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THE FIBONACCI-CIRCULANT SEQUENCES IN THE BINARY POLYHEDRAL GROUPS

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Abstract. In 2017 Deveci et al. defined the Fibonacci-circulant sequences of the first and second kinds as shown, respectively:

$$x_n^1 = -x_{n-1}^1 + x_{n-2}^1 - x_{n-3}^1 \text{ for } n \geq 4, \text{ where } x_1^1 = x_2^1 = 0 \text{ and } x_3^1 = 1$$

and

$$x_n^2 = -x_{n-3}^2 - x_{n-4}^2 + x_{n-5}^2 \text{ for } n \geq 6, \text{ where } x_1^2 = x_2^2 = x_3^2 = x_4^2 = 0 \text{ and } x_5^2 = 1.$$

Also, they extended the Fibonacci-circulant sequences of the first and second kinds to groups. In this paper, we obtain the periods of the Fibonacci-circulant sequences of the first and second kinds in the binary polyhedral groups.

1. Introduction and Preliminaries

Let G be a finite j-generator group and let X be the subset of $\underbrace{G \times G \times G \times \cdots \times G}_{j}$ such that $(x_1, x_2, \ldots, x_j) \in X$ if and only if G is generated by $x_1, x_2, \ldots, x_j$. We call $(x_1, x_2, \ldots, x_j)$ a generating j-tuple for G.

Definition 1.1. ***(Deveci et al.*** [6]***).*** *For a generating j-tuple $(x_1, x_2, \ldots, x_j) \in X$, we define the Fibonacci-circulant orbits of the first and second kinds as follows, respectively:*

$$a_{n+3}^1 = \left(a_n^1\right)^{-1} \left(a_{n+1}^1\right) \left(a_{n+2}^1\right)^{-1}$$

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for $n \geq 1$, with initial conditions

$$\begin{cases} a_1^1 = (x_1)^{-1},\ a_2^1 = x_2,\ a_3^1 = x_3 & \text{if } j = 3, \\ a_1^1 = (x_1)^{-1},\ a_2^1 = (x_1)^{-1},\ a_3^1 = x_2 & \text{if } j = 2 \end{cases}$$

and

$$a_{n+3}^2 = \left(a_{n-2}^2\right)\left(a_{n-1}^2\right)^{-1}\left(a_n^2\right)^{-1}$$

for $n \geq 3$, with initial conditions

$$\begin{cases} a_1^2 = x_1,\ a_2^2 = x_2,\ a_3^2 = x_3,\ a_4^2 = x_4,\ a_5^2 = x_5 & \text{if } j = 5, \\ a_1^2 = x_1,\ a_2^2 = x_1,\ a_3^2 = x_2,\ a_4^2 = x_3,\ a_5^2 = x_4 & \text{if } j = 4, \\ a_1^2 = (x_1)^2,\ a_2^2 = x_1,\ a_3^2 = x_1,\ a_4^2 = x_2,\ a_5^2 = x_3 & \text{if } j = 3, \\ a_1^2 = (x_1)^3,\ a_2^2 = (x_1)^2,\ a_3^2 = x_1,\ a_4^2 = x_1,\ a_5^2 = x_2 & \text{if } j = 2. \end{cases}$$

For a j-tuple $(x_1, x_2, \ldots, x_j) \in X$, the Fibonacci-circulant orbits of the first and second kinds are denoted by $F^1_{(x_1,x_2,\ldots,x_j)}(G)$ and $F^2_{(x_1,x_2,\ldots,x_j)}(G)$, respectively.

A sequence of group elements is *periodic* if, after a certain point, it consists only of repetitions of a fixed subsequence. The number of elements in the repeating subsequence is called the *period of the sequence*. For example, the sequence $a, b, c, d, e, b, c, d, e, b, c, d, e, \cdots$ is periodic after the initial element a and has period 4. A sequence of group elements is *simply periodic* with period k if the first k elements in the sequence form a repeating subsequence. For example, the sequence $a, b, c, d, e, f, a, b, c, d, e, f, a, b, c, d, e, f, \cdots$ is simply periodic with period 6.

Theorem 1.2. ***(Deveci et al.*** [6]***)*** *The Fibonacci-circulant orbits of the first and second kinds of a finite group are simply periodic.*

In [6], the lengths of the periods of the orbits $F^1_{(x_1,x_2,\ldots,x_j)}(G)$ and $F^2_{(x_1,x_2,\ldots,x_j)}(G)$ were denoted by $LF^1(G; x_1, x_2, \ldots, x_j)$ and $LF^2(G; x_1, x_2, \ldots, x_j)$, respectively.

Definition 1.3. *The binary polyhedral group $\langle l, m, n \rangle$, for $l, m, n > 1$, is defined by the presentation*

$$\left\langle x, y, z : x^l = y^m = z^n = xyz \right\rangle.$$

When $l = 2$, then we obtain for $\langle 2, m, n \rangle$ the presentation

$$\left\langle y, z : y^m = z^n = (yz)^2 \right\rangle.$$

The binary polyhedral group $\langle l, m, n \rangle$ is finite if and only if the number $k = lmn\left(\frac{1}{l} + \frac{1}{m} + \frac{1}{n} - 1\right) = mn + nl + lm - lmn$ is positive. Its order is $4lmn/k$.

For more and detailed information on these groups one may consult [2, 4].
The Linear recurrence sequences in groups were firstly studied by Wall [15] who calculated the periods of the Fibonacci sequence in cyclic groups. Recently, many authors have studied some special linear recurrence sequences in groups see, for example, [1, 3, 5, 7, 8, 9, 10, 11, 12, 13, 14, 16]. In this paper, we discuss the lengths of the periods of the Fibonacci-circulant orbits of the first and second kinds

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of the binary polyhedral groups $\langle n,2,2\rangle$,$\langle 2,n,2\rangle$,$\langle 2,2,n\rangle$, for any n. We consider binary polyhedral groups as 3-generator groups.

2. Main Results and Proofs

Theorem 2.1. *Let G_n, $(n \geq 2)$ be one the binary polyhedral groups $\langle n,2,2\rangle$, $\langle 2,n,2\rangle$ and $\langle 2,2,n\rangle$ in the 3-generator cases. For the generating 3-tuple (x,y,z), the lengths of the periods of the Fibonacci-circulant orbits of the first kind of the group G_n are as follows:*

$$LF^1(G_n;x,y,z) = \begin{cases} 4n, & n \text{ is even,} \\ 8n, & n \text{ is odd.} \end{cases}$$

Proof. It is clear that for the generating 3-tuple (x,y,z), the Fibonacci-circulant orbits of the first kind of the group G_n have the following form:

$$a_1^1 = x^{-1},\ a_2^1 = y,\ a_3^1 = z \text{ and } a_{n+3}^1 = \left(a_n^1\right)^{-1}\left(a_{n+1}^1\right)\left(a_{n+2}^1\right)^{-1} \text{ for } n \geq 1.$$

Let us consider the group $\langle n,2,2\rangle = \left\langle x,y,z : x^n = y^2 = z^2 = xyz\right\rangle$ where, $|x| = 2n$, $|y| = |z| = 4$, Moreover,

$$\begin{aligned} y^2 &= z^2 & &\Rightarrow y^{-1}z = yz^{-1} \\ z^2 &= xyz & &\Rightarrow z = xy,\ \ z^{-1}y = x \\ x^n &= y^2 = z^2 & &\Rightarrow \left[x,y^2\right] = \left[x,z^2\right] = e. \end{aligned}$$

Then, the sequence $F^1_{(x,y,z)}(\langle n,2,2\rangle)$ is:

$$x^{-1},\ y,\ z,\ e,\ x^{-1},\ y^{-1},\ yx,\ x^{n+2},\ x^{-1},\ yx^4,\ x^{-7}y,\ \ldots.$$

The Fibonacci-circulant orbit of the first kind of the group $\langle n,2,2\rangle$ can be said to form layers of length eight. By using the above comments the sequence becomes:

$$\begin{aligned} &a_1^1 = x^{-1},\ a_2^1 = y,\ a_3^1 = z, \ldots, \\ &a_9^1 = x^{-1},\ a_{10}^1 = yx^4,\ a_{11}^1 = x^{-7}y, \ldots, \\ &a_{8i+1}^1 = x^{-1},\ a_{8i+2}^1 = yx^{4i\cdot\lambda_1},\ a_{8i+3}^1 = x^{-4i\cdot\lambda_2+1}y, \ldots, \end{aligned}$$

where λ_1 and λ_2 are positive integers with $(\lambda_1,\lambda_2) = 1$. So, we need the smallest $i \in N$ such that $4i = 2nv$ for $v \in N$.

If n is even, then $i = \frac{n}{2}$. Thus, $8i = 4n$ and $LF^1(\langle n,2,2\rangle;x,y,z) = 4n$.

If n is odd, then $n = i$. Thus, $8i = 8n$ and $LF^1(\langle n,2,2\rangle;x,y,z) = 8n$. □

The proofs for the groups $\langle 2,n,2\rangle$ and $\langle 2,2,n\rangle$ are similar to the above and are omitted.

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Theorem 2.2. *Let G_n, $(n \geq 2)$ be one the binary polyhedral groups $\langle n, 2, 2\rangle$, $\langle 2, n, 2\rangle$ and $\langle 2, 2, n\rangle$ in the 3-generator cases. For the generating 3-tuple (x, y, z), the lengths of the periods of the Fibonacci-circulant orbits of the second kind of the group G_n are as follows:*

$$LF^2(G_n; x, y, z) = \begin{cases} 7n, & n \equiv 0 mod 4, \\ 14n, & n \equiv 2 mod 4, \\ 28n, & otherwise. \end{cases}$$

Proof. It is clear that for the generating 3-tuple (x, y, z), the Fibonacci-circulant orbits of the second kind of the group G_n in the following form:

$a_1^2 = x^2,\ a_2^2 = x,\ a_3^2 = x,\ a_4^2 = y,\ a_5^2 = z$ and $a_{n+3}^2 = a_{n-2}^2 \left(a_{n-1}^2\right)^{-1} \left(a_n^2\right)^{-1}$ for $n \geq 3$.

To prove the assertion for the group $\langle 2, n, 2\rangle$ we choose a direct hand calculation method. Note that, in this group we have It is important to note that $|x| = |z| = 4$, $|y| = 2n$, $x = yz$, $z = xy = y^{-1}x$ and $y = x^{-1}z = xz^{-1}$. Then the sequence $F^2_{(x,y,z)}(\langle 2, n, 2\rangle)$ is:

$$\begin{aligned} &x^2,\ x,\ x,\ y,\ z,\ e,\ y^{-1},\ y^{-3}x,\ y^4,\ y^{n+3}, \\ &e,\ x^{-1},\ x^{-1}y^{-4},\ y^{n+1},\ y^3x^{-1},\ y^{n-4},\ y^{n-5}, \\ &y^{n+2},\ x,\ x^{-1}y^6,\ y^{n-1},\ x^{-1}y^7,\ y^{n+8},\ y^7, \\ &y^n,\ x,\ xy^{-8},\ y,\ y^7x,\ y^{-8},\ y^{-9},\ y^2,\ x^{-1}, \\ &xy^{10},\ y^{-1},\ y^{-11}x,\ y^{12},\ y^{n+11},\ e,\ x^{-1}, \\ &x^{-1}y^{-12},\ y^{n+1},\ y^{11}x^{-1},\ y^{n-12},\ y^{n-13}, \\ &y^{n+2},\ x,\ x^{-1}y^{14},\ y^{n-1},\ y^{-15}x^{-1},\ y^{n+16}, \\ &y^{15},\ y^n,\ x,\ xy^{-16},\ y,\ y^{15}x, \ldots. \end{aligned}$$

The sequence can be said to form layers of length twenty eight. Using the above, the sequence becomes:

$$\begin{aligned} &a_1^2 = x^2,\ a_2^2 = x,\ a_3^2 = x,\ a_4^2 = y,\ a_5^2 = z, \ldots, \\ &a_{29}^2 = x^2,\ a_{30}^2 = x,\ a_{31}^2 = xy^{-8},\ a_{32}^2 = y,\ a_{33}^2 = y^7x, \ldots, \\ &a_{57}^2 = x^2,\ a_{58}^2 = x,\ a_{59}^2 = xy^{-16},\ a_{60}^2 = y,\ a_{61}^2 = y^{15}x, \ldots, \\ &a_{28i+1}^2 = x^2,\ a_{28i+2}^2 = x,\ a_{28i+3}^2 = xy^{-8i\cdot\lambda_1},\ a_{28i+4}^2 = y,\ a_{28i+5}^2 = y^{8i\lambda_2-1}x, \ldots, \end{aligned}$$

where λ_1 and λ_2 are positive integers with $(\lambda_1, \lambda_2) = 1$. So, we need the smallest $i \in N$ such that $8i = 2nv$ for $v \in N$.

If $n \equiv 0 mod 4$, then $i = \frac{n}{4}$. Thus, $28i = 7n$ and $LF^2(\langle 2, n, 2\rangle; x, y, z) = 7n$.

If $n \equiv 2 mod 4$, then $i = \frac{n}{2}$. Thus, $28i = 14n$ and $LF^2(\langle 2, n, 2\rangle; x, y, z) = 14n$.

If n is odd, then $n = i$. Thus, $28i = 28n$ and $LF^2(\langle 2, n, 2\rangle; x, y, z) = 28n$. □

The proofs for the groups $\langle n, 2, 2\rangle$ and $\langle 2, 2, n\rangle$ are similar to the above and are omitted.

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