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ON THE POWER GRAPHS OF ELEMENTARY ABELIAN AND EXTRA SPECIAL p-GROUPS

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Abstract. For a given odd prime p, we investigate the power graphs of three classes of finite groups: the elementary abelian groups of exponent p, and the extra special groups of exponents p or p^2. We show that these power graphs are Eulerian for every p. As a corollary, we describe two classes of non-isomorphic groups with isomorphic power graphs. In addition, we prove that the clique graphs of the power graphs of two considered classes are complete.

1. Introduction

The present paper deals with power graphs considered, for example in [4, 5, 7, 15, 19]. The investigation of graphs associated to groups is a large and important research area (cf. [3], [8], [2],[16] ,[9] ,[10]). Graphs of this kind are significant because they have valuable applications in mathematics and computer science (cf., for instance, [12, 13, 14]). The directed power graph of a group was introduced by Kelarev and Quinn [6]. As explained in the survey [1], the definition given in [6] also covers all undirected graphs and applies to semigroups. The undirected power graphs were also considered by Chakrabarty, Ghosh and Sen [3]. In particular, it is an interesting problem to find classes of groups where each group can be identified by its power graph. Following the survey [1], we recall the definition of an undirected power graph $P(S)$ for any algebraic structure S with a power-associative binary operation. The vertex set of $P(S)$ is S and two vertices x and y are adjacent if and only if $x = y^m$ or $y = x^m$, for some integer $m \geq 2$. For a prime p, we consider three classes of groups:

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$G_1 = Z_p \times Z_p \times \cdots \times Z_p$ (the direct product of n copies), G_2 (the special p-group of order p^3 and exponent p^2), and G_3 (the special p-group of order p^3 and exponent p).

Constructions of these groups are based on the direct and semidirect product of semigroups (and of groups). The definition of a direct product is well-known. We follow [17, 20] to recall the definition of semidirect product of two semigroups and then describe its analogue for groups. For two semigroups S, T and a homomorphism $\phi : T \to End(S)$ the semidirect product of S by T, denoted by $S \rtimes_\phi T$ is a semigroup consistsing the ordered pairs (s, t) where $s \in S$ and $t \in T$ such that the multiplication is defined by:

$$(s,t)(s',t') = (s\phi_t(s'), tt'),\ \phi(t) = \phi_t \in End(S),$$

for all $s, s' \in S$ and $t, t' \in T$. Letting S and T be groups and substituting $Aut(S)$ (the automorphism group of the group S) for $End(S)$, the constructed group will be denoted by $S \rtimes T$ and is called the $semi-direct\ product$ of the group S by the group T. We recall the definitions of $Eulerian$ and $semi-Eulerian$ graphs as well. A connected graph is called $Eulerian$ or $semi-Eulerian$ if every vertex is of even degree, either exactly two vertices are of odd degrees. The *clique graph* of a connected graph G is denoted by $K(G)$. Every vertex of $K(G)$ is a maximal complete subgraph of G and two vertices of $K(G)$ are adjacent if and only if the corresponding complete subgraphs in G have precisely one vertex in common.

Finally we have to mention the notion of presentation of groups. For a useful and prolific information on the presentation theory of groups, one may consult [11, 18]. The group G_1 is a non-cyclic abelian group of order p^n. However, the groups G_2 and G_3 are non-abelian groups of order p^3, known as extra special p-groups of exponents p and p^2, respectively. Recall the groups G_1, G_2, and G_3 by their presentations,

$$\begin{aligned} G_1 &= \langle a_1, a_2, \ldots, a_n \mid a_i^p = 1, a_i a_j = a_j a_i,\ 1 \le i, j \le n \rangle, \\ G_2 &= \langle a, b \mid a^{p^2} = b^p = 1, b^{-1}ab = a^{p+1} \rangle, \\ G_3 &= \langle a, b, c \mid a^p = b^p = c^p = 1, ac = ca, bc = cb, c = [a, b] \rangle, \end{aligned}$$

where the notation $[a, b]$ is used for the commutator $a^{-1}b^{-1}ab$. Note that, the employed automorphism in constructing of G_2 is $\theta : Z_{p^2} \to Z_{p^2}$ where $\theta(a) = a^{p+1}$. Our main results concerning the power graphs are:

Theorem 2.3 *For every odd prime p and every integer $n \ge 2$, $P(G_1)$ is an Eulerian graph.*

Theorem 2.7 *For every odd prime p, $P(G_2)$ is an Eulerian graph.*

Theorem 2.8 *For every odd prime p, $P(G_3)$ is an Eulerian graph.*

Corollary 2.9 *For every odd prime p,*

(i) The non-isomorphic groups G_3 and G_1 (for $n = p^2 + p + 1$) have the isomorphic power graphs.

(ii) The Clique graphs of $P(G_1)$ and $P(G_3)$ are complete graphs.

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2. The proofs

First of all, we follow [6, 7] and recall two results on the power graphs of the abelian groups.

Lemma 2.1. [6, Theorem 2.12] *For a finite group G, $P(G)$ is complete if and only if G is the cyclic group of order p^m, for a prime p and a positive integer m.*

Lemma 2.2. [7, Theorem 5] *Let G be a finite group of order p_1q_1 where p_1 and q_1 are primes and $p_1 > q_1$. Then*

(i) G is cyclic if and only if $P(G) \simeq (K_{p_1-1} \cup K_{q_1-1}) + K_{\phi(p_1q_1)+1}$, ($\phi$ is the well-known Eulerian function).

(ii) G is non-cyclic if and only if $P(G) \simeq K_1 + (pK_{q_1-1} \cup K_{p_1-1})$.

Theorem 2.3. *For every odd prime p and every integer $n \geq 2$, $P(G_1)$ is an Eulerian graph.*

Proof. First we give the proof for $n = 2$. The group G_1 is abelian and may be rewritten as the union of $p^2 - 3p + 5$ cyclic groups $\langle a \rangle$, $\langle b \rangle$, $\langle ab \rangle$ and $\langle a^i b^j \rangle$ where $1 \leq i \neq j \leq p - 1$. The intersection of any two groups is the identity group and each group is of order p. This shows that $P(Z_p \times Z_p)$ is a connected graph. Moreover, the corresponding vertices of each group constitute a complete graph of order p. Since each group is of order p then the corresponding connected component is the complete graph K_p. This proves that the degree of each vertex of the graph $P(Z_p \times Z_p)$ is even. Consequently, $P(Z_p \times Z_p)$ is an Eulerian graph.

Now let $n \geq 3$. Then $G_1 \simeq H_1 \times H_2 \times \cdots \times H_n$, where $H_i = \langle a_i \rangle \simeq Z_p$. As well as in the case $n = 2$, the group G_1 may be rewritten as a union of a finite number of cyclic groups,

$$\begin{array}{ll}
\langle a_i \rangle, & (i = 1, 2, \ldots, n), \\
\langle a_i a_j \rangle, & (1 \leq i \neq j \leq n), \\
\langle (a_i)^\alpha (a_j)^\beta \rangle, & (1 \leq i \neq j \leq n), (1 \leq \alpha \neq \beta \leq p - 1) \\
\langle a_i a_j a_k \rangle, & (1 \leq i \neq j \neq k \leq n), \\
\langle (a_i)^\alpha (a_j)^\beta (a_k)^\gamma \rangle, & (1 \leq i \neq j \neq k \leq n), (1 \leq \alpha \neq \beta \neq \gamma \leq p - 1) \\
\vdots &
\end{array}$$

Each group is of order p and constitutes a complete subgraph of $P(G_1)$. Hence, each vertex of $P(G_1)$ is of even degree; therefore, $P(G_1)$ is an Eulerian graph. □

As a graph illustration $P(G_1)$ is the union of complete graphs each of degree p such that all have only one vertex in common without any coinciding edges. Note that as a conventional notation in the graph theory, the *inner product* of two graphs H_1 and H_2 denoted by $H_1 \cdot H_2$ is the union of two graphs such that only one vertex of H_1 coincides with just one vertex of the graph H_2 without any coinciding edges. The inner product of more than two graphs may be defined as the union of them where all of the graphs have a unique common vertex. As examples, the graphs $P(Z_5^2) = K_5 \cdot K_5$ and

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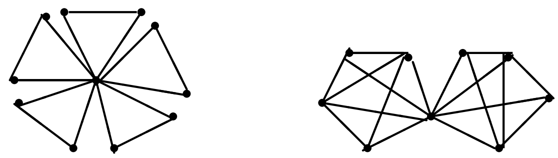
$P(Z_3^5) = K_3 \cdot K_3 \cdot K_3 \cdot K_3 \cdot K_3$ are:



$P(Z_3^5)$ $P(Z_5^2)$

Before proving the Theorem 2.7, we need to prove certain preliminaries on the group G_2. To represent the elements of G_2 in classical forms, for every i and j, where $1 \leq i \leq p^2-1$ and $1 \leq j \leq p-1$ we define the sequence $\{\alpha_m\}$ of numbers by $\alpha_1 = 0,\ \alpha_m = \frac{1}{2}ijpm(p-1)(m-1)$, $m \geq 2$.

Lemma 2.4. *For a given odd prime p, every element of G_2 may be written in the form $a^k b^\ell$, ($0 \leq k \leq p^2-1$ and $0 \leq \ell \leq p-1$). Moreover, $(a^i b^j)^m = (a^{mi} b^{mj}).a^{\alpha_m}$ for every $m \geq 2$.*

Proof. We use the relations of G_2. The relation $b^{-1}ab = a^{p+1}$ yields $b^{-p}ab = a^{p(p+1)}$ then we get $ba^p = a^p b$ and $ba = (ab)a^{p(p+1)}$. This relation gives us in turn the following relations

$$ba^i = (a^i b).a^{i(p^2-p)}, \tag{1}$$
$$b^j a = (ab^j).a^{j(p^2-p)}, \tag{2}$$
$$b^j a^i = (a^i b^j).a^{ij(p^2-p)}, \tag{3}$$

for every $i = 1, 2, \ldots, p^2-1$ and $j = 1, 2, \ldots, p-1$. To complete the proof, we use induction on m. Let $m = 2$ then, $\alpha_2 = ij(p)(p-1)$ and,

$$(a^i b^j)^2 = a^i (b^j a^i) b^j = a^i (a^i b^j a^{\alpha_2}) b^j = (a^{2i} b^{2j}) a^{\alpha_2}.$$

So by induction hypothesis we get:

$$\begin{aligned}
(a^i b^j)^{m+1} &= (a^i b^j)(a^{mi} b^{mj} a^{\alpha_m}), \\
&= a^i (b^j a^{mi}) b^{mj} a^{\alpha_m}, \\
&= a^i (a^{mi} b^j) b^{mj} a^{mij(p^2-p)}) b^{mj} a^{\alpha_m}, \text{ (by (3),)} \\
&= a^{(m+1)i} b^{(m+1)j}.a^{mij(p^2-p)+\alpha_m}, \text{ (for, } a^p b = ba^p), \\
&= a^{(m+1)i} b^{(m+1)j}.a^{\alpha_{m+1}}.
\end{aligned}$$

This holds because of the equality

$$mij(p^2-p) + \alpha_m = mij(p^2-p) + \frac{1}{2} ij(p^2-p)(m)(m-1) = \alpha_{m+1}.$$

This completes the proof. □

We are now ready to find the degrees of vertices of the graph $P(G_2)$. Note that the element $a^i b^j$ corresponds to a vertex and when we speak of $deg(a^i b^j)$, the "degree of $a^i b^j$", we mean the "degree of the corresponding vertex to the element $a^i b^j$".

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Lemma 2.5. *For every odd prime p, $deg(a^ib^j) = p^2-1$, for every $i = 1, 2, \dots, p^2-1$ and $j = 1, 2, \dots, p-1$.*

Proof. By Lemma 2.3 we get $(a^ib^j)^m = a^{mi}b^{mj}.a^{\frac{1}{2}pmij(p-1)(m-1)}$. First of all note that if $m_1 \neq m_2$ then $(a^ib^j)^{m_1} \neq (a^ib^j)^{m_2}$. Otherwise, the relations $b^{m_1j} = b^{m_2j}$ and $a^{m_1i+\frac{1}{2}pm_1ij(p-1)(m_1-1)} = a^{m_2i+\frac{1}{2}pm_2ij(p-1)(m_2-1)}$ yield the equations:

$$m_2j \equiv m_1j \ (mod\ p)$$
$$m_2i + \tfrac{1}{2}pm_2ij(p-1)(m_2-1) \equiv m_1i + \tfrac{1}{2}pm_1ij(p-1)(m_1-1) \ (mod\ p^2).$$

Since j is co-prime to p then $m_2 \equiv m_1 \ (mod\ p)$ and the second equation gives us $m_2i \equiv m_1i \ (mod p^2)$. Hence, p divides i, a contradiction.

To find the degrees of the vertices we consider different possible cases for $m \leq p^2$.

Case 1. If $m = p, 2p, 3p, \dots, (p-1)p$, then $(ab)^m = a^m$, i.e.; ab is adjacent with $p-1$ different vertices $a^p, a^{2p}, \dots, a^{(p-1)p}$.

Case 2. If $2 \leq m \leq p-1$ then the vertex a^ib^j is adjacent with $p-2$ different vertices.

Case 3. For every $k = 1, 2, \dots, p-1$ when $p+k \leq m \leq kp-1$, a^ib^j is adjacent with $p-1$ different vertices. So, there are exactly $(p-1)^2$ different vertices adjacent with a^ib^j in this case.

Case 4. Finally, when $m = p^2$, the vertex a^ib^j is adjacent with just one vertex (indeed, the vertex 1).

For m greater than p^2, $(a^ib^j)^m$ will produce the repeated elements because of the relation $(a^ib^j)^{p^2} = 1$. Consequently $deg(a^ib^j) = (p-1) + (p-2) + (p-1)^2 + 1 = p^2 - 1$, as required. □

On the basis of the facts above, we conclude that

$$(a^ib^j)^m = \begin{cases} 1, & \text{if } m \equiv\ 0(mod\ p) \text{ and } i \equiv\ 0(mod\ p) \\ a^{im}, & \text{if } m \equiv\ 0(mod\ p) \text{ and } i \neq p, 2p, \dots, (p-1)p, \\ a^{im}b^{jm}, & \text{if } m \neq p, 2p, \dots, (p-1)p \text{ and } i \equiv\ 0(mod\ p) \\ a^{im}b^{jm}.a^{\alpha_m}, & \text{otherwise.} \end{cases}$$

So the group G_2 may be written as the union of the following sets:

$$\{a^p, a^{2p}, \dots, a^{(p-1)p}\} \cup \{a^i \mid i \neq k.p, k = 1, 2, \dots, p-1\},$$
$$\{b^j \mid j = 0, 1, 2, \dots, p-1\},$$
$$\{a^{im}b^{jm} \mid m \neq k.p, k = 1, 2, \dots, p-1\}.$$

This decomposition gives us the following Lemma.

Lemma 2.6. *For every $j = 1, 2, \dots, p-1$ and every $i = 1, 2, \dots, p^2-1$, $deg(b^j) = p-1$ and* $deg(a^i) = \begin{cases} (p-1)(p+2), & i \in \{p, 2p, \dots, (p-1)p\}, \\ p^2-1, & \textit{otherwise.} \end{cases}$

Proof. Since $\{b^j \mid j = 0, 1, 2, \dots, p-1\}$ is a cyclic group of order p then by Lemma 2.1, its power graph is the complete graph K_p as a subgraph of $P(G_2)$. Hence, $deg(b^j) = p-1$, for ever j.

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Each element a^{kp} of the set $\{a^p, a^{2p}, \ldots, a^{(p-1)p}\}$ is adjacent with every element of each of two classes $\{1, a, \ldots, a^{p^2-1}\}$ and $\{a^{kp}b^j \mid j = 1, 2, \ldots, p-1\}$ then $deg(a^{kp}) = p - 1 + p^2 - 1$, and each element a^i where $i \neq p, 2p, \ldots, (p-1)p$, is adjacent with $p^2 - 1$ vertices of the set $\{1, a, \ldots, a^{p^2-1}\}$. This completes the proof. □

Theorem 2.7. *For every odd prime p, $P(G_2)$ is an Eulerian graph.*

Proof. Lemmas 2.4 and 2.5 show that the degree of every vertex of the power graph $P(G_2)$ is even. Hence, $P(G_2)$ is an Eulerian graph for every odd prime p. □

For every odd prime p, $P(G_3)$ is an Eulerian graph.

Theorem 2.8. *For every odd prime p, $P(G_3)$ is an Eulerian graph*

Proof. Each element of the group G_3 is of order p and this group is the union of$p^2 + p + 1$ subgroups:

$$\begin{aligned}
&\langle a\rangle, \langle b\rangle, \langle c\rangle, \\
&\langle ab^j\rangle, \langle ac^k\rangle, \langle bc^k\rangle, \quad (1 \leq j, k \leq p-1), \\
&\langle ab^j c^k\rangle, \qquad\qquad\quad (1 \leq j, k \leq p-1).
\end{aligned}$$

Indeed, the relation $b^{-1}ab = ac$ together with $ac = ca$ gives us the relation $b^{-j}a^ib^j = a^ic^{ij}$ which is equivalent to $b^ja^i = (a^ib^j)c^{p-ij}$. Since then, we conclude that every element of the group G_3 is in the form $a^ib^jc^k$ such that $0 \leq i, j, k \leq p-1$.

The intersection of any two subgroups is the identity element, and each subgroup is the cyclic group Z_p. Hence, by considering any non-identity element of G_3, its corresponding vertex is of degree $p-1$ in the graph $P(G_3)$. This means that every vertex of the graph $P(G_3)$ is of even degree and then $P(G_3)$ is an Eulerian graph. □

Corollary 2.9. *For every odd prime p,*
(i) The non-isomorphic groups G_3 and G_1 (for $n = p^2 + p + 1$) have the isomorphic power graphs.
(ii) The Clique graphs of $P(G_1)$ and $P(G_3)$ are complete graphs.

Proof. Evidently, the group G_1 is abelian and G_3 is not. So they are non-isomorphic groups. However, Theorems 2.3 and 2.7 show that the power graph of each one is the inner product of $p^2 + p + 1$ copies of Z_p. This completes the proof of (i).

To prove (ii) again we use the results of Theorems 2.3 and 2.7. Indeed, each of the graphs $P(G_3)$ and $P(G_1)$ (for every $n \geq 2$) is an inner product of complete graphs. So the Clique graphs of the graphs $P(G_3)$ and $P(G_1)$ are complete. □

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