CHARACTERIZATION OF FINITE GROUPS WITH A UNIQUE NON-NILPOTENT PROPER SUBGROUP

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ABSTRACT. We characterize finite non-nilpotent groups $G$ with a unique non-nilpotent proper subgroup. We show that $|G|$ has at most three prime divisors. When $G$ is supersolvable we find the presentation of $G$ and when $G$ is non-supersolvable we show that either $G$ is a direct product of an Schmidt group and a cyclic group or a semi direct product of a $p$-group by a cyclic group of prime power order.

1. Introduction

Throughout the paper all groups are assumed to be finite. Our notation and terminology are standard taken mainly from [8]. In particular the size of a finite group $G$ is shown by $|G|$. The center, the derived subgroup, and the Frattini subgroup of $G$ are denoted by $Z(G)$, $G'$, and $\Phi(G)$, respectively. For $x, g \in G$, $x^g := g^{-1}xg$ is the conjugate of $x$ by $g$; and $C_G(x)$ is the centralizer $x$ in $G$. For a subgroup $M$ of $G$, the centralizer of $M$ in $G$ is denoted by $C_G(M)$. The symbol $G = Y \rtimes X$ indicates that $G$ is a split extension (semi direct product) of a normal subgroup $Y$ of $G$ by a complement $X$.

Let $\mathcal{X}$ be a class of groups. A group $G$ that is not in $\mathcal{X}$ but each proper subgroup of $G$ is contained in $\mathcal{X}$, is said to be an $\mathcal{X}$-critical group or a minimal non-$\mathcal{X}$-group. Some people have studied the structure of $\mathcal{X}$-critical groups for a various classes of groups. For example Miller and Moreno [4]
considered the class of abelian groups, $\mathcal{A}$, and studied $\mathcal{A}$-critical groups, also called minimal non-abelian groups, and showed that such groups are solvable. Also they showed that if a $\mathcal{A}$-critical group $G$ is nilpotent, then it is a $p$-group and $|G : \Phi(G)| = |G : Z(G)| = p^2$. After that Rédei [5] continued the study of minimal non-abelian groups and classified such groups completely. We classified finite groups having a unique non-abelian proper subgroup in [11].

Schmidt [10] considered the class of nilpotent groups, $\mathcal{N}$, and studied $\mathcal{N}$-critical groups which also called minimal non-nilpotent groups (or Schmidt groups). Rédei [6] completely classified finite Schmidt groups. Itô [2] considered the minimal non-$p$-nilpotent groups for $p$ a prime and proved that such groups are just Schmidt groups. Ballester-Bolinches and Esteban-Romero [1], considered the class of supersolvable groups, $\mathcal{U}$, and studied the structure of $\mathcal{U}$-critical groups which also called minimal non-supersolvable groups.

Let $n$ be a positive integer. We say that a group $G$ is an $n$-$\mathcal{X}$-critical group, if $G \notin \mathcal{X}$ and has exactly $n - 1$ proper subgroups that are not belong to $\mathcal{X}$ and other subgroups belong to $\mathcal{X}$. Therefore a minimal non-$\mathcal{X}$-group is just a 1-$\mathcal{X}$-critical group.

Russo [9] studied 2-$\mathcal{N}$-critical groups, where $\mathcal{N}$ is the class of nilpotent groups, in infinite case and obtained some results in finite case. Also by the main result in [12], one can see that every finite $n$-$\mathcal{N}$-critical groups with $n \leq 21$ is solvable. In this paper we classify finite 2-$\mathcal{N}$-critical groups, i.e., finite non-nilpotent groups with a unique proper non-nilpotent subgroup. We show that each finite supersolvable 2-$\mathcal{N}$-critical group is a 2-$\mathcal{A}$-critical group.

First we characterize finite supersolvable 2-$\mathcal{N}$-critical groups:

**Theorem A.** Let $G$ be a finite non-nilpotent supersolvable group. Then $G$ has a unique non-nilpotent proper subgroup if and only if $G$ is one of the following groups.

1. $\langle a, b, c \mid a^p = b^q = c^{q^m} = 1, [a, b] = [a, c] = [b, c^q] = 1, b^c = a^i b^j \rangle$, where $p$ and $q$ are distinct prime numbers $0 \leq i, j \leq p - 1$, $j^q \equiv 1 \pmod{p}$ and if $i \neq 0$, then $1 + j + \cdots + j^{q - 1} \equiv 0 \pmod{p}$,
2. $\langle a, b \mid a^p = b^q^{m+1} = 1, [a, b^q] = 1, a^b = a^i \rangle$, where $q \mid p - 1$, $0 \leq i \leq p - 1$, $i \neq 1$, $m \geq 1$ and $i^{q^2} \equiv 1 \pmod{p}$,
3. $\langle a, b, c \mid a^p = b^{q^m} = c^r = 1, [a, c] = [b, c] = [a, b^q] = 1, a^b = a^i \rangle$, where $p$, $q$ and $r$ are distinct prime numbers, $0 \leq i \leq p - 1$ and $i^q \equiv 1 \pmod{p}$.

By virtue of [11] and using Theorem A, we have the following corollary.

**Corollary 1.** Let $G$ be a finite supersolvable 2-$\mathcal{N}$-critical group. Then it is a 2-$\mathcal{A}$-critical group. The converse is true if the unique non-abelian subgroup of $G$ is non-nilpotent.

Next we characterize 2-$\mathcal{N}$-critical non-supersolvable groups (in what follows $p$, $q$ and $r$ are prime numbers):

**Theorem B.** Let $G$ be a finite non-supersolvable group. Then $G$ is 2-$\mathcal{N}$-critical if and only if it is one of the following groups.
(I) $H \times \mathbb{Z}_r$, where $H$ is a non-supersolvable Schmidt group whose order is coprime to $r$.

(II) $H \times \mathbb{Z}_p$, where $H$ is a non-supersolvable Schmidt group of order $p^n q^m$, $q \nmid p - 1$. Also the Sylow $p$-subgroup of $H$ is an irreducible $Q$-module over the field of $p$ elements, $Q$ is cyclic and $|Q : C_Q(P)| = q$.

(III) $PQ$, where $P = G'$ is a minimal normal subgroup of $G$, $|P| = p^n$, and $Q = \langle c \rangle$ is of order $q^2$, the order of $p$ modulo $q^2$ being $n$. Furthermore $G'\langle c^q \rangle$ is an Schmidt group and $G'$ is a $(c^q)$-irreducible module.

(IV) $G'QL$, where $G'$ is a non-abelian special $p$-group of order $2l$, $Q$ is cyclic of order $q^m$, the order of $p$ modulo $q$ being $2l$. Also $G' \cap L = \Phi(G')$, $|L| = p|\Phi(G')|$, $[L, Q] = 1$, $G'/\Phi(G')$ is a faithful irreducible $Q$-module and $[\Phi(G'), Q] = 1$. Furthermore $|G'| \leq p^l$.

(V) $G'Q$, where $G'$ is a non-abelian special $p$-group of rank $2l$, $Q = \langle c \rangle$ is cyclic of order $q^{m+1}$, the order of $p$ modulo $q^2$ being $2l$. Also $G'\langle c^q \rangle$ is an Schmidt group, $G'/\Phi(G')$ is a faithful irreducible $Q$-module and $[\Phi(G'), Q] = 1 = [P, c^q]$. Furthermore $|G'| \leq p^l$.

2. Proofs

We begin by stating the following Theorem on the structure of finite Schmidt groups.

**Theorem 2.** [10, 5] Let $G$ be a finite Schmidt group. Then $G = P \times Q$, where $P$ is a Sylow $p$-subgroup and $Q = \langle z \rangle$ is a cyclic Sylow $q$-subgroup of order $q^r > 1$. Furthermore $Z(G) = \Phi(G) = \Phi(P) \times \langle z^q \rangle$; $G' = P$, $P' = G'' = \Phi(P)$, and one of the following cases hold:

1. $q$ does not divide $p - 1$ and $P$ is an irreducible $Q$-module over the field of $p$ elements with kernel $\langle z^q \rangle$ in $Q$. The subgroup $P$ is elementary abelian minimal normal $p$-subgroup of order $p^l$ where $l$ is the order of $p$ modulo $q$.

2. $P$ is a non-abelian special $p$-group, $|P/\Phi(P)| = p^{2m}$, $|P'| \leq p^m$, the order of $p$ modulo $q$ being $2m$, $z$ induces an automorphism in $P$ such that $P/\Phi(P)$ is a faithful irreducible $Q$-module, and $z$ centralizes $\Phi(P)$.

3. $q$ divides $p - 1$, $P = \langle a \rangle$ is cyclic of order $p$, and $a^z = a^i$, where $i$ is the least primitive $q$-th root of unity modulo $p$.

Note that Schmidt groups satisfying (1) and (2) of Theorem 2 are non-supersolvable and Schmidt groups satisfying (3) are supersolvable. In fact if $G$ is supersolvable and satisfies (1), then since $P$ is a minimal normal subgroup $G$, Theorem 5.4.7 of [8] implies that $|P| = p$. It follows that $q \mid p - 1$, a contradiction. Also if $G$ satisfies (2) and is supersolvable, then $P/\Phi(P)$ is a minimal normal subgroup of supersolvable Schmidt group $G/\Phi(P)$, which implies that $P$ is cyclic, a contradiction. Finally if $G$ satisfies (3), then $1 < P < G$ is a normal series with cyclic factors and so $G$ is supersolvable.

Let $G$ be a finite non-nilpotent group with a unique non-nilpotent proper subgroup $H$. It is readily seen that $H$ is an Schmidt group. Furthermore $H$ is a maximal and characteristic subgroup of $G$.

We have the following simple Lemma.
Lemma 3. Let $G$ be a finite non-nilpotent group with a unique non-nilpotent proper subgroup $H$. If $H$ is supersolvable, then $G$ is supersolvable.

Proof. Since $H$ is a normal maximal subgroup of $G$, $G/H$ is cyclic of prime order. Since $H$ is a supersolvable Schmidt group, from Theorem 2, we have $|H| = pq^m$ and a Sylow $p$-subgroup $P$ of $H$ is cyclic. Now $1 < P < H < G$ is a normal series with cyclic factors and so $G$ is supersolvable. 

Proof Theorem A. First we prove that all groups (1) – (3) have exactly one non-nilpotent proper subgroup.

Owing to [11, Theorems E, F and D], the groups (1) – (3) are non-nilpotent supersolvable $2$-$\mathcal{A}$-critical and so are $2$-$\mathcal{N}$-critical groups.

Now we prove the converse of the Theorem. Let $H$ be the only non-nilpotent proper subgroup of $G$. By Theorem 2, $|H| = pq^m$ and $q | p - 1$. We need to consider three cases $|G : H| \in \{p, q, r\}$, where $r$ is a prime distinct from $p$ and $q$.

Case 1. $|G : H| = p$. Then $|G| = p^2q^m$. Every proper subgroup of $G$ distinct from $H$ is the direct product of its Sylow subgroups, as such a subgroup is nilpotent. Noticing that each Sylow subgroup of $G$ is abelian, it follows that each non-trivial proper subgroup of $G$ distinct from $H$ is abelian, which implies that $G$ is a $2$-$\mathcal{A}$-critical group. So by [11, Theorem E], $G$ is the group (1).

Case 2. $|G : H| = q$. Then $|G| = pq^{m+1}$. Let $P$ be a Sylow $p$-subgroup and $Q$ be a Sylow $q$-subgroup of $G$. Let $Q_1 = \langle a \rangle$ be a Sylow $q$-subgroup of $H$. We claim that $Q$ is cyclic. Suppose for a contradiction that $Q$ is non-cyclic. Hence $Q$ is metacyclic and so $Q = \langle a, b \rangle$. Since $P \langle ab \rangle$ and $P \langle b \rangle$ are nilpotent, $[P, ab] = [P, b] = 1$. So $[P, a] = 1$, a contradiction. Therefore $Q$ is cyclic and consequently $G$ is a $2$-$\mathcal{A}$-critical group. Thus, according to [11, Theorem F], $G$ is the group (2).

Case 3. $|G : H| = r \notin \{p, q\}$. Then $|G| = pq^mr$. As each non-trivial proper subgroup of $G$ distinct from $H$ is nilpotent, it is the direct product of its Sylow subgroups. Noticing that each Sylow subgroup of $G$ is abelian, it follows that each non-trivial proper subgroup distinct from $H$ is abelian. This yields that $G$ is a $2$-$\mathcal{A}$-critical group. Thus, according to [11, Theorem D], $G$ is the group (3).

Proof Theorem B. First we prove that all groups of (I)-(V) are $2$-$\mathcal{A}$-$\mathcal{N}$-critical. Let $G$ be the group (I). As each subgroup of $G$ is of the form $A \times B$, where $A$ is a subgroup of $H$ and $B$ is a subgroup of $\mathbb{Z}_r$ and $H$ is an Schmidt group, we infer that $G$ is a $2$-$\mathcal{A}$-$\mathcal{N}$-critical group.

Let $G$ be the group (II). Put $H = G' \times \mathbb{Z}_{q^m}$ and suppose that $|H| = p^nq^m$, where $n, m$ are positive integers. We show that each maximal subgroup $M$ of $G$ distinct from $H$ is nilpotent and so $G$ is $2$-$\mathcal{N}$-critical. First suppose that $M$ is normal in $G$. If $|M| = |H| = p^nq^m$, then $|M \cap H| = p^{n-1}q^m$. Since $H$ and $M$ are normal maximal subgroups of $G$, $G' \subseteq M \cap H$, which is impossible. Therefore $|M| \neq |H|$ and hence $|M| = p^{n+1}q^{m-1}$. As $\Phi(G) \subseteq M$ and $q^{m-1} | |\Phi(G)|$, a Sylow $q$-subgroup of $M$ is normal in $M$. Thus $M$ is the direct product of its Sylow subgroups and so $M$ is nilpotent. Next, suppose that $M$ is non-normal in $G$. If $|M| = p^iq^m$, where $0 \leq i \leq n$, then $|M \cap H| = p^{i-1}q^m$. Noticing that $M \cap H$ is a nilpotent normal subgroup of $M$, it follows that a Sylow $q$-subgroup of $M$
is normal in $M$. Since $F(G) \not\subseteq M$, $|F(G) \cap M| = p^iq^{m-1}$ and so a Sylow $p$-subgroup of $M$ is normal in $M$. It follows that $M$ is nilpotent.

If $|M| = p^{j+1}q^j$, where $0 \leq j \leq m - 1$, then since $\Phi(G) \subseteq M$, we have $q^{m-1} \mid |M|$. It follows that each Sylow subgroup of $M$ is normal in $M$ and so $M$ is nilpotent. Therefore $G$ is a 2-$\mathcal{N}$-critical group.

Let $G$ be the group (III). We put $H = G' \rtimes \langle c^g \rangle$. Then a similar argument shows that every maximal subgroup of $G$ distinct from $H$ is nilpotent and thus $G$ is 2-$\mathcal{N}$-critical.

Let $G$ be the group (IV). We put $H = G'Q$. Clearly $G$ is a non-nilpotent normal subgroup of $G$. Since $H \cap L = \Phi(G')$ and $G = HL$, we have

$$1 \neq |G : H| = |HL : H| = |L : H \cap L| \leq |L : \Phi(G')| = p.$$ 

Since $H = G'Q$ it follows that $q \nmid |G : H| = p$ and so $|G : H| = p$.

Now a similar argument to (II) shows that every maximal subgroup of $G$ distinct from $H$ is nilpotent, which implies that $G$ is a 2-$\mathcal{N}$-critical group.

Finally, assume that $G$ is the group (V). Let $Q = \langle c \rangle$. Similar to group (II) we can see that $H = G'\langle c^g \rangle$, is the unique non-nilpotent proper subgroup of $G$.

Now we prove the converse of the Theorem. Suppose that $G$ is non-supersolvable with a unique non-nilpotent proper subgroup $H$. By virtue of Lemma 3, $H$ is non-supersolvable.

As $H$ is an Schmidt group, $|H| = p^nq^m$ and it is one of the groups of Theorem 2. Since $H$ is non-supersolvable, it can not be of type (3) of Theorem 2. Suppose that $P_1$ and $Q_1$ are Sylow $p$-subgroup and Sylow $q$-subgroup of $H$, respectively, where $Q_1 = \langle a \rangle$ is cyclic. Let $P$ and $Q$ be Sylow subgroups of $G$, where $P_1 \leq P$ and $Q_1 \leq Q$. Note that since $P_1$ is a characteristic subgroup of $H$, it follows that $P_1 \unlhd G$. If $Q$ is non-cyclic, then it is meta-cyclic so we can write $Q = \langle a, b \rangle$. Since $P_1\langle ab \rangle$ and $P_1\langle b \rangle$ are proper subgroups of $G$ distinct from $H$, they are nilpotent. Thus $[P_1, ab] = [P_1, b] = 1$, and so $[P_1, a] = 1$. Hence $H = P\langle a \rangle$ is abelian, which is a contradiction. Therefore $Q$ is cyclic. In the following we assume that $Q = \langle c \rangle$.

We claim that $P$ is normal in $G$. If $|G : H| \neq p$, then $P = P_1 \unlhd G$. If $|G : H| = p$, then $P \unlhd G$. Otherwise, since every non-normal maximal subgroups of $G$ are nilpotent, $G$ has a normal Sylow subgroup with nilpotent quotient [7, Theorem 1]. Since $Q$ is non-normal and $H$ is non-nilpotent, we must have $P \unlhd G$. Therefore in any case, $P$ is normal in $G$.

Let $|G : H| = r \notin \{p, q\}$. Then $|G| = p^nq^mr$. Since $G$ is solvable, by 9.1.7 of [8], there exists a Hall $p'$-subgroup $T$ of $G$. Since $T$ is a proper subgroup of $G$ distinct from $H$, it is nilpotent, and hence $T = Q_2 \times R_1$, where $|R_1| = r$ and $Q_2 = Q^g$, for some $g \in G$. Now $R := R_1^{g^{-1}}$ is a Sylow $r$-subgroup of $G$ and $1 = [Q_2, R_1] = [Q^g, R_1] = [Q, R]^{g^{-1}}$. Hence $[Q, R] = 1$. Also since $P$ is normal in $G$, $PR$ is a subgroup of $G$ and so is nilpotent. Hence $[P, R] = 1$. It follows that $R \subseteq Z(G)$. Therefore $G \cong H \times \mathbb{Z}_r$, where $H$ is an Schmidt group, the group mentioned in (I).

Now we assume that $|G : H| \in \{p, q\}$. Now two cases occur:
Case 1. \(H\) is of type (1) of Theorem 2. Then \(P_1\) is an elementary abelian irreducible \(Q_1\)-module. Since \(|Q_1 \cap C_H(P_1)| = q^{n-1}\), we have \(|Q_1 : C_{Q_1}(P_1)| = q\). As \(|G/P_1| \in \{pq^m, q^{m+1}\}\) and \(q \nmid p - 1\), so \(G/P_1\) is cyclic. Thus \(G' \subseteq P_1\). On the other hand, by Theorem 2, \(H' = P_1\). Therefore \(G' \subseteq P_1 = H' \subseteq G'\) and so \(G' = P_1\).

If \(|G : H| = p\), then \(Q = Q_1\). Since \(\Phi(P) < P_1\) and \(P_1\) is an irreducible \(Q_1\)-module, \(\Phi(P) = 1\). Since \(P_1\) is a \(Q\)-invariant subgroup of \(P\) and the action of \(Q\) on \(P\) is coprime, by 8.4.5 of \([3]\), \(P_1\) has a \(Q\)-invariant complement \(L\) in \(P\). Since \(LQ\) is nilpotent, \([L, Q] = 1\). Therefore \(G = P_1Q \times L = H \times L\), which is the group (II).

If \(|G : H| = q\), then \(P = P_1\). In this case \(H = PQ_1, q \nmid p - 1\). Since \(Q_1 = \langle c^q \rangle\) acts fixed point freely on \(P\), by 8.1.12 of \([3]\), \(H\) is a Frobenius group. Hence \(\langle c^q \rangle = Z(H) = 1\) and thus \(G = P \times Q \cong P \times \mathbb{Z}_{q^2}\).

Let \(X\) be the set of subgroups of order \(p\) of \(P\). Since \(Q_1\) acts irreducibly on \(P\), it follows that \(Q\) acts irreducibly on \(P\). Hence \(Q\) acts irreducibly on \(X\). In particular by orbit-stabilizer Theorem, \(q^2\) divides \(|X| = (p^q - 1)/p - 1\). Therefore \(G\) is the group (III).

Case 2. \(H\) is the group of type (2) of Theorem 2. In this case \(q \nmid p - 1\). Now similar to the Case 1, we have \(G' = P_1\). If \(|G : H| = p\), then since \(\Phi(P) < P_1\), we have \(\Phi(P) = \Phi(P_1)\). Since \(P/\Phi(P)\) is elementary abelian, \(P_1/\Phi(P)\) has a complement in \(P/\Phi(P)\). Now, 8.4.5 of \([3]\) yields that \(P_1/\Phi(P)\) has a \(Q\)-invariant complement in \(P/\Phi(P)\). Thus \(P/\Phi(P) = P_1/\Phi(P) \times L/\Phi(P)\), where \(L/\Phi(P)\) is a subgroup of \(P/\Phi(P)\). Since \(LQ\) is nilpotent, \([L, Q] = 1\). Therefore \(G = HL = G'QL, G' \cap L = \Phi(G')\) and \(G\) is the group (IV).

Finally assume that \(|G : H| = q\). Then \(P = P_1\) and \(H = G'(c^q)\) is an Schmidt group and \(\langle c^q \rangle\) acts irreducibly on \(G'/\Phi(G')\). Hence \(\langle c \rangle\) also acts irreducibly on \(G'/\Phi(G')\). Since \(P/\langle c' q^2 \rangle\) and \(\Phi(G')Q\) are proper subgroups distinct from \(H\), they are nilpotent, so \([P, c^q] = \Phi(G'), Q] = 1\). Now similar to the Case (1), the order of \(p\) modulo \(q^2\) being \(2l\) and \(|G''| \leq p^l\). So \(G\) is the group (V).

\[\square\]

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