in \mathrm{Irr}(N)$. Then $I_G(\theta) = \{g \in G \mid \theta^g = \theta\}$ is the inertia group of θ in G.

Proposition 2.2. [5, Theorems 6.2, 6.8, 11.29] *Let $N \trianglelefteq G$ and let $\chi \in \mathrm{Irr(G)}$. Let θ be an irreducible constituent of χ_N and suppose that $\theta_1 = \theta, \dots, \theta_t$ are the distinct conjugates of θ in G. Then $\chi_N = e\sum_{i=1}^{t} \theta_i$, where $e = [\chi_N, \theta]$ and $t = |G : I_G(\theta)|$. Also $\theta(1) \mid \chi(1)$ and $\chi(1)/\theta(1) \mid |G : N|$.*

Proposition 2.3. [16, Lemma] *Let G be a nonsolvable group. Then G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K/H is a direct product of isomorphic nonabelian simple groups and $|G/K| \mid |\mathrm{Out}(K/H)|$.*

Let G be a group and let $\pi \subseteq \rho(G)$, where $\rho(G)$ is the set of all prime divisors of irreducible character degrees of G. For solvable groups, Pálfy [12] showed that there is always an edge between

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two primes in π whenever $|\pi| \geq 3$. For arbitrary groups, Moretó and Tiep [11] proved that a similar conclusion holds provided that $|\pi| \geq 4$. We summarize their results in the following proposition.

Proposition 2.4. *Let G be a group and $\pi \subseteq \rho(G)$.*

- **(i):** ([12, Theorem]) *If G is solvable and $|\pi| \geq 3$, then there exist primes $p, q \in \pi$ and $\chi \in \mathrm{Irr}(G)$ such that $pq \mid \chi(1)$.*
- **(ii):** ([11, Main Theorem]) *If $|\pi| \geq 4$, then there exists $\chi \in \mathrm{Irr}(G)$ such that $\chi(1)$ is divisible by two distinct primes in π.*

Proposition 2.5. [17, Lemma 2] *Let G be a finite solvable group of order $p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_n}$, where $p_1, p_2, \ldots, p_n$ are distinct primes. If $(kp_n + 1) \nmid p_i^{\alpha_i}$, for each $i \leq n-1$ and $k > 0$, then the Sylow p_n-subgroup is normal in G.*

3. A characterization of Janko groups J_1, J_3 and J_4

In this section we attempt to characterize the Janko groups J_1, J_3 and J_4 by their orders and the largest irreducible character degree.

Lemma 3.1. *Let G be a finite group. Then $G \cong J_1$ if and only if $|G| = |J_1|$ and $L(G) = L(J_1)$.*

Proof. Let G be a finite group such that $|G| = |J_1|$ and $L(G) = L(J_1)$. We first assert that G is nonsolvable. Let G be a solvable group of order $2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19$ and let $\chi \in \mathrm{Irr}(G)$ be such that $\chi(1) = L(J_1) = 209$. Let H be a Hall subgroup of G of order $2^3 \cdot 3 \cdot 5 \cdot 11 \cdot 19$ and $H_G = \bigcap_{g \in G} H^g \leq H$. Then $G/H_G \hookrightarrow S_7$. By checking the Atlas [2], we see that the orders of solvable subgroups of S_7 which are divisible by 7 are $7, 14, 21$ and 42. Thus $|H_G|$ is one of $2^3 \cdot 3 \cdot 5 \cdot 11 \cdot 19$, $2^2 \cdot 3 \cdot 5 \cdot 11 \cdot 19$, $2^3 \cdot 5 \cdot 11 \cdot 19$, $2^2 \cdot 5 \cdot 11 \cdot 19$. Let $\theta \in \mathrm{Irr}(H_G)$ be such that $[\chi_{H_G}, \theta] \neq 0$. If $|H_G| = 2^3 \cdot 3 \cdot 5 \cdot 11 \cdot 19$, then $|G/H_G| = 7$. Since $\chi(1)/\theta(1) \mid |G : H_G| = 7$, it follow that $\theta(1) = 209$, but $43681 = \theta(1)^2 < 25080 = |H_G|$, a contradiction. For the rest possibilities of $|H_G|$, we can conclude contradiction by the same argument. Hence G is nonsolvable and thus by Proposition 2.3, G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K/H is a direct product of isomorphic nonabelian simple groups and $|G/K| \mid |\mathrm{Out}(K/H)|$. By the classification of finite simple groups in [2], it follows that K/H is isomorphic to A_5, $L_2(11)$, $L_2(7)$ or J_1. If $K/H \cong A_5$, then since $|\mathrm{Out}(A_5)| = 2$, it follows that $|G/K| \mid 2$. Thus $|H| = 2 \cdot 7 \cdot 11 \cdot 19$ or $|H| = 7 \cdot 11 \cdot 19$. Let $\theta \in \mathrm{Irr}(H)$ be such that $[\chi_H, \theta] \neq 0$. Since $\chi(1)/\theta(1) \mid |G : H|$, we have $\theta(1) = 209$ and either $43681 = \theta(1)^2 < 2926$ or $43681 \leq \theta(1)^2 < 1463$, a contradiction. Secondly, if $K/H \cong L_2(7)$, then since $|\mathrm{Out}(L_2(7))| = 2$, it follows that $|G/K| \mid 2$. Thus $|H| = 5 \cdot 11 \cdot 19$. Therefore H is solvable and by Proposition 2.5, $\mathbf{O}_{19}(G) \neq 1$. Now by Proposition 2.1, we get a contradiction. if $K/H \cong L_2(11)$, then since $|\mathrm{Out}(L_2(11))| = 2$, it follows that $|G/K| \mid 2$. Thus either $|H| = 2 \cdot 7 \cdot 19$ or $|H| = 7 \cdot 19$. Therefore H is solvable and by Proposition 2.5, $\mathbf{O}_{19}(G) \neq 1$. Now by Proposition 2.1, we get a contradiction. Thus $K/H \cong J_1$ and so $|H| = 1$. Therefore $G = K$ and so $G \cong J_1$. □

Lemma 3.2. *Let G be a finite group. Then $G \cong J_3$ if and only if $|G| = |J_3|$ and $L(G) = L(J_3)$.*

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Proof. Let G be a finite group such that $|G| = |J_3|$ and $L(G) = L(J_3)$. We first show that G is nonsolvable. Let G be a solvable group of order $2^7 \cdot 3^5 \cdot 5 \cdot 17 \cdot 19$ and $\chi \in \mathrm{Irr}(G)$ be such that $\chi(1) = L(J_3) = 2 \cdot 3^4 \cdot 19$. By Proposition 2.5, we see that the Sylow 19-subgroup of G is normal in G. Let N be the Sylow 19-subgroup of G. Then by Proposition 2.1, $\chi(1) \mid |G/N|$, a contradiction. Hence G is nonsolvable and by Proposition 2.3, G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K/H is a direct product of isomorphic nonabelian simple groups and $|G/K| \mid |\mathrm{Out}(K/H)|$. By the classification of finite simple groups in [2], it follows that K/H is isomorphic to A_5, A_6, $U_4(2)$, $L_2(17)$, $L_2(19)$, $L_2(16)$ or J_3. If $K/H \cong A_5$, then since $|\mathrm{Out}(A_5)| = 2$, it follows that $|G/K| \mid 2$. Thus $|H| = 2^t \cdot 3^4 \cdot 17 \cdot 19$, where $t = 4$ or $t = 5$. Let $\theta \in \mathrm{Irr}(H)$ be such that $[\chi_H, \theta] \neq 0$. By Proposition 2.2, we have $3^3 \cdot 19 \mid \theta(1)$. If H is nonsolvable, then by Proposition 2.3, H has a normal series $1 \trianglelefteq A \trianglelefteq B \trianglelefteq H$ such that B/A is a direct product of isomorphic nonabelian simple groups and $|H/B| \mid |\mathrm{Out}(B/A)|$. By [2], it follows that $B/A \cong L_2(17)$. Thus either $|A| = 3^2 \cdot 19$ or $|A| = 2 \cdot 3^2 \cdot 19$. Let N be a Sylow 19-subgroup of A. By Proposition 2.5, we have $N \trianglelefteq A$. But Ito's theorem yields that $\theta(1) \mid |A/N|$, a contradiction. Also if H is solvable, then by Proposition 2.5, $\mathbf{O}_{19}(G) \neq 1$, a contradiction. If $K/H \cong A_6$, then since $|\mathrm{Out}(A_6)| = 4$, it follows that $|G/K| \mid 4$. Thus $|H| = 2^t \cdot 3^3 \cdot 17 \cdot 19$, where $t \in \{2, 3, 4\}$. If either $|H| = 2^2 \cdot 3^3 \cdot 17 \cdot 19$ or $|H| = 2^3 \cdot 3^3 \cdot 17 \cdot 19$, then H is solvable and by Proposition 2.5, $\mathbf{O}_{19}(G) \neq 1$, a contradiction. Thus we may suppose that $|H| = 2^4 \cdot 3^3 \cdot 17 \cdot 19$. Let $\theta \in \mathrm{Irr}(H)$ be such that $[\chi_H, \theta] \neq 0$. By Proposition 2.2, we have $3^2 \cdot 19 \mid \theta(1)$. If H is nonsolvable, then by Proposition 2.3, H has a normal series $1 \trianglelefteq A \trianglelefteq B \trianglelefteq H$ such that B/A is a direct product of isomorphic nonabelian simple groups and $|H/B| \mid |\mathrm{Out}(B/A)|$. By [2], it follows that $B/A \cong L_2(17)$. Thus $|B| = 2^4 \cdot 3^3 \cdot 17 \cdot 19$ and so $|A| = 3 \cdot 19$. Let N be a Sylow 19-subgroup of A. Clearly $N \trianglelefteq A$ and hence $\theta(1) \mid |A/N|$, a contradiction. Also if H is solvable, then by Proposition 2.5, $\mathbf{O}_{19}(G) \neq 1$, a contradiction. If $K/H \cong U_4(2)$, then since $|\mathrm{Out}(U_4(2))| = 2$, it follows that $|G/K| \mid 2$. Thus $|H| = 2^t \cdot 3 \cdot 17 \cdot 19$, where $t \in \{0, 1\}$. Thus H is solvable and by Proposition 2.5, $\mathbf{O}_{19}(G) \neq 1$, a contradiction. If $K/H \cong L_2(17)$, then since $|\mathrm{Out}(L_2(17))| = 2$, it follows that $|G/K| \mid 2$. Thus $|H| = 2^t \cdot 3^3 \cdot 5 \cdot 19$, where $t \in \{2, 3\}$. Let $\theta \in \mathrm{Irr}(H)$ be such that $[\chi_H, \theta] \neq 0$. Since $\chi(1)/\theta(1) \mid |G : H|$, it follow that $3^2 \cdot 19 \mid \theta(1)$. If H is solvable, then $\mathbf{O}_{19}(G) \neq 1$, a contradiction. Thus H is nonsolvable and so H has a normal series $1 \trianglelefteq A \trianglelefteq B \trianglelefteq H$ such that B/A is a direct product of isomorphic nonabelian simple groups and $|H/B| \mid |\mathrm{Out}(B/A)|$. First suppose that $|H| = 2^3 \cdot 3^3 \cdot 5 \cdot 19$. By the classification of finite simple groups, it follows that $B/A \cong A_5$, A_6 or $L_2(19)$. If $B/A \cong A_5$, then either $|A| = 2 \cdot 3^2 \cdot 19$ or $|A| = 3^2 \cdot 19$. Therefore A is solvable and so we can see that $\mathbf{O}_{19}(G) \neq 1$, a contradiction. Also if $B/A \cong A_6$, then we can see that $|A| = 3 \cdot 19$. Therefore $\mathbf{O}_{19}(G) \neq 1$, a contradiction. If $B/A \cong L_2(19)$, then either $|A| = 2 \cdot 3$ or $|A| = 3$. Thus A has a normal subgroup of order 3, say M. Suppose $\beta \in \mathrm{Irr}(M)$ be such that $[\theta_M, \beta] \neq 0$. Then $3^2 \cdot 19 \mid \theta(1) = e \cdot t \cdot \beta(1)$. Observe that $\mathbf{C}_H(M) \leq I_H(\beta)$, so

$$t = |H : I_H(\beta)| \leq |H : \mathbf{C}_H(M)| \leq |\,\mathrm{Aut}(M)| = 2.$$

Therefore $3^2 \cdot 19 \mid e$ and thus $3^4 \cdot 19^2 \leq |H : M| = 2^2 \cdot 3^2 \cdot 5 \cdot 19$, a contradiction. Now suppose that $|H| = 2^2 \cdot 3^3 \cdot 5 \cdot 19$. Thus $B/A \cong A_5$ or $L_2(19)$. Now by the same argument as above, we have a contradiction.

If $K/H \cong L_2(19)$, then since $|\mathrm{Out}(L_2(19))| = 2$, it follows that $|G/K| \mid 2$. Thus either $|H| = 2^5 \cdot 3^3 \cdot 17$ or $|H| = 2^4 \cdot 3^3 \cdot 17$. First suppose that $|H| = 2^5 \cdot 3^3 \cdot 17$. Let $\theta \in \mathrm{Irr}(H)$ be such that $[\chi_H, \theta] \neq 0$. Since $\chi(1)/\theta(1) \mid |G : H| = 2^2 \cdot 3^2 \cdot 5 \cdot 19$, it follows that $3^2 \mid \theta(1)$. If H is solvable, then $\mathbf{O}_{17}(G) \neq 1$. Thus H is nonsolvable and so H has a normal series $1 \trianglelefteq A \trianglelefteq B \trianglelefteq H$ such that B/A is a direct product of isomorphic nonabelian simple groups and $|H/B| \mid |\mathrm{Out}(B/A)|$. By the classification of finite simple groups, we have $B/A \cong L_2(17)$. Thus $|H/B| \mid 2$. Thus either $|A| = 2 \cdot 3$ or $|A| = 3$. In each case, A has a normal subgroup of order 3, say M. Thus $|H/M| = 2^5 \cdot 3^2 \cdot 17$. Suppose that H/M is nonsolvable. By checking the finite simple K_5-groups, we see that H/M has a normal subgroup of order 2, say T/M. Also $H/M/T/M \cong L_2(17)$. Therefore $H/M = T/M \times L_2(17)$, and hence $\mathrm{cd}(H/M) = \{1, 9, 16, 17, 18\}$. But $2^5 \cdot 3^2 \cdot 17 \neq 1 + 9^2 + 16^2 + 17^2 + 18^2$, a contradiction. Now suppose that $|H| = 2^4 \cdot 3^3 \cdot 17$. If H is solvable, then $\mathbf{O}_{17}(G) \neq 1$, a contradiction.

Thus H is nonsolvable and H has a normal series $1 \trianglelefteq A \trianglelefteq B \trianglelefteq H$ such that B/A is a direct product of isomorphic nonabelian simple groups and $|H/B| \mid |\mathrm{Out}(B/A)|$. By [2], we see that $B/A \cong L_2(7)$. Thus $|H/B| \mid 2$. It follows that $|A| = 3^2$ and hence $|H/A| = 2^4 \cdot 3^2 \cdot 17$. Now by [2], we can see that H/A is solvable and hence H is solvable, a contradiction.

Finally if $K/H \cong L_2(16)$, then since $|\mathrm{Out}(L_2(16))| = 4$, it follows that $|G/K| \mid 4$. Thus $|H| = 2^t \cdot 3^4 \cdot 19$, where $t \in \{1, 2, 3\}$. Let $\theta \in \mathrm{Irr}(H)$ be such that $[\chi_H, \theta] \neq 0$. First suppose that $|H| = 2^3 \cdot 3^4 \cdot 19$. Since $\chi(1)/\theta(1) \mid |G : H| = 2^4 \cdot 3 \cdot 5 \cdot 17$. It follows that $3^3 \cdot 19 \mid \theta(1)$. By [2], we can see that H is a solvable group. By Proposition 2.5, we have $\mathbf{O}_{19}(G) \neq 1$, a contradiction. Now suppose that either $|H| = 2 \cdot 3^4 \cdot 19$ or $|H| = 2^2 \cdot 3^4 \cdot 19$. Again by [2], we can see that H is a solvable group. By Proposition 2.5, we have $\mathbf{O}_{19}(G) \neq 1$, a contradiction. Therefore $K/H \cong J_3$ and so $G \cong J_3$. □

Lemma 3.3. *Let G be a finite group. Then $G \cong J_4$ if and only if $|G| = |J_4|$ and $L(G) = L(J_4)$.*

Proof. Let G be a finite group such that $|G| = |J_4|$ and $L(G) = L(J_4)$. We first show that G is nonsolvable. Let G be a solvable group and let $\chi \in \mathrm{Irr}(G)$ be such that $\chi(1) = 3 \cdot 11^3 \cdot 23 \cdot 29 \cdot 31 \cdot 37$. By Proposition 2.5, we see that the Sylow 37-subgroup of G is normal in G, Say N. Thus by Proposition 2.1, $\chi(1) \mid |G/N|$, a contradiction. Hence, G is nonsolvable and G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K/H is a direct product of isomorphic nonabelian simple groups and $|G/K| \mid |\mathrm{Out}(K/H)|$. By [2], we see that K/H is isomorphic to one of the following groups: $A_5, A_6, L_2(7), L_2(8), A_7, U_3(3), M_{11}, A_8, L_3(4), L_2(11), L_2(23), L_2(29), L_2(31), L_2(32), M_{12}, M_{22}, L_5(2), M_{23}, U_3(11), M_{24}, L_2(43)$ and J_4. If $K/H \cong A_5$, then since $|\mathrm{Out}(A_5)| = 2$, it follows that $|G/K| \mid 2$. Thus either $|H| = 2^{19} \cdot 3^2 \cdot 7 \cdot 11^3 \cdot 23 \cdot 29 \cdot 31 \cdot 37 \cdot 43$ or $|H| = 2^{18} \cdot 3^2 \cdot 7 \cdot 11^3 \cdot 23 \cdot 29 \cdot 31 \cdot 37 \cdot 43$. First suppose that $|H| = 2^{19} \cdot 3^2 \cdot 7 \cdot 11^3 \cdot 23 \cdot 29 \cdot 31 \cdot 37 \cdot 43$. Let $\theta \in \mathrm{Irr}(H)$ be such that $[\chi_H, \theta] \neq 0$. By Proposition 2.2, we have $11^3 \cdot 23 \cdot 29 \cdot 31 \cdot 37 \mid \theta(1)$. If H is solvable, then $\mathbf{O}_{37}(G) \neq 1$, a contradiction. Thus H is nonsolvable and so H has a normal series $1 \trianglelefteq A \trianglelefteq B \trianglelefteq H$ such that B/A is a direct product of isomorphic nonabelian simple groups and $|H/B| \mid |\mathrm{Out}(B/A)|$. By the classification of finite simple groups, B/A is isomorphic to one of the following groups:

$$L_2(8), L_2(23), L_2(32), L_2(43), L_2(7).$$

If $B/A \cong L_2(7)$, then either $|A| = 2^{16} \cdot 3 \cdot 11^3 \cdot 23 \cdot 29 \cdot 31 \cdot 37 \cdot 43$ or $|A| = 2^{15} \cdot 3 \cdot 11^3 \cdot 23 \cdot 29 \cdot 31 \cdot 37 \cdot 43$. First let $|A| = 2^{16} \cdot 3 \cdot 11^3 \cdot 23 \cdot 29 \cdot 31 \cdot 37 \cdot 43$. If A is solvable, then $\mathbf{O}_{37}(A) \neq 1$, a contradiction. Thus A is nonsolvable and A has a normal series $1 \trianglelefteq N \trianglelefteq M \trianglelefteq A$ such that M/N is a direct product of isomorphic nonabelian simple groups and $|A/M| \mid |\mathrm{Out}(M/N)|$. By [2], we see that $M/N \cong L_2(23)$. Thus $|A/M| \mid 2$ and so either $|N| = 2^{13} \cdot 11^2 \cdot 29 \cdot 31 \cdot 37 \cdot 43$ or $|N| = 2^{12} \cdot 11^2 \cdot 29 \cdot 31 \cdot 37 \cdot 43$. By checking the simple groups, we see that N is solvable group and hence $\mathbf{O}_{37}(G) \neq 1$, a contradiction. If $M/N \cong L_2(32)$, then $|N| = 2^{11} \cdot 11^2 \cdot 23 \cdot 29 \cdot 37 \cdot 43$. By checking the simple groups, we see that N is solvable and so $\mathbf{O}_{37}(N) \neq 1$, a contradiction. When B/A is isomorphic to other cases, by similar arguments we get a contradiction. Also when K/H is isomorphic to other cases except J_4, by similar arguments we get a contradiction. Therefore $K/H \cong J_4$ and so $G \cong J_4$. □

4. A characterization of simple K_4 groups

In this section we invest some K_4-groups. We prove that these groups can be uniquely determined by their character degree graphs and orders.

Lemma 4.1. *Let G be a finite group of order $2^3 \cdot 3 \cdot 11 \cdot 23$ such that $\Gamma(G) = \Gamma(L_2(23))$. Then $G \cong L_2(23)$.*

Proof. By [2], we know that $\mathrm{cd}(L_2(23)) = \{1, 11, 22, 23, 24\}$. Thus $\Gamma(G)$ has two connected components. By Pálfy's theorem, G is nonsolvable. Thus by Proposition 2.3, G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K/H is a direct product of isomorphic nonabelian simple groups and $|G/K| \mid |\mathrm{Out}(K/H)|$. By the classification of finite simple groups and [2], it follows that K/H is isomorphic to $L_2(23)$. Thus $H = 1$ and hence $G \cong L_2(23)$. □

Lemma 4.2. *Let G be a group of order $2^3 \cdot 3 \cdot 5^2 \cdot 13$ such that $\Gamma(G) = \Gamma(L_2(25))$. Then $G \cong L_2(25)$.*

Proof. By [2], we know that $\mathrm{cd}(L_2(25)) = \{1, 13, 24, 25, 26\}$. Therefore $\Gamma(G)$ has two connected components. By Pálfy's theorem, G is nonsolvable. So G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K/H is a direct product of isomorphic nonabelian simple groups and $|G/K| \mid |\mathrm{Out}(K/H)|$. By the classification of finite simple groups and [2], it follows that K/H is isomorphic to either A_5 or $L_2(25)$. If $K/H \cong A_5$, then $|\mathrm{Out}(A_5)| = 2$. If $|G/K| = 1$, then $|K| = 2^3 \cdot 3 \cdot 5^2 \cdot 13$. Thus $|H| = 2 \cdot 5 \cdot 13$ and hence H is a solvable group. Now by Proposition 2.5, the Sylow 13-subgroup of G is normal and thus $\mathbf{O}_{13}(G) \neq 1$. Hence by Proposition 2.1, we get a contradiction. If $|G/K| = 2$, then $|K| = 2^2 \cdot 3 \cdot 5^2 \cdot 13$. Thus $|H| = 5 \cdot 13$. So $\mathbf{O}_{13}(G) \neq 1$, a contradiction. Thus K/H is isomorphic to $L_2(25)$. Therefore $H = 1$ and hence $G \cong L_2(25)$. □

Lemma 4.3. *Let G be a group of order $2^4 \cdot 3 \cdot 23 \cdot 47$ such that $\Gamma(G) = \Gamma(L_2(47))$. Then $G \cong L_2(47)$.*

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Proof. By [2], we know that $\mathrm{cd}(L_2(47)) = \{1, 23, 46, 47, 48\}$. Therefore $\Gamma(G)$ has two connected components. Also 47 is an isolated vertex of $\Gamma(G)$. Thus by Pálfy's theorem, G is a nonsolvable group and so G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K/H is a direct product of isomorphic nonabelian simple groups and $|G/K| \mid |\mathrm{Out}(K/H)|$. By the classification of finite simple groups and [2], it follows that K/H is isomorphic to $L_2(47)$. Thus $H = 1$ and hence $G \cong L_2(47)$. □

Lemma 4.4. *Let G be a finite group of order $2^4 \cdot 3^4 \cdot 5 \cdot 41$ such that $\Gamma(G) = \Gamma(L_2(81))$. Then $G \cong L_2(81)$.*

Proof. We see that $\Gamma(G)$ has vertex set $\{2, 3, 5, 41\}$. Also, we know that $\{2, 3\}$, $\{3, 41\}$, $\{5, 41\} \notin E(\Gamma(G))$, where $E(\Gamma(G))$ is the set of edges of $\Gamma(G)$. Let G be a solvable group. By Proposition 2.5, the Sylow 41-subgroup of G is normal and hence $\mathbf{O}_{41}(G) \neq 1$. Now by Ito's theorem we get a contradiction. Therefore G is nonsolvable and so G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K/H is a direct product of isomorphic nonabelian simple groups and $|G/K| \mid |\mathrm{Out}(K/H)|$. By the classification of finite simple groups and [2], it follows that K/H is isomorphic to A_5, A_6 or $L_2(81)$. If $K/H \cong A_5$, then $|\mathrm{Out}(A_5)| = 2$. If $|G/K| = 1$, then $|K| = 2^4 \cdot 3^4 \cdot 5 \cdot 41$. Thus $|H| = 2^2 \cdot 3^3 \cdot 41$. By checking the simple groups in [2], we can see that H is solvable. Thus by Proposition 2.5, $\mathbf{O}_{41}(G) \neq 1$, a contradiction. If $|G/K| = 2$, then $|K| = 2^3 \cdot 3^4 \cdot 5 \cdot 41$. Thus $|H| = 2 \cdot 3^3 \cdot 41$. By checking the simple groups in [2], we can see that H is solvable. Thus as above we have a contradiction. If $K/H \cong A_6$, then $|\mathrm{Out}(A_6)| = 4$. If $|G/K| = 1$, then $|K| = 2^4 \cdot 3^4 \cdot 5 \cdot 41$. Thus $|H| = 2 \cdot 3^2 \cdot 41$. Therefore H is solvable. By Proposition 2.5, $\mathbf{O}_{41}(G) \neq 1$, a contradiction. Also, if $|G/K| = 2$, then $|K| = 2^3 \cdot 3^4 \cdot 5 \cdot 41$. Thus $|H| = 3^2 \cdot 41$ and so H is solvable. Now by Proposition 2.5, $\mathbf{O}_{41}(G) \neq 1$, a contradiction. Also we can see that $|G/K| \neq 4$. Finally if $K/H \cong \mathrm{L}_2(81)$, then $H = 1$ and thus $G \cong L_2(81)$. □

Lemma 4.5. *Let G be a finite group of order $2^7 \cdot 3 \cdot 7^3 \cdot 43$ such that $\Gamma(G) = \Gamma(U_3(7))$. Then $G \cong U_3(7)$.*

Proof. We see that $\Gamma(G)$ has vertex set $\{2, 3, 7, 43\}$. Also, we know that $\Gamma(G)$ is a complete graph. If G is solvable, then by Proposition 2.5, the Sylow 43-subgroup of G is normal, a contradiction. Therefore G is nonsolvable and so G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K/H is a direct product of isomorphic nonabelian simple groups and $|G/K| \mid |\mathrm{Out}(K/H)|$. By the classification of finite simple groups and [2], it follows that K/H is isomorphic to either $L_2(7)$ or $U_3(7)$. If $K/H \cong L_2(7)$, then $|\mathrm{Out}(L_2(7))| = 2$. Thus either $|H| = 2^4 \cdot 7^2 \cdot 43$ or $|H| = 2^3 \cdot 7^2 \cdot 43$. Therefore H is solvable and by Proposition 2.5, H has normal Sylow 43-subgroup, a contradiction. Thus $K/H \cong U_3(7)$ and so $H = 1$. Therefore $G \cong U_3(7)$. □

Lemma 4.6. *Let G be a finite group of order $2^9 \cdot 3^4 \cdot 7 \cdot 19$ such that $\Gamma(G) = \Gamma(U_3(8))$. Then $G \cong U_3(8)$.*

Proof. We see that $\Gamma(G)$ has vertex set $\{2, 3, 7, 19\}$. Also, we know that $\Gamma(G)$ is a complete graph. Let G be a solvable group. By Proposition 2.5, the Sylow 19-subgroup of G is normal and hence $\mathbf{O}_{19}(G) \neq 1$. But by Ito's theorem, we get a contradiction. Therefore G is nonsolvable and so G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K/H is a direct product of isomorphic nonabelian simple

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groups and $|G/K| \mid |\mathrm{Out}(K/H)|$. By the classification of finite simple groups and [2], it follows that K/H is isomorphic to $L_2(7)$, $L_2(8)$, $U_3(3)$ or $U_3(8)$. If $K/H \cong L_2(7)$, then $|\mathrm{Out}(L_2(7))| = 2$. If $|G/K| = 1$, then $|K| = 2^9 \cdot 3^4 \cdot 7 \cdot 19$. Thus $|H| = 2^6 \cdot 3^3 \cdot 19$. By checking the simple groups in [2], we can see that H is solvable. Now by Proposition 2.5, $\mathbf{O}_{19}(G) \neq 1$, a contradiction. If $|G/K| = 2$, then $|K| = 2^8 \cdot 3^4 \cdot 7 \cdot 19$. Thus $|H| = 2^5 \cdot 3^3 \cdot 19$. By checking the simple groups in [2], we can see that H is solvable. Thus as above, we have a contradiction. If $K/H \cong L_2(8)$, then $|\mathrm{Out}(L_2(8))| = 3$. If $|G/K| = 1$, then $|K| = 2^9 \cdot 3^4 \cdot 7 \cdot 19$. Thus $|H| = 2^6 \cdot 3^2 \cdot 19$. Therefore H is solvable. Now by Proposition 2.5, $\mathbf{O}_{19}(G) \neq 1$, a contradiction. Also if $|G/K| = 3$, then $|K| = 2^9 \cdot 3^3 \cdot 7 \cdot 19$. Thus $|H| = 2^6 \cdot 3 \cdot 19$. So H is solvable. Now by Proposition 2.5, $\mathbf{O}_{19}(G) \neq 1$, a contradiction. If $K/H \cong U_3(3)$, then either $|H| = 2^3 \cdot 3 \cdot 19$ or $|H| = 2^4 \cdot 3 \cdot 19$. Now by a similar argument as above, we can get a contradiction. Finally if $K/H \cong U_3(8)$, then $H = 1$ and so $G \cong U_3(8)$. □

Lemma 4.7. *Let G be a finite group of order $2^6 \cdot 5 \cdot 7 \cdot 13$ such that $\Gamma(G) = \Gamma(\mathrm{Sz}(8))$. Then $G \cong \mathrm{Sz}(8)$.*

Proof. By [2], we know that $\mathrm{cd}(\mathrm{Sz}(8)) = \{1, 14, 35, 64, 65, 91\}$. We see that $\{2, 13\}$, $\{2, 5\} \notin E(\Gamma(G))$. If $\mathbf{O}_5(G) \neq 1$, then $\mathbf{O}_5(G)$ is a normal abelian Sylow 5-subgroup of G. Thus by Ito's theorem, $\chi(1) \mid |G : \mathbf{O}_5(G)|$ for all $\chi \in \mathrm{cd}(G)$, a contradiction. Similarly we can prove that $\mathbf{O}_7(G) = \mathbf{O}_{13}(G) = 1$. If G is solvable, then by Proposition 2.5, the Sylow 13-subgroup of G is normal, a contradiction. Therefore G is nonsolvable and so G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K/H is a direct product of isomorphic nonabelian simple groups and $|G/K| \mid |\mathrm{Out}(K/H)|$. By the classification of finite simple groups and [2], it follows that $K/H \cong \mathrm{Sz}(8)$. Thus $H = 1$ and so $G \cong \mathrm{Sz}(8)$. □

Lemma 4.8. *Let G be a finite group of order $2^{10} \cdot 5^2 \cdot 31 \cdot 41$ such that $\Gamma(G) = \Gamma(\mathrm{Sz}(32))$. Then $G \cong \mathrm{Sz}(32)$.*

Proof. We see that $\Gamma(G)$ has vertex set $\{2, 5, 31, 41\}$. Also, we know that $\{2, 5\}$, $\{2, 41\} \notin E(\Gamma(G))$. Let G be a solvable group. By Proposition 2.5, the Sylow 41-subgroup of G is normal and hence $\mathbf{O}_{41}(G) \neq 1$. Now by Ito's theorem, we get a contradiction. Therefore G is nonsolvable and so G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K/H is a direct product of isomorphic nonabelian simple groups and $|G/K| \mid |\mathrm{Out}(K/H)|$. By the classification of finite simple groups and [2], it follows that K/H is isomorphic to $\mathrm{Sz}(32)$. Thus $H = 1$ and so $G \cong \mathrm{Sz}(32)$. □

Lemma 4.9. *Let G be a finite group of order $2^5 \cdot 3^6 \cdot 5^2 \cdot 73$ such that $\Gamma(G) = \Gamma(U_3(9))$. Then $G \cong U_3(9)$.*

Proof. We see that $\Gamma(G)$ has vertex set $\{2, 3, 5, 73\}$. Also, we know that $\{3, 5\} \notin E(\Gamma(G))$. Let G be a solvable group. By Proposition 2.5, the Sylow 73-subgroup of G is normal and hence $\mathbf{O}_{73}(G) \neq 1$. Now by Ito's theorem, we get a contradiction. Therefore G is nonsolvable and so G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K/H is a direct product of isomorphic nonabelian simple groups and $|G/K| \mid |\mathrm{Out}(K/H)|$. By the classification of finite simple groups and [2], it follows that K/H is isomorphic to A_5, $A_5 \times A_5$, A_6 or $U_3(9)$. If $K/H \cong A_5$, then $|\mathrm{Out}(A_5)| = 2$. Thus $|H| = 2^t \cdot 3^5 \cdot 5 \cdot 73$,

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where $t \in \{2, 3\}$. If H is solvable, then by Proposition 2.5, $\mathbf{O}_{73}(G) \neq 1$, a contradiction. So we may suppose that H is nonsolvable. First suppose that $|H| = 2^3 \cdot 3^5 \cdot 5 \cdot 73$. By Proposition 2.3, H has a normal series $1 \trianglelefteq A \trianglelefteq B \trianglelefteq H$ such that B/A is a direct product of isomorphic nonabelian simple groups and $|H/B| \mid |\mathrm{Out}(B/A)|$. By the classification of finite simple groups and [2], it follows that either $B/A \cong A_5$ or A_6. If $B/A \cong A_5$, then $|H/B| \mid 2$. Therefore either $|A| = 2 \cdot 3^4 \cdot 73$ or $|A| = 3^4 \cdot 73$. Hence A is solvable and by Proposition 2.5, $\mathbf{O}_{73}(G) \neq 1$, a contradiction. Similarly if $|H| = 2^2 \cdot 3^5 \cdot 5 \cdot 73$, we get a contradiction.

If $K/H \cong A_5 \times A_5$, then $\mathrm{Out}(A_5 \times A_5) = \mathrm{Out}(A_5) \wr S_2$. Thus $|\mathrm{Out}(A_5 \times A_5)| = 8$. So $|H| = 2^t \cdot 3^5 \cdot 5 \cdot 73$, where $t \in \{0, 1, 2, 3\}$. If H is solvable, then by Proposition 2.5, $\mathbf{O}_{73}(G) \neq 1$, a contradiction. Thus we may suppose that H is nonsolvable and so H has a normal series $1 \trianglelefteq A \trianglelefteq B \trianglelefteq H$ such that B/A is a direct product of isomorphic nonabelian simple groups and $|H/B| \mid |\mathrm{Out}(B/A)|$. First suppose that $|H| = 2^2 \cdot 3^5 \cdot 5 \cdot 73$. By the classification of finite simple groups and [2], it follows that $B/A \cong A_5$. If $B/A \cong A_5$, then $|H/B| \mid 2$. Therefore $|A| = 3^4 \cdot 73$. Hence A is solvable and by Proposition 2.5, $\mathbf{O}_{73}(G) \neq 1$, a contradiction. Similarly if either $|H| = 3^5 \cdot 5 \cdot 73$ or $|H| = 2 \cdot 3^5 \cdot 5 \cdot 73$, we get a contradiction.

If $K/H \cong A_6$, then $|\mathrm{Out}(A_6)| = 4$. Thus either $|H| = 2^2 \cdot 3^4 \cdot 5 \cdot 73$ or $|H| = 2 \cdot 3^4 \cdot 5 \cdot 73$ or $|H| = 3^4 \cdot 5 \cdot 73$. First suppose that $|H| = 2^2 \cdot 3^4 \cdot 5 \cdot 73$. If H is solvable, then by Proposition 2.5, $\mathbf{O}_{73}(G) \neq 1$, a contradiction. Thus we may suppose that H is nonsolvable and so H has a normal series $1 \trianglelefteq A \trianglelefteq B \trianglelefteq H$ such that B/A is a direct product of isomorphic nonabelian simple groups and $|H/B| \mid |\mathrm{Out}(B/A)|$. By the classification of finite simple groups and [2], it follows that $B/A \cong A_5$. Thus $|H/B| \mid 2$ and so $|A| = 3^3 \cdot 73$. Hence A is solvable and by Proposition 2.5, $\mathbf{O}_{73}(G) \neq 1$, a contradiction. Similarly if either $|H| = 2 \cdot 3^4 \cdot 5 \cdot 73$ or $|H| = 3^4 \cdot 5 \cdot 7$, we get a contradiction. Thus K/H is isomorphic to $U_3(9)$. So $H = 1$ and hence $G \cong U_3(9)$. □

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