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CHARACTERIZATION OF FINITE GROUPS WITH A UNIQUE NON-NILPOTENT PROPER SUBGROUP

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Abstract. We characterize finite non-nilpotent groups G with a unique non-nilpotent proper subgroup. We show that $|G|$ has at most three prime divisors. When G is supersolvable we find the presentation of G and when G is non-supersolvable we show that either G is a direct product of an Schmidt group and a cyclic group or a semi direct product of a p-group by a cyclic group of prime power order.

1. Introduction

Throughout the paper all groups are assumed to be finite. Our notation and terminology are standard taken mainly from [8]. In particular the size of a finite group G is shown by $|G|$. The center, the derived subgroup, and the Frattini subgroup of G are denoted by $Z(G)$, G', and $\Phi(G)$, respectively. For $x, g \in G$, $x^g := g^{-1}xg$ is the conjugate of x by g; and $C_G(x)$ is the centralizer x in G. For a subgroup M of G, the centralizer of M in G is denoted by $C_G(M)$. The symbol $G = Y \rtimes X$ indicates that G is a split extension (semi direct product) of a normal subgroup Y of G by a complement X.

Let $\mathscr{X}$ be a class of groups. A group G that is not in $\mathscr{X}$ but each proper subgroup of G is contained in $\mathscr{X}$, is said to be an $\mathscr{X}$- critical group or a minimal non-$\mathscr{X}$-group. Some people have studied the structure of $\mathscr{X}$-critical groups for a various classes of groups. For example Miller and Moreno [4]

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considered the class of abelian groups, $\mathscr{A}$, and studied $\mathscr{A}$-critical groups, also called minimal non-abelian groups, and showed that such groups are solvable. Also they showed that if a $\mathscr{A}$-critical group G is nilpotent, then it is a p-group and $|G : \Phi(G)| = |G : Z(G)| = p^2$. After that Rédei [5] continued the study of minimal non-abelian groups and classified such groups completely. We classified finite groups having a unique non-abelian proper subgroup in [11].

Schmidt [10] considered the class of nilpotent groups, $\mathscr{N}$, and studied $\mathscr{N}$-critical groups which also called minimal non-nilpotent groups (or Schmidt groups). Rédei [6] completely classified finite Schmidt groups. Itô [2] considered the minimal non-p-nilpotent groups for p a prime and proved that such groups are just Schmidt groups. Ballester-Bolinches and Esteban-Romero [1], considered the class of supersolvable groups, $\mathscr{U}$, and studied the structure of $\mathscr{U}$-critical groups which also called minimal non-supersolvable groups.

Let n be a positive integer. We say that a group G is an n-$\mathscr{X}$-critical group, if $G \notin \mathscr{X}$ and has exactly $n-1$ proper subgroups that are not belong to $\mathscr{X}$ and other subgroups belong to $\mathscr{X}$. Therefore a minimal non-$\mathscr{X}$-group is just a 1-$\mathscr{X}$-critical group.

Russo [9] studied 2-$\mathscr{N}$-critical groups, where $\mathscr{N}$ is the class of nilpotent groups, in infinite case and obtained some results in finite case. Also by the main result in [12], one can see that every finite n-$\mathscr{N}$-critical groups with $n \leq 21$ is solvable. In this paper we classify finite 2-$\mathscr{N}$-critical groups, i.e., finite non-nilpotent groups with a unique proper non-nilpotent subgroup. We show that each finite supersolvable 2-$\mathscr{N}$-critical group is a 2-$\mathscr{A}$-critical group.

First we characterize finite supersolvable 2-$\mathscr{N}$-critical groups:

Theorem A. Let G be a finite non-nilpotent supersolvable group. Then G has a unique non-nilpotent proper subgroup if and only if G is one of the following groups.

(1) $\langle a, b, c \mid a^p = b^p = c^{q^m} = 1,\ [a, b] = [a, c] = [b, c^q] = 1,\ b^c = a^i b^j\rangle$, where p and q are distinct prime numbers $0 \leq i, j \leq p-1$, $j^q \equiv 1 \pmod p$ and if $i \neq 0$, then $1 + j + \cdots + j^{q-1} \equiv 0 \pmod p$,

(2) $\langle a, b \mid a^p = b^{q^{m+1}} = 1,\ [a, b^{q^2}] = 1,\ a^b = a^i\rangle$, where $q \mid p-1$, $0 \leq i \leq p-1$, $i \neq 1$, $m \geq 1$ and $i^{q^2} \equiv 1 \pmod p$,

(3) $\langle a, b, c \mid a^p = b^{q^m} = c^r = 1,\ [a, c] = [b, c] = [a, b^q] = 1,\ a^b = a^i\rangle$, where p, q and r are distinct prime numbers, $0 \leq i \leq p-1$ and $i^q \equiv 1 \pmod p$.

By virtue of [11] and using Theorem A, we have the following corollary.

Corollary 1. Let G be a finite supersolvable 2-$\mathscr{N}$-critical group. Then it is a 2-$\mathscr{A}$-critical group. The converse is true if the unique non-abelian subgroup of G is non-nilpotent.

Next we characterize 2-$\mathscr{N}$-critical non-supersolvable groups (in what follows p, q and r are prime numbers):

Theorem B. Let G be a finite non-supersolvable group. Then G is 2-$\mathscr{N}$-critical if and only if it is one of the following groups.

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(I) $H \times \mathbb{Z}_r$, where H is a non-supersolvable Schmidt group whose order is coprime to r.

(II) $H \times \mathbb{Z}_p$, where H is a non-supersolvable Schmidt group of order $p^n q^m$, $q \nmid p-1$. Also the Sylow p-subgroup of H is an irreducible Q-module over the field of p elements, Q is cyclic and $|Q : C_Q(P)| = q$.

(III) PQ, where $P = G'$ is a minimal normal subgroup of G, $|P| = p^n$, and $Q = \langle c \rangle$ is of order q^2. The order of p modulo q^2 being n. Furthermore $G'\langle c^q \rangle$ is an Schmidt group and G' is a $\langle c^q \rangle$-irreducible module.

(IV) $G'QL$, where G' is a non-abelian special p-group of rank $2l$, Q is cyclic of order q^m, the order of p modulo q being $2l$. Also $G' \cap L = \Phi(G')$, $|L| = p|\Phi(G')|$, $[L, Q] = 1$, $G'/\Phi(G')$ is a faithful irreducible Q-module and $[\Phi(G'), Q] = 1$. Furthermore $|G''| \leq p^l$.

(V) $G'Q$, where G' is a non-abelian special p-group of rank $2l$, $Q = \langle c \rangle$ is cyclic of order q^{m+1}, the order of p modulo q^2 being $2l$. Also $G'\langle c^q \rangle$ is an Schmidt group, $G'/\Phi(G')$ is a faithful irreducible Q-module and $[\Phi(G'), Q] = 1 = [P, c^{q^2}]$. Furthermore $|G''| \leq p^l$.

2. **Proofs**

We begin by stating the following Theorem on the structure of finite Schmidt groups.

Theorem 2. [10, 5] Let G be a finite Schmidt group. Then $G = P \rtimes Q$, where P is a Sylow p-subgroup and $Q = \langle z \rangle$ is a cyclic Sylow q-subgroup of order $q^r > 1$. Furthermore $Z(G) = \Phi(G) = \Phi(P) \times \langle z^q \rangle$; $G' = P$, $P' = G'' = \Phi(P)$, and one of the the following cases hold:

(1) q does not divide $p-1$ and P is an irreducible Q-module over the field of p elements with kernel $\langle z^q \rangle$ in Q. The subgroup P is elementary abelian minimal normal p-subgroup of order p^l where l is the order of p modulo q.

(2) P is a non-abelian special p-group, $|P/\Phi(P)| = p^{2m}$, $|P'| \leq p^m$, the order of p modulo q being $2m$, z induces an automorphism in P such that $P/\Phi(P)$ is a faithful irreducible Q-module, and z centralizes $\Phi(P)$.

(3) q divides $p-1$, $P = \langle a \rangle$ is cyclic of order p, and $a^z = a^i$, where i is the least primitive q-th root of unity modulo p.

Note that Schmidt groups satisfying (1) and (2) of Theorem 2 are non-supersolvable and Schmidt groups satisfying (3) are supersolvable. In fact if G is supersolvable and satisfies (1), then since P is a minimal normal subgroup G, Theorem 5.4.7 of [8] implies that $|P| = p$. It follows that $q \mid p-1$, a contradiction. Also if G satisfies (2) and is supersolvable, then $P/\Phi(P)$ is a minimal normal subgroup of supersolvable Schmidt group $G/\Phi(P)$, which implies that P is cyclic, a contradiction. Finally if G satisfies (3), then $1 \lhd P \lhd G$ is a normal series with cyclic factors and so G is supersolvable.

Let G be a finite non-nilpotent group with a unique non-nilpotent proper subgroup H. It is readily seen that H is an Schmidt group. Furthermore H is a maximal and characteristic subgroup of G.

We have the following simple Lemma.

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Lemma 3. Let G be a finite non-nilpotent group with a unique non-nilpotent proper subgroup H. If H is supersolvable, then G is supersolvable.

Proof. Since H is a normal maximal subgroup of G, G/H is cyclic of prime order. Since H is a supersolvable Schmidt group, from Theorem 2, we have $|H| = pq^m$ and a Sylow p-subgroup P of H is cyclic. Now $1 \lhd P \lhd H \lhd G$ is a normal series with cyclic factors and so G is supersolvable. □

Proof Theorem A. First we prove that all groups $(1)-(3)$ have exactly one non-nilpotent proper subgroup.

Owing to [11, Theorems E, F and D], the groups $(1)-(3)$ are non-nilpotent supersolvable 2-$\mathscr{A}$-critical and so are 2-$\mathscr{N}$-critical groups.

Now we prove the converse of the Theorem. Let H be the only non-nilpotent proper subgroup of G. By Theorem 2, $|H| = pq^m$ and $q \mid p-1$. We need to consider three cases $|G : H| \in \{p, q, r\}$, where r is a prime distinct from p and q.

Case 1. $|G : H| = p$. Then $|G| = p^2q^m$. Every proper subgroup of G distinct from H is the direct product of its Sylow subgroups, as such a subgroup is nilpotent. Noticing that each Sylow subgroup of G is abelian, it follows that each proper subgroup of G distinct from H is abelian, which implies that G is a 2-$\mathscr{A}$-critical group. So by [11, Theorem E], G is the group (1).

Case 2. $|G : H| = q$. Then $|G| = pq^{m+1}$. Let P be a Sylow p-subgroup and Q be a Sylow q-subgroup of G. Let $Q_1 = \langle a\rangle$ be a Sylow q-subgroup of H. We claim that Q is cyclic. Suppose for a contradiction that Q is non-cyclic. Hence Q is metacyclic and so $Q = \langle a, b\rangle$. Since $P\langle ab\rangle$ and $P\langle b\rangle$ are nilpotent, $[P, ab] = [P, b] = 1$. So $[P, a] = 1$, a contradiction. Therefore Q is cyclic and consequently G is a 2-$\mathscr{A}$-critical group. Thus, according to [11, Theorem F], G is the group (2).

Case 3. $|G : H| = r \notin \{p, q\}$. Then $|G| = pq^m r$. As each non-trivial proper subgroup of G distinct from H is nilpotent, it is the direct product of its Sylow subgroups. Noticing that each Sylow subgroup of G is abelian, it follows that each non-trivial proper subgroup distinct from H is abelian. This yields that G is a 2-$\mathscr{A}$-critical group. Thus, according to [11, Theorem D], G is the group (3). □

Proof Theorem B. First we prove that all groups of (I)-(V) are 2-$\mathscr{N}$-critical. Let G be the group (I). As each subgroup of G is of the form $A \times B$, where A is a subgroup of H and B is a subgroup of $\mathbb{Z}_r$ and H is an Schmidt group, we infer that G is a 2-$\mathscr{N}$-critical group.

Let G be the group (II). Put $H = G' \rtimes \mathbb{Z}_{q^m}$ and suppose that $|H| = p^nq^m$, where n, m are positive integers. We show that each maximal subgroup M of G distinct from H is nilpotent and so G is 2-$\mathscr{N}$-critical. First suppose that M is normal in G. If $|M| = |H| = p^nq^m$, then $|M \cap H| = p^{n-1}q^m$. Since H and M are normal maximal subgroups of G, $G' \subseteq M \cap H$, which is impossible. Therefore $|M| \neq |H|$ and hence $|M| = p^{n+1}q^{m-1}$. As $\Phi(G) \subseteq M$ and $q^{m-1} \mid |\Phi(G)|$, a Sylow q-subgroup of M is normal in M. Thus M is the direct product of its Sylow subgroups and so M is nilpotent. Next, suppose that M is non-normal in G. If $|M| = p^iq^m$, where $0 \leq i \leq n$, then $|M \cap H| = p^{i-1}q^m$. Noticing that $M \cap H$ is a nilpotent normal subgroup of M, it follows that a Sylow q-subgroup of M

is normal in M. Since $F(G) \nsubseteq M$, $|F(G)\cap M| = p^i q^{m-1}$ and so a Sylow p-subgroup of M is normal in M. It follows that M is nilpotent.

If $|M| = p^{n+1}q^j$, where $0 \le j \le m-1$, then since $\Phi(G) \subseteq M$, we have $q^{m-1} \mid |M|$. It follows that each Sylow subgroup of M is normal in M and so M is nilpotent. Therefore G is a 2-$\mathscr{N}$-critical group.

Let G be the group (III). We put $H = G' \rtimes \langle c^q\rangle$. Then a similar argument shows that every maximal subgroup of G distinct from H is nilpotent and thus G is 2-$\mathscr{N}$-critical.

Let G be the group (IV). We put $H = G'Q$. Clearly G is a non-nilpotent normal subgroup of G. Since $H \cap L = \Phi(G')$ and $G = HL$, we have

$$1 \ne |G : H| = |HL : H| = |L : H \cap L| \le |L : \Phi(G')| = p.$$

Since $H = G'Q$ it follows that $q \nmid |G : H| = p$ and so $|G : H| = p$.

Now a similar argument to (II) shows that every maximal subgroup of G distinct from H is nilpotent, which implies that G is a 2-$\mathscr{N}$-critical group.

Finally, assume that G is the group (V). Let $Q = \langle c\rangle$. Similar to group (II) we can see that $H = G'\langle c^q\rangle$, is the unique non-nilpotent proper subgroup of G.

Now we prove the converse of the Theorem. Suppose that G is non-supersolvable with a unique non-nilpotent proper subgroup H. By virtue of Lemma 3, H is non-supersolvable.

As H is an Schmidt group, $|H| = p^n q^m$ and it is one of the groups of Theorem 2. Since H is non-supersolvable, it can not be of type (3) of Theorem 2. Suppose that P_1 and Q_1 are Sylow p-subgroup and Sylow q-subgroup of H, respectively, where $Q_1 = \langle a\rangle$ is cyclic. Let P and Q be Sylow subgroups of G, where $P_1 \le P$ and $Q_1 \le Q$. Note that since P_1 is a characteristic subgroup of H, it follows that $P_1 \trianglelefteq G$. If Q is non-cyclic, then it is meta-cyclic so we can write $Q = \langle a, b\rangle$. Since $P_1\langle ab\rangle$ and $P_1\langle b\rangle$ are proper subgroups of G distinct from H, they are nilpotent. Thus $[P_1, ab] = [P_1, b] = 1$, and so $[P_1, a] = 1$. Hence $H = P\langle a\rangle$ is abelian, which is a contradiction. Therefore Q is cyclic. In the following we assume that $Q = \langle c\rangle$.

We claim that P is normal in G. If $|G : H| \ne p$, then $P = P_1 \trianglelefteq G$. If $|G : H| = p$, then $P \trianglelefteq G$. Otherwise, since every non-normal maximal subgroups of G are nilpotent, G has a normal Sylow subgroup with nilpotent quotient [7, Theorem 1]. Since Q is non-normal and H is non-nilpotent, we must have $P \trianglelefteq G$. Therefore in any case, P is normal in G.

Let $|G : H| = r \notin \{p, q\}$. Then $|G| = p^n q^m r$. Since G is solvable, by 9.1.7 of [8], there exists a Hall p'-subgroup T of G. Since T is a proper subgroup of G distinct from H, it is nilpotent, and hence $T = Q_2 \times R_1$, where $|R_1| = r$ and $Q_2 = Q^g$, for some $g \in G$. Now $R := R_1^{g^{-1}}$ is a Sylow r-subgroup of G and $1 = [Q_2, R_1] = [Q^g, R_1] = [Q, R]^{g^{-1}}$. Hence $[Q, R] = 1$. Also since P is normal in G, PR is a subgroup of G and so is nilpotent. Hence $[P, R] = 1$. It follows that $R \subseteq Z(G)$. Therefore $G \cong H \times \mathbb{Z}_r$, where H is an Schmidt group, the group mentioned in (I).

Now we assume that $|G : H| \in \{p, q\}$. Now two cases occur:

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Case 1. H is of type (1) of Theorem 2. Then P_1 is an elementary abelian irreducible Q_1-module. Since $|Q_1 \cap C_H(P_1)| = q^{m-1}$, we have $|Q_1 : C_{Q_1}(P_1)| = q$. As $|G/P_1| \in \{pq^m, q^{m+1}\}$ and $q \nmid p-1$, so G/P_1 is cyclic. Thus $G' \subseteq P_1$. On the other hand, by Theorem 2, $H' = P_1$. Therefore $G' \subseteq P_1 = H' \subseteq G'$ and so $G' = P_1$.

If $|G : H| = p$, then $Q = Q_1$. Since $\Phi(P) < P_1$ and P_1 is an irreducible Q_1-module, $\Phi(P) = 1$. Since P_1 is a Q-invariant subgroup of P and the action of Q on P is coprime, by 8.4.5 of [3], P_1 has a Q-invariant complement L in P. Since LQ is nilpotent, $[L, Q] = 1$. Therefore $G = P_1Q \times L = H \times L$, which is the group (II).

If $|G : H| = q$, then $P = P_1$. In this case $H = PQ_1$, $q \nmid p-1$. Since $Q_1 = \langle c^q \rangle$ acts fixed point freely on P, by 8.1.12 of [3], H is a Frobenius group. Hence $\langle c^{q^2} \rangle = Z(H) = 1$ and thus $G = P \rtimes Q \cong P \rtimes \mathbb{Z}_{q^2}$. Let X be the set of subgroups of order p of P. Since Q_1 acts irreducibly on P, it follows that Q acts irreducibly on P. Hence Q acts irreducibly on X. In particular by orbit-stabilizer Theorem, q^2 divides $|X| = (p^n - 1)/p - 1$. Therefore G is the group (III).

Case 2. H is the group of type (2) of Theorem 2. In this case $q \nmid p-1$. Now similar to the Case 1, we have $G' = P_1$. If $|G : H| = p$, then since $\Phi(P) < P_1$, we have $\Phi(P) = \Phi(P_1)$. Since $P/\Phi(P)$ is elementary abelian, $P_1/\Phi(P)$ has a complement in $P/\Phi(P)$. Now, 8.4.5 of [3] yields that $P_1/\Phi(P)$ has a Q-invariant complement in $P/\Phi(P)$. Thus $P/\Phi(P) = P_1/\Phi(P) \times L/\Phi(P)$, where $L/\Phi(P)$ is a subgroup of $P/\Phi(P)$. Since LQ is nilpotent, $[L, Q] = 1$. Therefore $G = HL = G'QL$, $G' \cap L = \Phi(G')$ and G is the group (IV).

Finally assume that $|G : H| = q$. Then $P = P_1$ and $H = G'\langle c^q \rangle$ is an Schmidt group and $\langle c^q \rangle$ acts irreducibly on $G'/\Phi(G')$. Hence $\langle c \rangle$ also acts irreducibly on $G'/\Phi(G')$. Since $P\langle c^{q^2} \rangle$ and $\Phi(G')Q$ are proper subgroups distinct from H, they are nilpotent, so $[P, c^{q^2}] = [\Phi(G'), Q] = 1$. Now similar to the Case (1), the order of p modulo q^2 being $2l$ and $|G''| \leq p^l$. So G is the group (V). □

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