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INTEGRAL FORMS IN VERTEX OPERATOR ALGEBRAS, A SURVEY

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Abstract. We give a brief survey of recent work on integral forms in vertex operator algebras (VOAs).

1. Introduction

The definition of a vertex operator algebra (abbreviated VOA) is too long to give here. We refer the reader to a standard reference for VOA theory: [7], definition on page 244. A few main points: $V = \bigoplus_{n\geq 0} V_n$, a graded vector space in characteristic 0 with each $dim(V_i)$ finite; vacuum element $\mathbf{1} \in V_0$, Virasoro element $\omega \in V_2$; a linear monomorphism $Y : V \to End(V)[[z^{-1}, z]]$, written $Y(v, z) = \sum_{n\in\mathbb{Z}} v_n z^{-n-1}$.

For each $n \in \mathbb{Z}$, there is a product $a, b \mapsto a_n b$ (meaning the result of the endomorphism a_n applied to b), giving a ring (V, n^{th}).

A vertex algebra (VA) is a generalization of VOA to graded modules over a commutative rings of scalars. It has a vacuum element but not necessarily a Virasoro element. Our examples will be over finite fields or the integers.

Definition 1.1. An integral form in an algebra in characteristic 0 is the $\mathbb{Z}$-span of a basis which is closed under the product.

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Example 1.2. (1) $Mat_{n\times n}(\mathbb{Z}) \leq Mat_{n\times n}(\mathbb{C})$; (2) In a simple finite dimensional complex Lie algebra, the $\mathbb{Z}$-span of a Chevalley basis is an integral form.

Example 1.3. [7] For an even integral lattice L, there is a lattice type VOA V_L which, as a graded vector space, has shape $\mathbb{S}(\hat{H})\otimes\mathbb{C}[L]$. Here, $\mathbb{S}$ means symmetric algebra of a vector space, $H := \mathbb{C}\otimes_{\mathbb{Z}} L$ and $\hat{H}$ means $H_1 \oplus H_2 \oplus \dots$ where H_k is a copy of H declared to have degree k. Finally $\mathbb{C}[L]$ is the group algebra of the abelian group L, with basis e^α, for $\alpha \in L$.

Example 1.4. [7] The Moonshine VOA $V^\natural$ is a twisted version of V_Λ, where Λ is the Leech lattice (rank 24, determinant 1, minimum norm 4). It has $Aut(V^\natural) \cong \mathbb{M}$, the Monster.

Definition 1.5. An *integral form* R in a vertex operator algebra $V = \bigoplus_{n\in\mathbb{Z}} V_n$ with a nondegenerate symmetric bilinear form is the $\mathbb{Z}$-span of a basis which is closed under all the VOA products and for all n, $R \cap V_n$ is an integral form of the vector space V_n; also R must contain the vacuum element and a positive integer multiple of the Virasoro element.

So, an integral form in a VOA is a vertex algebra over the ring of integers.

With Chongying Dong, we [5, 6] studied the following question. Given a finite group G in $Aut(V)$, is there an integral form in V which is stable under G? We gave some general sufficient conditions.

Our main applications: (1) for lattice type VOAs, (L of rank r) there is a G-invariant integral form where G has the form $2^r.O(L)$, a downward extension of the isometry group $O(L)$ of the lattice L by an elementary abelian group of order 2^r. (One description of the integral form: it is generated as a vertex algebra over the integers by the $\mathbb{Z}$-span of the components of $Y(e^\alpha, z)\mathbf{1}$ for $\alpha \in L$.)

(2) For the Moonshine VOA $V^\natural$, we proved that there is a Monster-invariant integral form. (This is created by a kind of averaging, and is not (yet) described explicitly.)

The recent preprint of Carnahan [2] proves existence of a Monster-invariant integral form in which every graded component has determinant 1. This form is not given explicitly.

Remark 1.6. We learned after our proof was written, that in the 80s, Borcherds had asserted the existence of an integral form for lattice type VOAs. Borcherds also observed that there is a Monster-invariant $\mathbb{Z}[\frac{1}{2}]$-form in $V^\natural$ but claimed nothing about a form over $\mathbb{Z}$.

If J is an integral form in a VOA, it inherits a symmetric bilinear form from the VOA. We say J is *lattice integral* if $\langle x, y\rangle \in \mathbb{Z}$ for all $x, y \in J$. It is unclear when the restriction of the form to $J \times J$ is integral-valued (or a multiple by some positive integer is integral valued).

We gave two sufficient conditions to prove lattice integrality. (1) We showed that it is integral valued whenever the integral form J is generated by quasi-primary vectors (in VOA theory, this means vectors annihilated by a certain operator $L(1)$). This criterion applies to the Monster-invariant form we built earlier. (2) We gave an averaging-type argument.

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2. **Classical lattice type VOAs**

This section represents joint work with Ching Hung Lam [9, 10]. Consider the case of lattice type VOA where L is a root lattice of type ADE, and let $V := V_L$. Then $(V_1, 0^{th})$ is a copy of the Lie algebra associated to L. *If J is our integral form, $J \cap V_1$ is the $\mathbb{Z}$-lattice spanned by a Chevalley basis! So, this J is spanned as an abelian group by a set of elements which generalizes "Chevalley basis".* It turns out that a Chevalley group can be defined on V_L with the standard generators $x_r(\pm 1)$ fixing the integral form.

We can also take any commutative associative ring R and form $R \otimes J$, a vertex algebra over R, called *the classical VA of type L over R*. We also get an action of the Chevalley group of type L over R on $R \otimes J$ as VA automorphisms.

When R is a field we get all Chevalley groups of types ADE (with graph outer automorphisms) as full automorphism groups of these VAs over R. We also defined VA over R for types BCGF. This gives the Steinberg variations (twisted Chevalley groups) acting on VAs and being essentially the full automorphism groups.

We would like to find a series of VAs whose automorphism groups are essentially the Ree and Suzuki groups but have not (yet) done so.

Remark 2.1. Our construction gives infinite dimensional graded modules for each Chevalley group and Steinberg variation over its field of definition. These modules may be a good opportunity for study of representation theory (*Ext*, indecomposables, etc.).

3. **The degree 2 component of a VOA**

Given a VOA $V = \oplus_{i \geq 0} V_i$, the k-th product gives a bilinear map $V_i \times V_j \longrightarrow V_{i+j-k-1}$. So, V_n under the $(n-1)^{th}$ product is a finite dimensional algebra, denoted $(V_n, (n-1)^{th})$.

In addition, (a) if $dim(V_0) = 1$, $(V_1, 0^{th})$ is a Lie algebra; (b) if $dim(V_0) = 1$ and $dim(V_1) = 0$, then $(V_2, 1^{st})$ is a *commutative algebra* with a symmetric, associative form $(ab, c) = (a, bc)$. Algebras as in (b) are sometimes called *Griess algebras* [11].

There are many examples of algebras (b) with finite automorphism groups, e.g., [13, 14], [3, 4]. The 196884-dimension algebra used to construct the Monster occurs this way in the Moonshine VOA $V^\natural$ [7].

Now suppose $dim(V_0) = 1$ and that $e \in V_2$ is a conformal vector [11] of central charge $\frac{1}{2}$ and that the subVOA generated by e is simple. Miyamoto [11] showed that e gives $t_e \in Aut(V)$ of order 1 or 2 (if order 2, t_e is called a *Miyamoto involution*).

In the special case of a *dihedral VOA* (a VOA which is generated by a pair of such conformal vectors e, f, as above), the degree 2 algebra (on the homogeneous component of degree 2, with the first product) has integral forms. Those which are *maximal* integral forms and invariant under the group $\langle t_e, t_f \rangle$ were classified in the thesis of Greg Simon (U Michigan, 2016)[15]. In most cases, t_e and t_f have order 2 so that $\langle t_e, t_f \rangle$ is a dihedral group. For the nine types of dihedral VOAs (classified by

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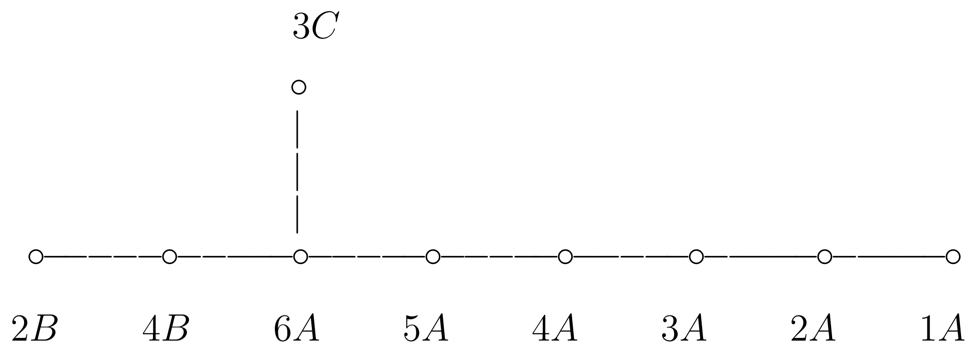
Sakuma [12]; they correspond to nodes of the *extended E_8-diagram*, displayed below), there is just one maximal invariant form in all cases but $2A$, in which case there are three.

(1)



This classification of maximal invariant forms in V_2 does not (yet) extend to invariant forms in the entire dihedral VOA.

4. Modular Moonshine of Borcherds and Ryba

Borcherds and Ryba wrote several articles [1, 16] about Modular Moonshine (positive characteristic) which imitated the story of the Monster and the graded representation $V^\natural$ and modular forms, but for smaller sporadic groups.

They discussed an interesting case. In $\mathbb{M}$, take g a $3C$-element; then $C(g) = \langle g \rangle \times S$, where $S \cong F_3$ a sporadic simple group of order $2^{15}3^{10}5^37^213 \cdot 19 \cdot 31$ (Thompson's group).

Borcherds and Ryba used K, Borcherds's $\mathbb{Z}[\frac{1}{2}]$-form in $V^\natural$, then considered its 0-th Tate cohomology group

$$\hat{H}^0(\langle g \rangle, K) := K^g/(1+g+g^2)K,$$

(notation K^g means the fixed points of g in K). This inherits structure to make a VA over $\mathbb{F}_3$. It looked like the classical E_8 type VA over $\mathbb{F}_3$, but nonzero terms occur only in degrees $0, 3, 6, \dots$ and have respective dimensions $1, 248, \dots$, just like for the genuine E_8 VA in degrees $0, 1, 2, \dots$ There was no obvious isomorphism (which triples degree of the grading) between these two VAs over $\mathbb{F}_3$.

To prove existence of an isomorphism, Lam and RLG adapted a covering idea of Frohardt-Griess [8], which is illustrated in the following example.

Example 4.1. F algebraically closed field of characteristic 3. The Lie algebra $a_2(F)$ has a 1-dimensional central ideal, Z. While $Aut(a_2(F))$ is $PGL(2,F){:}2$, $Aut(a_2(F)/Z) \cong G_2(F)$. Our proof takes the Lie algebra $d := d_4(F)$ and graph automorphism γ of order 3, then considers

$$0 < (1+\gamma+\gamma^2)d < d^\gamma < d, \quad \text{dimensions } 0,\ 7,\ 14,\ 28.$$

The group $G_2(F)$ acts on each subobject and on the 7-dimensional quotient Lie algebra $d^\gamma/(1+\gamma+\gamma^2)d$. One can see inside d (look at the long roots) a copy of $a_2(F)$ which maps onto $d^\gamma/(1+\gamma+\gamma^2)d$;

the image is isomorphic to $a_2(F)/Z$. Therefore $Aut(a_2(F)/Z)$ contains a copy of $G_2(F)$. This containment is equality.

Lam and RLG took M, the standard integral form for V_{E_8}, and a sublattice M' of K (integral form in $V^\natural$) which "covered" the Tate cohomology group $K^g/(1+g+g^2)K$. (*Roughly,* M' is the $\mathbb{Z}$-span of the image of M under a map suggested by $x \mapsto x \otimes x \otimes x$ for $x \in EE_8 \cong \sqrt{2}E_8$; think of the containment of lattices $EE_8 \perp EE_8 \perp EE_8 \leq \Lambda$, the Leech lattice). This led to an isomorphism.

An application of the Borcherds-Ryba theory is a new proof that the group F_3 of Thompson embeds in $E_8(3)$ (first proof 1974, by Thompson and P. Smith, used a study of Dempwolff decompositions and computer work). This VA viewpoint gives a nontrivial homomorphism of $C(g)/\langle g\rangle \cong F_3$ into the group $E_8(3)$ without knowing much about the structure of $C(g)$.

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Files of certain articles may be found at http://www.math.lsa.umich.edu/∼rlg/.

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