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FAITHFUL REAL REPRESENTATIONS OF GROUPS OF F-TYPE

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Abstract. Groups of F-type were introduced in [B. Fine and G. Rosenberger, Generalizing Algebraic Properties of Fuchsian Groups, *London Math. Soc. Lecture Note Ser.*, **159** (1991) 124–147.] as a natural algebraic generalization of Fuchsian groups. They can be considered as the analogs of cyclically pinched one-relator groups where torsion is allowed. Using the methods In [B. Fine. M. Kreuzer and G. Rosenberger, Faithful Real Representations of Cyclically Pinched One-Relator Groups, *Int. J. Group Theory*, **3** (2014) 1–8.] we prove that any hyperbolic group of F-type has a faithful representation in $PSL(2,\mathbb{R})$. From this we also obtain that a cyclically pinched one-relator group has a faithful real representation if and only if it is hyperbolic. We further survey the many nice properties of groups of F-type.

1. Introduction

A **group of F-type** is a group with a presentation of the form

$$G = \langle a_1, a_2, \ldots, a_m, b_1, b_2, \ldots, b_n; a_1^{e_1} = \cdots = a_m^{e_m} = b_1^{f_1} = \cdots = b_n^{f_n} = 1, u = v\rangle$$

where $1 \leq n, m$, $e_i = 0$ or $e_i \geq 2$ for $i = 1, 2, \ldots, m$, $f_j = 0$ or $f_j \geq 2$ for $j = 1, 2, \ldots, n$, $u = u(a_1, a_2, \ldots, a_m)$ is cyclically reduced and of infinite order in the free product on $a_1, a_2, \ldots, a_m$ and $v = v(b_1, b_2, \ldots, b_n)$ is cyclically reduced and of infinite order in the free product on $b_1, b_2, \ldots, b_n$. In more general language they can be described as cyclically pinched one-relator products of cyclics. Hence cyclically pinched one-relator groups (see [8]) are groups of F-type.

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Let G be a group of F-type, then it can be possible that G is already decomposable as a free product of finitely many cyclic groups. This is exactly the case if $\langle a_1, \ldots, a_m \rangle$is decomposable as a free product $\langle x_1 \rangle \star \cdots \star \langle x_m \rangle$ with $x_1 = u$ or if $\langle b_1, \ldots, b_n \rangle$ is decomposable as a free product $\langle y_1 \rangle \star \cdots \star \langle y_n \rangle$ with $y_1 = v$. Free products of finitely many cyclic groups are considered as well understood (see [8]). Especially such a free product of finitely many cyclic groups always has a faithful representation into $PSL(2, \mathbb{R})$. Also it is SQ-universal and has a non-abelian free subgroup of finite index unless it is infinite cyclic or isomorphic to the infinite dihedral group $\mathbb{Z}_2 \star \mathbb{Z}_2$. Hence for the remainder of this paper we assume that group of F-type does not decompose as a free product of cyclic groups.

Groups of F-type were introduced by Fine and Rosenberger in [9] as natural algebraic generalizations of Fuchsian groups via presentations. Any finitely generated co-compact Fuchsian group of geometric rank > 2, is a group of F-type via its Poincare presentation. In [9] it was shown that any group of F-type such that u and v are not proper powers in the respective free products has a faithful representation into $PSL(2, \mathbb{C})$ and from this many nice properties of groups of F-type were obtained. We will describe these in section 3. In [16] it was shown that a cyclically pinched one-relator group in which u and v are not proper powers has a faithful representation into $PSL(2, \mathbb{R})$. In this paper we extend this to all hyperbolic groups of F-type.

2. Faithful Real Representations of Hyperbolic Groups of F-type

In [16] it was shown that any cyclically pinched one-relator group where u and v are not proper powers has a faithful representation in $PSL(2, \mathbb{R})$. Here we give our main result which extends this to hyperbolic groups of F-type. It follows from this theorem that a cyclically pinched one-relator has a faithful representation in $PSL(2, \mathbb{R})$ if and only if it is hyperbolic.

Recall that a **cyclically pinched one-relator group** is a group with a presentation of the following form

$$G = < a_1, \ldots, a_m, b_1, \ldots, b_n; u = v >$$

where $1 \neq u = u(a_1, \ldots, a_m)$ is a cyclically reduced, non-primitive (not part of a free basis) word in the free group F_1 on $a_1, \ldots, a_m$ and $1 \neq v = v(b_1, \ldots, b_n)$ is a cyclically reduced, non-primitive word in the free group F_2 on $b_1, \ldots, b_n$. Hence this is a group of F-type that is torsion-free.

Clearly such a group is the free product of the free groups on $a_1, \ldots, a_m$ and $b_1, \ldots, b_n$ respectively amalgamated over the cyclic subgroups generated by u and v. If either u or v is not a proper power in the respective free groups on $a_1, \ldots, a_m$ and $b_1, \ldots, b_n$ then from results of Bestvina and Feighn [6], Kharlampovich and Myasnikov [20] and Juhasz and Rosenberger [18] the resulting group is hyperbolic (see [8] and the above references). From Juhasz and Rosenberger [18] we get an extension to groups of F-type.

Theorem 2.1. [18] *Let G be a group of F-type as in the definition. G is hyperbolic unless u is a proper power or a product of two elements of order 2 and v also is a proper power or a product of two elements of order 2.*

Hyperbolic surface groups are cyclically pinched one-relator groups and general hyperbolic cyclically pinched one-relator groups share many properties with surface groups. It is well-known that hyperbolic surface groups have faithful representations in $PSL(2,\mathbb{R})$. In [9] it was shown that a group of F-type has a faithful representation in $PSL(2,\mathbb{C})$ where u and v are not proper powers. Here we extend this to faithful real representations of hyperbolic groups of F-type.

We first need the following lemma.

Lemma 2.2. *Let $F \subset SL(2,\mathbb{R})$ (or $SL(2,\mathbb{C})$) be a free group. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in F$, $B = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \in F$ with $t \neq 1$ and $\langle A, B\rangle$ non-cyclic for the subgroup generated by A and B. Then $a \neq 0$, $b \neq 0$, $c \neq 0$ and $d \neq 0$.*

Proof. First of all $t \neq -1$ because otherwise $B^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. We assume first that $b = 0$. Then $ad = 1$ and $ABA^{-1}B^{-1} = \begin{pmatrix} 1 & 0 \\ cd(1-t^{-2}) & 1 \end{pmatrix}$. This means that $\langle A, B\rangle$ is solvable, and therefore $\langle A, B\rangle$ is cyclic because F is free which gives a contradiction. Hence $b \neq 0$. Analogously $c \neq 0$. Now assume that $a = 0$. Then $bc = -1$ and $ABA^{-1} = \begin{pmatrix} t^{-1} & 0 \\ cd(t-t^{-1}) & t \end{pmatrix}$ which, by the arguments above, means that $\langle ABA^{-1}, B\rangle$ is solvable, and therefore $\langle A, B\rangle$ is cyclic which gives a contradiction. Hence $a \neq 0$. Analogously we get $d \neq 0$. □

Remark 2.3. *Lemma* 2.2 also follows easily from the well known general fact: Let G be a torsion-free group whose abelian subgroups are all cyclic. Then all solvable subgroups of G are cyclic. Examples for such groups G are the torsion-free hyperbolic groups.

The next theorem concerns hyperbolic cyclically pinched one-relator groups .

Theorem 2.4. *Let $G = \langle a_1, a_2, \ldots, a_m, b_1, b_2, \ldots, b_n; u = v\rangle$, $n \geq 1$, $m \geq 1$ and $u = u(a_1, a_2, \ldots, a_n)$ a non-trivial, not primitive element in the free group $F_1 = \langle a_1, a_2, \ldots, a_n\rangle$ and $v = v(b_1, b_2, \ldots, b_n)$ a non-trivial, not primitive element in the free group $F_2 = \langle b_1, b_2, \ldots, b_n\rangle$.*

Let either u or v be not a proper power in the respective free group.

Then there exists a faithful representation $\phi : G \to PSL(2,\mathbb{R})$.

Proof. We prove the existence of a faithful representation into $SL(2,\mathbb{R})$. Under the given conditions G is hyperbolic. Since a hyperbolic cyclically pinched one-relator group is centerless this faithful representation can be extended to a faithful representation into $PSL(2,\mathbb{R})$.

We first let $n, m \geq 2$. We have that not both u and v are proper powers. Hence, without loss of generality, we may assume that u is not a proper power in F_1. Let $v = v_1^p$, $p \geq 1$, $v_1 \in F_2$, and v_1 not a proper power in F_2. We first embed F_1 into a free group $H_1 = \langle a, b; \ \rangle$ of rank 2 and F_2 into a free group $H_2 = \langle c, d; \ \rangle$ also of rank 2. It follows that u, v are both non trivial and not primitive in H_1, H_2 respectively.

We now consider $H_1 = \langle a, b; \ \rangle$. Choose $A, B \in SL(2, \mathbb{R})$ with $tr(A) = x > 2$ an algebraic number and $tr(B) = y > 2$ also an algebraic number.

Then $tr(AB) = r$, and we will choose r later in a suitable manner.
The map $a \mapsto A$, $b \mapsto B$ defines a homomorphism $\phi_1 : H_1 \to SL(2, \mathbb{R})$. Let $U = \phi_1(u)$ be the image of u. Then $tr(U)$ is a nonconstant polynomial $f(r)$ with coefficients in $\mathbb{Z}[x, y]$, and the highest coefficient is positive. Moreover all the coefficients are algebraic numbers.

We now argue analogously for $H_2 = \langle c, d; \ \rangle$. Choose $C, D \in SL(2, \mathbb{R})$ with $tr(C) = z > 2$ an algebraic numbers and $tr(D) = w > 2$ an algebraic number. We let $s = tr(CD)$ and, just as for r, we will later choose s in a suitable manner.

As before, the map $c \mapsto C$, $d \mapsto D$ defines a homomorphism $\phi_2 : H_2 \to SL(2, \mathbb{R})$. Let $V = \phi_2(v)$, $V_1 = \phi_2(v_1)$ the image of v, v_1, respectively.

Recall that $V = V_1^p$, $p \geq 1$. As before, $tr(V_1)$ is a nonconstant polynomial $g_1(s)$ in s with coefficients in $\mathbb{Z}[z, w]$; and the highest coefficient is positive. Moreover all the coefficients are algebraic numbers. Recall that $V = V_1^p$. We define the Tschebyscheff-polynomials $S_\nu(x)$ by $S_0(x) = 0$, $S_1(x) = 1$ and $S_\nu(x) = xS_{\nu-1}(x) - S_{\nu-2}(x)$ for $\nu \geq 2$. Then

$$V = V_1^p = S_p(g_1(s))V_1 - S_{p-1}(g_1(s))E_2$$

where $E_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Then

$$tr(V) = tr(V_1)S_p(g_1(s)) - 2S_{p-1}(g_1(s)) = g(s).$$

We make the following observations

(1) $f(x) \to \infty$ if $x \to \infty$ and
(2) $g(x) \to \infty$ if $x \to \infty$.

If we choose a sufficiently large transcendental number $t > 4$ then by the intermediate value theorem there exist $r \in \mathbb{R}$ and $s \in \mathbb{R}$ such that $f(r) = t = g(s)$. The real numbers r, s have to be transcendental because the polynomials $f(x)$ and $g(x)$ have algebraic coefficients (if r were algebraic then $f(r)$ would also be algebraic). After a suitable conjugation of $\phi_1(H_1)$ and $\phi_2(H_2)$ in $SL(2, \mathbb{R})$ we may assume that

$$U = \begin{pmatrix} t_1 & 0 \\ 0 & t_1^{-1} \end{pmatrix} = V, \quad t = t_1 + t_1^{-1},$$

$$V_1 = \begin{pmatrix} t_2 & 0 \\ 0 & t_2^{-1} \end{pmatrix}, \quad t_1 = t_2^p,$$

with t_1, t_2 real transcendental numbers (recall that $t > 4$). We have the following facts

(a) $\langle A, B \rangle$ is a free group of rank 2 because r is transcendental.

(*b*) $\langle C, D\rangle$ is a free group of rank 2 because s is transcendental.

Therefore ϕ_1 and ϕ_2 are monomorphisms and hence embeddings of the respective free groups in $SL(2,\mathbb{R})$. F_1 is a subgroup of $H_1 = \langle a, b\rangle$ and F_2 is a subgroup of $H_2 = \langle c, d\rangle$. Hence

$$\phi_1 \mid_{F_1} : F_1 \to SL(2,\mathbb{R}) \quad \text{and} \quad \phi_2 \mid_{F_2} : F_2 \to SL(2,\mathbb{R})$$

are embeddings with

$$\phi_1 \mid_{F_1} (u) = U = V = \phi_2 \mid_{F_2} (v) \text{ and } \phi_2|_{F_2}(v_1) = V_1.$$

We remark that V_1 is not a proper power in $\phi_2 \mid_{F_2} (F_2)$, it could be a proper power in $\phi_2(H_2)$ but this possibility we did not use.

Then the combination of $\phi_1 \mid_{F_1}$ and $\phi_2 \mid_{F_2}$ defines a homomorphism

$$\phi : G \to SL(2,\mathbb{R}) \quad \text{with} \quad \phi \mid_{F_1} = \phi_1 \mid_{F_1},\ \phi \mid_{F_2} = \phi_2 \mid_{F_2}.$$

From the homomorphism $\phi : G \to SL(2,\mathbb{R})$ we get an injective homomorphism $\rho : G \to SL(2,\mathbb{R})$.

This we may seen as follows.

Let $tr(G) = \{tr(\varphi(g)) \mid g \in G\}$. We choose a real transcendental number τ which is not algebraic over $K = \mathbb{Q}(tr(G))$. Now define

$T = \begin{pmatrix} \tau & 0 \\ 0 & \tau^{-1} \end{pmatrix}$, and a homomorphism $\rho : G \to SL(2,\mathbb{R})$ by $\rho(g) = T\phi(g)T^{-1}$ if $g \in F_1$ and $\rho(g) = \phi(g)$ if $g \in F_2$.

Since $\rho(u) = \phi(u) = U = V$ this does indeed give a homomorphism. For $g \in G$ we write g_{ij} for the entry of $\rho(g)$ in the ij-th position, that is, $\rho(g) = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$. By Lemma 2.2 we have $g_{ij} \neq 0$ if $g \in F_1 \setminus \langle u\rangle$ or $g \in F_2 \setminus \langle v_1\rangle$ for all ij since F_1, F_2 are non abelian and u and v_1 are not proper powers in F_1 and F_2, respectively.

We now show that ρ is injective.

Every element $g \in G$ is conjugated either to an element of F_1 or F_2 or to an element of the form $g = x_1y_1x_2y_2 \cdots x_ky_k$ with $k \geq 1$ and $x_i \in F \setminus \langle u\rangle$ and $y_i \in F_2 \setminus \langle v\rangle$ for $i = 1, 2, \ldots, k$.

To prove that ρ is injective we may assume that $g = x_1y_1x_2y_2 \cdots x_ky_k$ as above.

We claim that g_{11} is a Laurent polynomial of degree $2d_1 \geq 0$ (in τ over $K(t_2)$), g_{12} is a Laurent polynomial of degree $2d_2 \geq 0$, where $max(d_1, d_2) > 0$, g_{21} is a Laurent polynomial of degree $2d_3$ with $d_3 \leq min(d_1, d_2)$ and g_{22} is a Laurent polynomial of degree $2d_4$ with $d_4 \leq min(d_1, d_2)$. Moreover $d_3, d_4 < d_1$ if $d_1 = d_2$.

We prove the claim by induction on k.

If $k = 1$ then $g = xy$ with $x = x_1$, $y = y_1$. Recall that

$$T\phi(x)T^{-1} = \begin{pmatrix} \tau & 0 \\ 0 & \tau^{-1} \end{pmatrix} \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} \begin{pmatrix} \tau^{-1} & 0 \\ 0 & \tau \end{pmatrix}$$

with $\alpha_{ij} \neq 0$ for all ij because ϕ is injective on F_1 (see Lemma 2.2).

First let $y \notin \langle v_1 \rangle$. Then $y_{11} \neq 0 \neq y_{12}$, $y_{21} \neq 0 \neq y_{22}$ by Lemma 2.2, and

$$g_{11} = \alpha_{11} y_{11} + \alpha_{12} y_{21} \tau^2,$$
$$g_{12} = \alpha_{11} y_{12} + \alpha_{12} y_{22} \tau^2,$$
$$g_{21} = \alpha_{22} y_{21} + \alpha_{21} y_{11} \tau^{-2},$$
$$g_{22} = \alpha_{22} y_{22} + \alpha_{21} y_{12} \tau^{-2},$$

and the claim holds here.

Now, let $y_1 \in \langle v_1 \rangle$. Then

$$\rho(y_1) = \begin{pmatrix} t_2^{\alpha} & 0 \\ 0 & t_2^{-\alpha} \end{pmatrix}$$

for some integer $\alpha \neq 0$ (recall that $y_1 \notin \langle v \rangle$). Then

$$g_{11} = \alpha_{11} t_2^{\alpha},$$
$$g_{12} = \alpha_{12} t_2^{-\alpha} \tau^2,$$
$$g_{21} = \alpha_{21} t_2^{\alpha} \tau^{-2},$$
$$g_{22} = \alpha_{22} t_2^{-\alpha},$$

and the claim also holds here. This proves the claim for $k = 1$.

Now let $k \geq 2$. We write $g = xy$ with $x = x_1 y_1$ and $y = x_2 y_2 x_3 y_3 \cdots x_k y_k$. By the induction hypothesis and the case $k = 1$ already proven the claim holds for both x and y. We now multiply $\rho(x)$ with $\rho(y)$ and get the overall claim. We just need to remark that, if $p \geq 2$, then $\rho(x_1)\rho(y_1)\rho(x_2)\rho(y_2) \cdots \rho(x_\ell)\rho(y_\ell) \neq E_2$, $\ell \geq 1$, where all $x_i \in F_1 \setminus \langle u \rangle$ and all $y_i \in \langle v_1 \rangle \setminus \langle v \rangle$. This easily follows by induction. In fact, since it already holds for $\ell = 1$, it is enough to show this for $\ell = 2$. The general case follows in an analogous manner if we factorize as above. Hence, let $p \geq 2$ and $\ell = 2$. Let

$$\rho(x_1)\rho(y_1) = \begin{pmatrix} \alpha_{11} t_2^{\alpha} & \alpha_{12} t_2^{-\alpha} \tau^2 \\ \alpha_{21} t_2^{\alpha} \tau^{-2} & \alpha_{22} t_2^{-\alpha} \end{pmatrix}$$

and

$$\rho(x_2)\rho(y_2) = \begin{pmatrix} \beta_{11} t_2^{\beta} & \beta_{12} t_2^{-\beta} \tau^2 \\ \beta_{21} t_2^{\beta} \tau^{-2} & \beta_{22} t_2^{-\beta} \end{pmatrix}$$

with $\alpha, \beta \neq 0$. We may suppose that $\alpha, \beta > 0$.

Assume that

$$\rho(x_1)\rho(y_1)\rho(x_2)\rho(y_2) = E_2 \tag{2.1}$$

The equation (1) now implies the following:

$$\alpha_{11}\beta_{11} t_2^{\alpha+\beta} + \alpha_{21}\beta_{12} t_2^{\alpha-\beta} = 1,$$

$$\alpha_{21}\beta_{12}t_2^{\alpha-\beta} + \alpha_{22}\beta_{22}t_2^{-\alpha-\beta} = 1,$$
$$\alpha_{12} = -\beta_{12}t_2^{\alpha-\beta},$$
$$\alpha_{21} = -\beta_{21}t_2^{\beta-\alpha}.$$

This then implies

$$\alpha_{11}\beta_{11}t_2^{\alpha+\beta} + \alpha_{12}\beta_{21} = 1$$

and

$$\alpha_{22}\beta_{22}t_2^{-\alpha-\beta} + \alpha_{12}\beta_{21} = 1.$$

On the other side from the determinant condition for $\rho(x_1)$ and $\rho(x_2)$ we get

$$\alpha_{11}\alpha_{22} + \alpha_{12}\beta_{21}t_2^{\beta-\alpha} = 1$$

and

$$\beta_{11}\beta_{22} + \alpha_{12}\beta_{21}t_2^{\alpha-\beta} = 1$$

because $\alpha_{12} = -\beta_{12}t^{\alpha-\beta}$ and $\alpha_{21} = -\beta_{21}t^{\beta-\alpha}$. But this contradicts

$$\alpha_{11}\beta_{11}t_2^{\alpha+\beta} + \alpha_{12}\beta_{21} = 1$$

and

$$\alpha_{22}\beta_{22}t^{-\alpha-\beta} + \alpha_{12}\beta_{21} = 1$$

because also necessarily

$$\alpha_{11}t_2^{\alpha} = \beta_{22}t_2^{-\beta} \text{ and } \alpha_{22}t_2^{-\alpha} = \beta_{11}t_2^{\beta}.$$

Therefore

$$\rho(x_1)\rho(y_1)\rho(x_2)\rho(y_2) \neq E_2.$$

Hence, the claim holds, and therefore $tr(\rho(g))$ is transcendental over $K(t_2)$ or $g_{12} \neq 0$ or $g_{21} \neq 0$ which proves that the homomorphism ρ is injective.

This proves the theorem for $m, n \geq 2$. Now, let $m = 1$ or $n = 1$. Again we assume that u is not a proper power in F_1.

Then $m \geq 2$ because otherwise u would be primitive in F_1.

Hence, let $m \geq 2$ and $n = 1$.

Then $F_2 = \langle v_1 \rangle$ and $v = v_1^p$ for some $p \geq 1$. We may apply the construction for F_1 as above. Then we get directly a homomorphism $\varphi : G \to SL(2, \mathbb{C})$ with $\varphi\mid_{F_1}$, $\varphi\mid_{F_2}$ embeddings and

$$\varphi\mid_{F_1}(u) = U = V = \begin{pmatrix} t_1 & 0 \\ 0 & t_1^{-1} \end{pmatrix} = \varphi\mid_{F_2}(v), \qquad V_1 = \begin{pmatrix} t_2 & 0 \\ 0 & t_2^{-1} \end{pmatrix}$$

with t_1, t_2 real transcendental numbers and $t_1 = t_2^p$ (recall that $\varphi\mid_{F_2} = \langle V_1 \rangle$).
If $g = x_1y_1x_2y_2 \cdots x_ky_k$ and all $x_i \in F_2 \setminus \langle u \rangle$, $y_i \in \langle v_1 \rangle \setminus \langle v \rangle$ then $\varphi(g) \neq E_2$ as above.

This proves Theorem 2.4.

□

We now extend this to hyperbolic groups of F-type. We first recall Theorem 2.1 by Juhasz and Rosenberger [18].

Theorem 2.5. [18] *Let G be a group of F-type as in the definition. G is hyperbolic unless u is a proper power or a product of two elements of order 2 and v also is a proper power or a product of two elements of order 2.*

As in the above proof we need the following lemma.

Lemma 2.6. *Let $F \subset PSL(2,\mathbb{R})$ (or $PSL(2,\mathbb{C})$) be a free product of cyclic groups. Let $A = \pm\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in F$, $B = \pm\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \in F$ an element of infinite order, and $\langle A, B\rangle$ non-cyclic for the subgroup generated by A and B.*

Then one of the following cases hold.

(1) $a \neq 0$, $b \neq 0$, $c \neq 0$ *and* $d \neq 0$.

(2) B *is a product of two elements of order* 2.

Moreover, if $B = B_1^p$, $p \geq 1$, for some $B_1 \in F$, then also B_1 is a product of two elements of order 2 if B is such a product.

Proof. The proof follows the lines of the proof of Lemma 2.2.

$\langle A, B\rangle$ is the free product of two cyclic groups. Since $\langle A, B\rangle$ is non-cyclic it is solvable if and only if $\langle A, B\rangle$ is isomorphic to the infinite dihedral group. Then necessarily $A = \pm\begin{pmatrix} 0 & \rho \\ \rho^{-1} & 0 \end{pmatrix}$ for some $\rho \in \mathbb{R}$ (or $\mathbb{C}$, respectively). If now $B = B_1^p, p \geq 1$, then $(AB)^2 = (AB_1)^2 = 1$. □

Theorem 2.7. *Let G be a hyperbolic group of F-type. Then G has a faithful representation in $PSL(2,\mathbb{R})$.*

Proof. Let G be given as in the definition. Let first u or v be a product of two elements of order 2. Without loss of generality, let u be a product of two elements of order 2.

Since G is hyperbolic, then v is neither a product of two elements of order 2 nor a proper power. Also, by Lemma 2.6, we may assume that u is not a proper power.

Then G has a faithful representation in $PSL(2,\mathbb{R})$ by [16]. Now, let neither u nor v be a product of two elements of order 2. Then at most one of u and v is a proper power. Using Lemme 2.6 we now get the result analogously as for Theorem 2.4. □

As a corollary we get the following:

Corollary 2.8. *A cyclically pinched one-relator group has a faithful representation in $PSL(2,\mathbb{R})$ if and only if it is hyperbolic.*

Proof. Let G be a hyperbolic cyclically pinched one-relator group. Then from Theorems 2.1 and 2.3 the group G has a faithful representation in $PSL(2,\mathbb{R})$.

Now suppose that G is a cyclically pinched one-relator group that has a faithful representation ϕ in $PSL(2,\mathbb{R})$. If G were not hyperbolic, then both u and v must be proper powers in the free groups on the generators they involve. Then $u = u_1^p$ and $v = v_1^q$ for some $u_1 \in F_1, v_1 \in F_2$ and integers $p, q > 1$. However in $PSL(2,\mathbb{R})$ two elements commute if and only if they have the same fixed points, considered as linear fractional transformations. It follows that $PSL(2,\mathbb{R})$ is commutative transitive (see[14]). Hence $\phi(u_1)$ and $\phi(v_1)$ commute, but in G the elements u_1 and v_1 do not commute. If follows that at least one of u or v must not be a proper power and therefore G is hyperbolic. □

Remark: Now let G be a group of F-type. Concerning the question if there exists a faithful representation $\phi : G \to PSL(2,\mathbb{R})$, we are left, up to symmetry, with the case that $u = xy$ with $x^2 = y^2 = 1$ and $v = v_1^p$ with p an integer $p > 1$. But in this case there does not exist such a ϕ because $(\phi(uy))^2 = (\phi(vy))^2 = 1$ implies that $(\phi(v_1y))^2 = 1$ but $(v_1y)^2 \neq 1$ in G.

3. On Groups of F-type

As mentioned, groups of F-type were introduced in [9] as a natural algebraic generalization of co-compact Fuchsian groups. In the case of geometric rank > 2 co-compact Fuchsian groups are groups of F-type via their Poincare presentations. The main focus was on the question: Given an algebraic property of a Fuchsian group how does it extend (if at all) to the more general class of groups of F-type. Using various techniques it was shown in [9] and [15] that most algebraic properties of Fuchsian groups extend in this more general context. This can be considered surprising since so much of the theory of Fuchsian groups depends on hyperbolic geometry and topology. In this section we summarize many of these results in order to highlight the generality of groups of F-type. Most of the proofs of these can be found in [9], [10] or the book [8].

From the faithful representations of hyperbolic groups of F-type into $PSL(2,\mathbb{R})$ we obtain the linearity properties of Fuchsian groups. Recall that a group is **commutative transitive** abbreviated **CT** if commutativity is transitive on nonidentity elements. Any subgroup of $PSL(2,\mathbb{R})$ is commutative transitive (see [14]).

Theorem 3.1. *Let G be a hyperbolic group of F-type Then:*

(1) G is virtually torsion-free

(2) G is residually finite and Hopfian

(3) G is commutative transitive

(4) If $e_i \geq 2$ then a_i has order exactly e_i. The analogous result holds for the b_j.

(5) Any element of finite order is conjugate to a power of some a_i or some b_j.

(6) Any finite subgroup of G is cyclic and conjugate to a subgroup of some $\langle a_i \rangle$ or some $\langle b_j \rangle$

(7) If G is a cyclically pinched one-relator group and if neither u nor v is in the commutator subgroup of its respective factor then G is free-by-cyclic [5].

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(10) In $m+n>2$ then G is SQ-universal. In particular G contains a non-abelian free group.

For proofs of these see [9], [15] or [8]. We note that (2) was proved originally under the restriction that either u or v is is not a proper power. Using results of Allenbey (see [3]) this restriction can be removed.

Closely tied to commutative transitivity is the concept of being CSA. A group G is **CSA** or **conjugately separated abelian** if maximal abelian subgroups are malnormal. These concepts have played a prominent role in the studies of fully residually free groups, limit groups and discriminating groups (see [14] and [13]. They also play a role in the solution to the Tarski problems. CSA always implies CT. In general the class of CSA groups is a proper subclass of the class of CT groups, however they are equivalent in the presence of residual freeness. For hyperbolic groups of F-type we have.

Theorem 3.2. *Let G be a hyperbolic group of F-type. Assume that G is torsion-free or has only odd torsion, that is, e_i is odd if $e_i \geq 2$ and f_j is odd if $f_j \geq 2$. Then G is CSA.*

Proof. We may assume that G is already a subgroup of $PSL(2,\mathbb{R})$. Let A be a maximal abelian subgroup of G. Since G is a subgroup of $PSL(2,\mathbb{R})$ any two elements a,b of A have, considered as linear fractional transformations, the same fixed points.

Assume that G is not CSA. Then there are non-trivial elements a,b in A and $x \in G \setminus A$ with $xax^{-1}=b$. This means that x permutes non-trivially the fixed points of a, which are also the fixed points of b. Hence, x must have order two. This gives a contradiction, and therefore G is CSA. □

Recall that linear groups satisfy the Tits alternative, that is they either contain a free subgroup of rank 2 or are virtually solvable. From linearity and an examination of the possible solvable cases we have for groups of F-type.

Theorem 3.3. *Let G be a group of F-type. The either G contains a free subgroup of rank 2 or G is solvable with one of the following presentations*

(1) $\langle a,b; a^2b^2=1\rangle$

(2) $\langle a,b,c; a^2=b^2=abc^2=1\rangle$

(3) $\langle a,b,c,d; a^2=b^2=c^2=d^2=abcd=1\rangle$

Further if G is not solvable then G is SQ-universal.

Note that the SQ-universality follows in the non-solvable cases because G has a subgroup of finite index that maps onto a free group of rank 2.

Recall that a group G is **conjugacy separable** if given any non-trivial $g,h \in G$ that are not conjugate then there exists a finite quotient G^* of G where the images of g and h are still not conjugate, Conjugacy separability implies residual finiteness. Further G is **subgroup separable** or LERF if given any subgroup $H \subset G$ and $g \notin H$ then there exists a finite quotient G^* of G with $g^* \notin H^*$ with g^*, H^* the images of g, H in G^*.

Using results of Allenby [3], Allenby and Tang [4] and Niblo [21] and Aab and Rosenberger [1] groups of F-type were shown to be both conjugacy separable and subgroup separable (see [10], [15] or [8].

Theorem 3.4. *Let G be a group of F-type. Then:*

(1) G is conjugacy separable

(2) G is subgroup separable.

Dahmani and Guiradel [7] proved that all hyperbolic groups have a solvable isomorphism problem. Sela [23] had proved this earlier for torsion-free hyperbolic groups. In general hyperbolic groups have solvable word problem and conjugacy problem. These then apply to the hyperbolic groups of F-type. Earlier Rosenberger [22] using Nielsen cancellation showed that the isomorphism problem is solvable in the class of cyclically pinched one-relator groups in general without the additional condition of hyperbolicity.

Theorem 3.5. *For a hyperbolic group of F-type*

(1) the word problem is solvable

(2) The conjugacy problem is solvable

(3) The isomorphism problem in the class of hyperbolic groups of F-type is solvable.

For Fuchsian groups the concept of an Euler characteristic plays a large role and by the Riemann-Hurwitz formula restricts by signature the type of Fuchsian subgroups a given Fuchsian group can contain (see [8] or [19]). In [17] it was shown that a group of F-type has a rational Euler characteristic that extends that of a Fuchsian group and also satisfies the Riemann-Hurwitz formula.

Theorem 3.6. *Let G be a group of F-type. then G has a rational Euler characteristic given by*

$$\chi(G) = 2 + \sum_{i=1}^{m} \alpha_i + \sum_{j=1}^{n} \beta_j$$

where $\alpha_i = -1$ if $e_i = 0$ and $\alpha_i = -1 + \frac{1}{e_i}$ if $e_i \geq 2$ and $\beta_j = -1$ if $f_j = 0$ and $\beta_j = -1 + \frac{1}{f_j}$ if $f_j \geq 2$. Further if H is a subgroup of finite index in G than $\chi(H)$ is defined and the Riemann-Hurwitz formula $\chi(H) = [G : H]\chi(G)$ holds.

Further G is of finite homological type with $vcd(G) \leq 2$.

In [17] there were further results on Euler characteristic for more general one-relator products of cyclics.

Further, there is a complete classification of two-generator subgroups of groups of F-type (see [15]). The structure of two-generator subgroups of groups of F-type is given by the following..

Theorem 3.7. *Let G be a group of F-type and let H be a non-cyclic two-generator subgroup of G. Then H is conjugate to a subgroup $\langle x, y\rangle$ satisfying one of the following two conditions:*

(1) $\langle x, y\rangle$ is a free product of cyclics.

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(2) x^t is in $\langle u\rangle = \langle v\rangle$ for some natural number t and $y^{-1}x^ty$ is in $\langle a_1,\dots,a_m\rangle$ or $y^{-1}x^ty$ is in $\langle b_1,\dots,b_n\rangle$.

If in addition G is hyperbolic then H is a free product of cyclics.

Finally, groups of F-type fall into the wider class of one-relator products of cyclics. For this class of groups various versions of the Freiheitsattz of Magnus have been proved see [11]). This can be extended to groups of F-type.

Theorem 3.8. *(Freiheitssatz for groups of F-type) Let G be a group of F-type and suppose that the relator uv involves all the generators. Then:*

(1) Any subset of $m+n-2$ of the given generators generates a free product of cyclics of the obvious orders.

(2) If both u and v are proper powers in the free product of the respective generators they involve, then any subset of $m+n-1$ of the given generators generates a free product of cyclics of the obvious orders.

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