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SMALL DOUBLING IN m-ENGEL GROUPS

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Abstract. We study some inverse problems of small doubling type in the class of m-Engel groups. In particular we investigate the structure of a finite subset S of a torsion-free m-Engel group if $|S^2| = 2|S| + b$, where $0 \leq b \leq |S| - 4$, for some values of b.

1. **Introduction**

Let G be a group written multiplicatively. If A, B, S are subsets of G, we write $AB = \{ab \mid a \in A, b \in B\}$, and $S^2 = SS = \{ab \mid a, b \in S\}$. If G is written additively, we write $A+B = \{a+b \mid a \in A, b \in B\}$ and $2S = \{a+b \mid a, b \in S\}$. We are interested in two kinds of problems.

Problem 1- Find information about $|S^2|$ in terms of $|S|$ if S is a finite subset of a group G.

This is **a direct problem**.

Problem 2 - Find information about the structure of a finite subset S of a group G if there is a bound for $|S^2|$.

This is **an inverse problem.**

These kinds of problems have been intensely studied in Additive Combinatorics.

In particular many authors investigated structural information on S if $|S^2|$ is very close to $2|S|$, these kinds of problems are called **inverse problems of small doubling type**.

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If S is a finite subset of the group of the integers it is easy to prove:

Theorem 1.1. *If S is a finite subset of the integers, $|S| = k$, then $|2S| \geq 2|S|-1$, and $|2S| = 2|S|-1$ if and only if there exist integers a, q such that $S = \{a, a+q, a+2q, \dots, a+(k-1)q\}$, i.e. S is an arithmetic progression of length k.*

G. A. Freiman did a more detailed investigation on subsets of the group of the integers, he proved the following results.

Theorem 1.2. [10, G. A. Freiman], [11, G. A. Freiman]
If S is a finite set of integers with $k \geq 3$ elements and

$$|2S| \leq 3k-4,$$

then there exist integers a, q such that $q > 0$ and $S \subseteq \{a, a+q, a+2q, \dots, a+(2k-4)q\}$.

Theorem 1.3. [10, G. A. Freiman], [11, G. A. Freiman] *Let S be a finite set of integers with $k \geq 3$ elements and suppose that $|2S| = 2k+b$, where $0 \leq b \leq k-4$. Then S is contained in an arithmetic progression of length $k+b+1$.*

More generally, if S is a finite subset of any torsion-free group, we have:

Theorem 1.4. [34, J. H. B Kemperman] *If S is a finite subset of any torsion-free group, then $|S^2| \geq 2|S|-1$.*

Theorem 1.5. [21, G. A. Freiman and B. M. Schein] *If S is a finite subset of any torsion-free group and $|S| = k$, then $|S^2| = 2|S|-1$ if and only if $S = \{a, aq, \dots, aq^{k-1}\}$, i.e. S is a geometric progression and either $aq = qa$ or $aqa^{-1} = q^{-1}$.*

Starting from the results true for the group of the integers, G. A. Freiman proposed the following conjecture.

Conjecture. (G. A. Freiman) If S is a finite set of a torsion-free group with $k \geq 3$ elements and $|S^2| \leq 3|S|-4$, then there exist x and a in G such that $S \subseteq \{x, xa, xa^2, \dots, xa^{2k-4}\}$, i.e. S is a subset of a geometric progression of length $2k-3$.

Notice that if G is moltiplicative group, to be a subset of a geometric progression means that S is contained in a coset of a cyclic subgroup of G.

More generally:

Conjecture. (G. A. Freiman) If S is a finite set of a torsion-free group with $k \geq 3$ elements and $|S^2| = 2|S|+b$, where $0 \leq b \leq k-4$, then S is a subset of a geometric progression of length $|S|+b+1$ (in particular S is contained in a coset of a cyclic subgroup of G).

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In Additive Combinatorics this conjecture is known as the Freiman $(3k-4)$-conjecture, and it has been investigated by many authors. The first positive answer is for abelian groups.

Theorem 1.6. [12, G. A. Freiman] *Freiman's conjecture is true if G is a torsion-free abelian group.*

Another positive result is the following theorem.

Theorem 1.7. [30, Y. O. Hamidoune, A. S. LLadó and O. Serra] *If S is a finite subset of any torsion-free group and $|S| = k \geq 4$, then $|S^2| \leq 2|S|$ if and only if there are elements x, r such that the following two conditions hold:*
(i) $rx = xr$,
(ii) $Sx = \{1, r, \ldots, r^k\} \setminus \{c\}$ where $c \in \{1, r\}$.

Therefore Freiman's conjecture is true if $b = 0$ and $|S| \geq 4$.

Together with G. A. Freiman and M. Herzog we proved the following result for orderable groups. Here an orderable group (a right orderable group) G is a group on which it is possible to define a total order $\leq$ such that, with $a, b, x, y \in G$, $x \leq y$ implies $axb \leq ayb$ ($x \leq y$ implies $xb \leq yb$). Notice that an orderable group is always torsion-free. More results on orderable groups can be found for example in the books [4] and [23].

Theorem 1.8. [14, G. A. Freiman, M. Herzog, P. Longobardi and M. Maj] *Let G be an orderable group and let S be a subset of G with $|S| = k \geq 3$.*

If $|S^2| \leq 3|S| - 3$, then $\langle S \rangle$ is abelian.

If $t = |S^2| \leq 3|S| - 4$, then there exist $x, g \in G$ such that $gx = xg$ and S is a subset of $\{x, xg, xg^2, ^{t-k}\}$.

Therefore Freiman's conjecture is true if G is an orderable group, in particular if G is a torsion-free nilpotent group.

Recently A. Abdollahi and F. Jafari proved the following result.

Theorem 1.9. [1, A. Abdollahi and F. Jafari] *Let G be a unique product group and S be a finite subset of G containing the identity element with $|S| \geq 7$. If $|S^2| \leq 2|S| + 1$ then $\langle S \rangle$ is abelian.*

Therefore Freiman's conjecture is true if G is a unique product group, $b = 1$ and $|S| \geq 7$. Recall that a group G has the unique product property (is a unique product group), if for every pair of finite non-empty subsets A, B of G, there exists an element $x \in AB$ such that $|\{(a, b) \in A \times B \mid x = ab\}| = 1$. Obviously a unique product group is torsion-free, it is easy to prove that every right orderable group is a unique product group (cfr. [8], [40], [41], [49] for more results on unique product groups).

Some other positive results are contained in the following theorems.

Theorem 1.10. [7, K. J. Böröczky, P. P. Pálfy, O. Serra] *Let S be a finite subset of any torsion-free group and $|S| = k \geq 6^6$. If $|S^2| = 2|S| + b$, where $0 \leq b \leq \frac{1}{2}|S|^{\frac{1}{6}} - 3$, then S is a subset of a progression of length $|S| + b + 1$.*

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Therefore Freiman's conjecture is true if G is a torsion-free group, $|S| \geq 6^6$, and $0 \leq b \leq \frac{1}{2}|S|^{\frac{1}{6}} - 3$.

Theorem 1.11. *(, [7, K. J. Böröczky, P. P. Pálfy and O. Serra]) Let S be a finite subset of a unique product group and $|S| = k \geq 6^3$. If $|S^2| = 2|S| + b$, where $0 \leq b \leq \frac{1}{2}|S|^{\frac{1}{3}} - \frac{3}{2}$, then S is a subset of a progression of length $|S| + b + 1$.*

Therefore Freiman's conjecture is true if G is a torsion-free group, $|S| \geq 6^3$, and $0 \leq b \leq \frac{1}{2}|S|^{\frac{1}{3}} - \frac{3}{2}$.

The aim of this paper is to study Freiman's conjecture assuming that G is an m-Engel group. Here, if m is a positive integer, an m-Engel group G is a group such that $[x,_m y] = 1$, for any $x, y \in G$, where $[x,_m y]$ is defined by putting $[x,_1 y] = [x, y] = x^{-1}y^{-1}xy$, and, by induction, if $i > 1$, $[x,_i y] = [[x,_{i-1} y], y]$. We also put $[x,_0 y] = x$.

We prove the following result that improves the bound in Theorem 1.11 if we add the hypothesis that G is an m-Engel group.

Theorem A. *Let G be a unique product m-Engel group, S a finite subset of G, $|S| = k \geq 6^2$. If $|S^2| = 2|S| + n$, where $0 \leq n \leq \frac{1}{2}|S|^{\frac{1}{2}} - 1$, then $\langle S \rangle$ is abelian and S is a subset of a progression of length $|S| + n + 1$.*

If we assume that G is a right-orderable m-Engel group, we could also improve the bound in Theorem A.

Theorem B. *Let G be a rigth-orderable m-Engel group, S a finite subset of G, $|S| = k \geq 3^2$. If $|S^2| = 2|S| + n$, where $0 \leq n \leq |S|^{\frac{1}{2}} - 3$, then $\langle S \rangle$ is abelian and S is a subset of a progression of length $|S| + n + 1$.*

For some values of $|S|$ we have some better bounds.

Theorem C. *Let G be a right-orderable m-Engel group, S a finite subset of G. Suppose $|S| = k$, $k \geq 20$.*

If $|S^2| \leq 2|S| + 2$, then $\langle S \rangle$ is abelian and S is a subset of a progression of length $|S| + 3$.

Theorem D. *Let G be a right-orderable m-Engel group, S a finite subset of G. Suppose $|S| = k$, $k > 30$.*

If $|S^2| \leq 2|S| + 3$, then $\langle S \rangle$ is abelian and S is a subset of a progression of length $|S| + 4$.

Notice that Theorems B, C and D immediately follow from Theorem 1.8 if it is positive the answer to the following Problem.

Problem 1 (well-known, see [3, Problem 2.25]) Is every right orderable m-Engel group nilpotent?

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Notice that the answer to Problem 1 is yes if $m = 2, 3, 4$, also the answer is yes if G is orderable [35, Y. Kim and A. H. Rhemtulla].

Theorems B, C and D are related with the following Problem:

Problem 2 (well-known, see [3, Problem 2.9]) Is every torsion-free m-Engel group right-orderable?

Of course the answer to Problem 2 is yes if $m = 2, 3, 4$, since in these cases G is nilpotent, and torsion-free nilpotent groups are orderable.

Finally we recall the following problem posed by P. A. Linnell.

Problem 3 (P. A. Linnell, see [3, Problem 2.34]) Is every unique product group right-orderable?

The paper consists of four sections. Section 1 is the Introduction, in Section 2 we recall, and we prove for the sake of completeness, some known results concerning torsion-free m-Engel groups, in Section 3 we prove three useful lemmas, finally in Section 4 we prove Theorems A, B, C, D.

Our notation is the usual one (see for example [42] or [52]). We will often make use of the following result:

Theorem 1.12. [34, J. H. B Kemperman] *If S, T are finite subsets of any torsion-free group, then $|ST| \geq |S| + |T| - 1$.*

In the proof of Theorem A we use some ideas contained in a paper by K. J. Böröczky, P. P. Pálfy and O. Serra (see [7]), in particular we will use the so-called isoperimetric method, see Hamidoune ([27], [28] and [29]).

For more recent results on small doubling problems in groups we refer to [2], [5], [6], [9], [13], [15]-[20], [22]-[26], [31]-[33], [36]-[39], [43]-[48], [50], [51].

2. Some results on torsion-free m-Engel groups

We start this section with the following three well-known results.

Proposition 2.1. *Let G be a torsion-free m-Engel group, $a, b \in G, X \subseteq G, n \in \mathbb{N}$.*

Then

$$(i)\ [a^n, b] = 1 \text{ implies } [a, b] = 1;$$

$$(ii)\ \langle a^n, X\rangle \text{ nilpotent implies } \langle a, X\rangle \text{ nilpotent.}$$

Proof. (i) Let s be the minimum positive integer such that $[b,_s a] = 1$. Suppose $s \geq 2$ and write $c = [b,_{s-2} a]$. Then $[c, a] \neq 1$, moreover we have $[c, a^n] = 1$ and $[c, a, a] = 1$, thus $[c, a]^n = [c, a^n] = 1$ and the contradiction $[c, a] = 1$, since G is torsion-free. Therefore $s = 1$ and (i) is proved.

(ii) Write $Y = \langle a^n, X\rangle$, $T = \langle a, X\rangle$ and suppose Y nilpotent of class k. By (i) we have $Z(Y) \subseteq Z(T)$. Assume by induction $Z_l(Y) \subseteq Z_l(T)$. Let $x \in Z_{l+1}(Y)$. Then $[x, y] \in Z_l(Y) \subseteq Z_l(T)$. Moreover $[x, a^n] \in Z_l(Y) \subseteq Z_l(T)$, thus $[x, a] \in Z_l(T)$ since $T/Z_l(T)$ is a torsion-free m-Engel group. Therefore $xZ_l(T) \in Z(T/Z_l(T)) = Z_{l+1}(T)/Z_l(T)$, thus $x \in Z_{l+1}(T)$. Hence $Z_v(Y) \subseteq Z_v(T)$ for every positive

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integer v. In particular we have $Y = Z_k(Y) \subseteq Z_k(T)$ and $T/Z_k(T)$ cyclic implies $T = Z_t(T)$. Thus (ii) is proved. □

Proposition 2.2. *Let G be an m-Engel group, $a, x \in G$. Then there exists $s > 0$ such that*

$$\langle a \rangle^{\langle x \rangle} = \langle a, a^x, a^{x^2}, \ldots, a^{x^s} \rangle.$$

Proof. See, for example [52, p. 4]. □

Proposition 2.3. *Let G be a torsion-free m-Engel group, $a, x \in G$. If $x^{-1}a^{\alpha}x \in \langle a \rangle$, for some positive integer α, then $\langle a, x \rangle$ is abelian.*

Proof. By induction on i, it is easy to show that $x^{-i}a^{\alpha^i}x^i \in \langle a \rangle$, for every positive integer i. Then $\langle a, a^{x^i} \rangle$ is abelian by Proposition 2.1, hence $\langle a \rangle^{\langle x \rangle}$ is abelian. Therefore $\langle a, x \rangle = \langle a \rangle^{\langle x \rangle}\langle x \rangle$ is soluble, then it is nilpotent, since it is a soluble m-Engel group (see for example [52, Theorem 2.2]). Assume by induction $\langle a, x \rangle$ nilpotent of class less or equal to 2, and write $x^{-1}a^{\alpha}x = a^{\beta}$. Then from $[a, x]^{\alpha} = [a^{\alpha}, x] = a^{\beta-\alpha}$ we get $a^{\beta-\alpha} \in Z(\langle a, x \rangle)$, thus either $\beta - \alpha = 0$ or $a \in Z(\langle a, x \rangle)$ since $\langle a, x \rangle / Z(\langle a, x \rangle)$ is either trivial or torsion-free. In the first case we obtain $x^{-1}a^{\alpha}x = a^{\alpha}$ and again $[a, x] = 1$ by Proposition 2.1. In any case $\langle a, x \rangle$ is abelian. □

Next Proposition is similar to [14, Proposition 2.4].

Proposition 2.4. *Let G be a torsion-free m-Engel group and let S be a subset of G of order k. If there exist $y \in G, x_1 \in S$ such that $yx_1 \neq x_1 y$, then $|yS \cup Sy| \geq k+1$. In particular there exists $w \in S$ such that $wy \notin yS$.*

Proof. Suppose, to the contrary, that $yS = Sy$. Then there exists $x_2 \in S$ such that $x_2 \neq x_1$ and $yx_1 = x_2 y$. Suppose that there exist $x_1, x_2, \ldots, x_t \in S$ such that

$$yx_1 = x_2 y$$
$$yx_2 = x_3 y$$
$$\cdots$$
$$yx_{t-1} = x_t y$$

where $x_i = x_j$ if and only if $i = j$. Since $yS = Sy$, there exists $x_{t+1} \in S$ such that

$$yx_t = x_{t+1} y.$$

We claim that $x_{t+1} \notin \{x_1, x_2, \ldots, x_t\}$. Indeed, if $x_{t+1} = x_u$ for some integer u, $1 \leq u \leq t$, then $x_t = y^{-1}x_{t+1}y = y^{-1}x_u y = y^{-2}x_{u+1}y^2 = \cdots = y^{-(t-u+1)}x_t y^{t-u+1}$ and hence $[x_t, y^{t-u+1}] = 1$. It follows by Proposition 2.1 that $yx_t = x_t y = yx_{t-1}$. But then $x_t = x_{t-1}$, a contradiction. This proves our claim. Since this procedure can be carried out indefinitely, we have reached a contradiction to the finiteness of S. Hence $yS \neq Sy$ and the proposition follows. □

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3. Some useful lemmas

Lemma 3.1. *Let G be a torsion-free group, S a finite subset of G, u an element of G.*

If $|\{1,u\}S| \leq |S| + t$, where t is a positive integer, then there exist a positive integer $l \leq t$ and elements $g_1, g_2, \dots, g_l \in S$ such that

$$S = (S \cap \langle u \rangle g_1) \dot{\cup} (S \cap \langle u \rangle g_2) \dot{\cup} \cdots \dot{\cup} (S \cap \langle u \rangle g_l).$$

Proof. Write $S_u = \{x \in S \mid ux \notin S\}$. Then $\{1,u\}S = S \dot{\cup} uS_u$. Then $|S_u| \leq t$, since $|\{1,u\}S| \leq |S| + t$. Write $\langle u \rangle S_u = \langle u \rangle g_1 \dot{\cup} \langle u \rangle g_2 \dot{\cup} \cdots \dot{\cup} \langle u \rangle g_l$, where $g_1, \dots, g_l \in S_u$ and hence $l \leq t$. Obviously $(S \cap \langle u \rangle g_1) \dot{\cup} (S \cap \langle u \rangle g_2) \dot{\cup} \cdots \dot{\cup} (S \cap \langle u \rangle g_l) \subseteq S$. Conversely, if $g \in S$ there exists an integer $\alpha \geq 0$ such that $u^\alpha g \in S$ and $u^{\alpha+1} g \notin S$, since S is finite. Then $u^\alpha g \in S_u$. Thus $u^\alpha g \in \langle u \rangle g_i$, for some $i \in \{1, \dots, l\}$, then $g \in S \cap \langle u \rangle g_i$. Therefore $S \subseteq (S \cap \langle u \rangle g_1) \dot{\cup} (S \cap \langle u \rangle g_2) \dot{\cup} \cdots \dot{\cup} (S \cap \langle u \rangle g_l)$. The lemma is proved. □

Lemma 3.2. *Let G be a torsion-free m-Engel group, S a finite subset of G such that $\langle S \rangle$ is not abelian. Suppose that $u, g_1, \dots, g_t$ are elements of G, $t \leq 3$, such that*

$$S = (S \cap \langle u \rangle g_1) \dot{\cup} (S \cap \langle u \rangle g_2) \dot{\cup} \cdots \dot{\cup} (S \cap \langle u \rangle g_t),$$

where every $S \cap \langle u \rangle g_i$ is non-empty.

$$If\ \ t = 1 \ \ then \ \ |S^2| = |S|^2,$$

$$if\ \ t = 2 \ \ and\ |S| \geq 12, \ then \ \ |S^2| \geq 3|S| - 3,$$

$$if\ \ t = 3 \ \ and\ |S| \geq 30, \ then \ \ |S^2| \geq 2|S| + |S|^{\frac{1}{2}},$$

$$if\ \ t = 3 \ \ and\ |S| \geq 36, \ then \ \ |S^2| \geq 2|S| + |S|^{\frac{1}{2}} + 1.$$

Proof. First notice that:

($\star$) $u^\alpha g_i u^\beta g_i = u^\gamma g_i u^\delta g_i$ for suitable integers $\alpha, \beta, \gamma, \delta$, with $\alpha \neq \gamma$ implies $[u, g_i] = 1$.

In fact, assuming $\alpha > \gamma$ we get $\alpha - \gamma > 0$ and $g_i^{-1} u^{\alpha-\gamma} g_i = u^{\delta-\beta} \in \langle u \rangle$ and $[u, g_i] = 1$ by Proposition 2.3.

Therefore, if $t = 1$, since $\langle S \rangle$ is not abelian, we have by ($\star$) that the products $u^\alpha g_1 u^\beta g_1$ are all different, thus $|S^2| = |S|^2$, as required.

Now suppose $t = 2$, thus $S = (S \cap \langle u \rangle g_1) \dot{\cup} (S \cap \langle u \rangle g_2)$. Write $S_1 = S \cap \langle u \rangle g_1$, $S_2 = S \cap \langle u \rangle g_2$, $s_1 = |S_1|$, $s_2 = |S_2|$. Then $s_1, s_2 \geq 1$ and we can suppose $s_1 \geq s_2$. Then from $s_1 + s_2 \geq 12$ we get $s_1 \geq 6$.

If $[u, g_1] \neq 1$, then $|S_1^2| = |S_1|^2$ by ($\star$), and we have $S^2 \supseteq S_1^2$, thus $|S^2| \geq s_1^2 \geq 3s_1 + 3s_1 \geq 3s_1 + 3s_2 = 3|S|$, as required.

Suppose $[u, g_1] = 1$. In this case $\langle u, g_1 \rangle$ is abelian, therefore $g_2 \notin \langle u, g_1 \rangle$ since S is not abelian, in particular $S_1^2 \cap S_1 S_2 = \emptyset = S_1 S_2 \cap S_2^2$. We also have $S_1^2 \cap S_2^2 = \emptyset$, otherwise from $u^\alpha g_2 u^\beta g_2 \in S_1^2$ for some integers α, β we get $u^\alpha g_2 u^\beta g_2 \in \langle u, g_1 \rangle$, $u^{\beta-\alpha} u^\alpha g_2 u^\beta g_2 \in \langle u, g_1 \rangle \subseteq C_G(u) \cap C_G(g_1)$, thus

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$(u^\beta g_2)^2 \in C_G(u) \cap C_G(g_1)$ and then $u^\beta g_2 \in C_G(u) \cap C_G(g_1)$ by Proposition 2.1. Then $\langle g_1, g_2, u\rangle$ is abelian and we have the contradiction $\langle S\rangle$ abelian. Therefore $S^2 \supseteq S_1^2 \dot\cup S_1S_2 \dot\cup S_2^2$ and $|S^2| \geq |S_1^2| + |S_1S_2| + |S_2^2| \geq 2s_1 - 1 + s_1 + s_2 - 1 + 2s_2 - 1 = 3s_1 + 3s_2 - 3 = 3|S| - 3$, as required.

Finally suppose $t = 3$, then $S = (S \cap \langle u\rangle g_1) \dot\cup (S \cap \langle u\rangle g_2) \dot\cup (S \cap \langle u\rangle g_3)$. Write $S_1 = S \cap \langle u\rangle g_1$, $S_2 = S \cap \langle u\rangle g_2$, $S_3 = S \cap \langle u\rangle g_3$, $s_1 = |S_1|$, $s_2 = |S_2|$, $s_3 = |S_3|$. Then $s_1, s_2, s_3 \geq 1$ and we can assume $s_1 \geq s_2 \geq s_3$.

First suppose $|S| \geq 30$. Then from $s_1 + s_2 + s_3 \geq 30$ we get $s_1 \geq 10$. Moreover we have $(s_1 - 4)^2 \geq 3s_1 \geq |S|$, thus $s_1 \geq |S|^{\frac{1}{2}} + 4$.

Arguing as before if $[u, g_1] \neq 1$ then $|S_1^2| = s_1^2$ and we have $|S^2| \geq |S_1^2| = s_1^2 \geq 10s_1 \geq 2s_1 + 2s_2 + 2s_3 + s_1 \geq 2|S| + |S|^{\frac{1}{2}}$. Then we can suppose $[u, g_1] = 1$. If $[u, g_2] \neq 1$, then arguing as before we have $S^2 \supseteq S_1^2 \dot\cup S_1S_2 \dot\cup S_2^2$ and

$$(\star\star)\ |S^2| \geq 2s_1 - 1 + s_1 + s_2 - 1 + s_2^2 = 3s_1 + s_2 + s_2^2 - 2.$$

If $s_2 \geq 3$, then $s_2^2 \geq s_2 + s_2 + s_2 \geq s_2 + 2s_3$ and from $(\star\star)$ we obtain $|S^2| \geq 2s_1 + 2s_2 + 2s_3 + s_1 - 2 \geq 2|S| + |S|^{\frac{1}{2}}$, as required. If $s_2 = 2$, then $s_2^2 = 2s_2$ and from $(\star\star)$ we get $|S^2| \geq 2s_1 + s_1 + 3s_2 - 2 \geq 2s_1 + 2s_2 + 2s_3 + s_1 - 4 \geq 2|S| + |S|^{\frac{1}{2}}$, as required. If $s_2 = 1$, then $s_3 = s_2 = 1$ and from $(\star\star)$ we get $|S^2| \geq 3s_1 = 2s_1 + s_1 + 2s_2 + 2s_3 - 4 \geq 2|S| + |S|^{\frac{1}{2}}$, as required. Then we can suppose $[u, g_1] = [u, g_2] = 1$. Now, if $[g_1, g_2] \neq 1$ and $S_1S_2 \cap S_2S_1 = \emptyset$ then $S^2 \supseteq S_1^2 \dot\cup S_1S_2 \dot\cup S_2S_1 \dot\cup S_2^2$ and $|S^2| \geq 2s_1 - 1 + s_1 + s_2 - 1 + s_2 + s_1 - 1 + 2s_2 - 1 \geq 4s_1 + 4s_2 - 4 \geq 2s_1 + 2s_2 + 2s_3 + s_1 - 4 \geq 2|S| + |S|^{\frac{1}{2}}$ as required. Finally assume either $[g_1, g_2] = 1$ or $S_1S_2 \cap S_2S_1 \neq \emptyset$. In the first case $\langle u, g_1, g_2\rangle$ is abelian, in the latter case we obtain $[g_1, g_2] \in \langle u\rangle \subseteq C_G(u) \cap C_G(g_1) \cap C_G(g_2)$ thus $\langle u, g_1, g_2\rangle$ is nilpotent of class 2. In any case $\langle u, g_1, g_2\rangle$ is nilpotent of class ≤ 2. Now, if $g_3 \in \langle u, g_1, g_2\rangle$ then $S \subseteq \langle u, g_1, g_2\rangle$, thus $\langle S\rangle$ is nilpotent of class 2 and in this case by Theorem 1.8 we have $|S^2| \geq 3|S| - 2 \geq 2|S| + |S|^{\frac{1}{2}}$, as required. Finally assume that $g_3 \notin \langle u, g_1, g_2\rangle$. Then in this case $(S_1 \cup S_2)^2 \cap S_1S_3 = \emptyset$ and $(S_1 \cup S_2)^2 \cap S_2S_3 = \emptyset$. If $\langle u, g_1, g_2\rangle$ is abelian we have also $S_1S_3 \cap S_2S_3 = \emptyset$, otherwise from $u^\alpha g_1 u^\beta g_3 = u^\gamma g_2 u^\delta g_3$ we get $u^{\alpha+\beta} g_1 = u^{\gamma+\delta} g_2$ from which $\langle u\rangle g_1 = \langle u\rangle g_2$ and $S_1 = S_2$ a contradiction. Thus in this case $|S^2| \geq |(S_1 \cup S_2)^2| + |S_1S_3| + |S_2S_3| \geq 2s_1 + 2s_2 - 1 + s_1 + s_3 - 1 + s_2 + s_3 - 1 \geq 2|S| + s_1 - 4 \geq 2|S| + |S|^{\frac{1}{2}}$, as required. Finally, if $\langle u, g_1, g_2\rangle$ is nilpotent of class 2, then $|(S_1 \cup S_2)^2| \geq 3|S_1 \cup S_2| - 2$, and from $S^2 \supseteq (S_1 \cup S_2)^2 \dot\cup S_1S_3$ we get $|S^2| \geq 3s_1 + 3s_2 - 2 + s_1 + s_3 - 1 \geq 2|S| + s_1 - 3 \geq 2|S| + |S|^{\frac{1}{2}}$, as required.

Now suppose $|S| \geq 36$. Then from $s_1 + s_2 + s_3 \geq 36$ we get $s_1 \geq 12$. Moreover we have $(s_1 - 5)^2 \geq 3s_1 \geq |S|$, thus $s_1 \geq |S|^{\frac{1}{2}} + 5$. Arguing exactly as in the previous paragraph we obtain $|S^2| \geq 2|S| + |S|^{\frac{1}{2}} + 1$.

The lemma is proved. □

Lemma 3.3. *Let G be a torsion-free m-Engel group. Suppose that A is a subset of G with $|A| = 3$ and A is not contained in a left coset of any cyclic subgroup of G. Let $d \geq 9$. For any subset B of G*

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of cardinality greater or equal to d we have

$$|AB| \geq |B| + d^{\frac{1}{2}}.$$

Proof. Write $|AB| = |B| + c$ and suppose by contradiction that $c < d^{\frac{1}{2}}$. Then $c^2 < d$. We may assume $A = \{1, u, v\}$ where $\langle u, v\rangle$ is not cyclic. Then $|\{1, u\}B| \leq |B| + c$ and, by Lemma 3.1,

$$B = (B \cap \langle u\rangle g_1)\dot{\cup}(B \cap \langle u\rangle g_2)\dot{\cup}\cdots\dot{\cup}(B \cap \langle u\rangle g_l),$$

where $l \leq c$. Since $|B| \geq d > c^2$ there exists $i \in \{1, \dots, l\}$ such that $|B \cap \langle u\rangle g_i| \geq c + 1$. We also have, by Lemma 3.1,

$$B = (B \cap \langle v\rangle h_1)\dot{\cup}(B \cap \langle v\rangle h_2)\dot{\cup}\cdots\dot{\cup}(B \cap \langle v\rangle h_s),$$

where $s \leq c$. Therefore there exist two different elements $u^\alpha g_i$, $u^\beta g_i$ of B in the same right coset of $\langle v\rangle$. Then $u^\alpha g_i g_i^{-1} u^{-\beta} \in \langle v\rangle$. Then $1 \neq u^{\alpha-\beta} \in \langle v\rangle$. Write $u^{\alpha-\beta} = v^\gamma$, then $\langle u, v\rangle$ is abelian by Proposition 2.1. From $\langle u, v\rangle/\langle v^\gamma\rangle$ finite we get $\langle u, v\rangle$ cyclic, since G is torsion-free, and a contradiction since we are assuming $\langle u, v\rangle$ not cyclic. The lemma is proved. □

Lemma 3.4. *Let G be a torsion-free m-Engel group, S a finite subset of G such that $\langle S\rangle$ is not abelian. Suppose that $a, g_1, \dots, g_t$ are elements of G such that*

$$S = (S \cap \langle a\rangle g_1)\dot{\cup}(S \cap \langle a\rangle g_2)\dot{\cup}\cdots\dot{\cup}(S \cap \langle a\rangle g_t).$$

$$If \quad t \leq \frac{\sqrt{|S|}}{2} \quad then \quad |S^2| \geq 2|S| + |S|^{\frac{1}{2}} + 1.$$

Proof. Without loss of generality we can suppose that t is minimum such that S is contained in $t \leq \frac{\sqrt{|S|}}{2}$ cosets of a cyclic subgroup of G. By Lemma 3.2 we can suppose $t \geq 4$. Write $S_i = (S \cap \langle a\rangle g_i)$, $s_i = |S_i|$. We can assume $s_1 \geq s_2, s_3, \dots, s_t$. Then

$$s_1 \geq 2|S|^{\frac{1}{2}}.$$

In fact, if $s_1 < 2|S|^{\frac{1}{2}}$, then $s_2, \dots, s_t \leq s_1$ implies $|S| < 2|S|^{\frac{1}{2}}\frac{\sqrt{|S|}}{2}$, a contradiction. We also can suppose

$$[a, g_1] = 1.$$

In fact, suppose $[a, g_1] \neq 1$. Then, arguing as in the proof of Lemma 3.2 we obtain $|S_1^2| = s_1^2$, thus $|S^2| \geq |S_1^2| \geq 4|S| = 2|S| + 2|S| \geq 2|S| + |S|^{\frac{1}{2}} + 1$, as required. In particular we can assume that

$$\langle S_1\rangle \textit{ is abelian.}$$

First suppose that $\langle S_1 \cup S_i\rangle$ is nilpotent for some $i > 1$. Without loss of generality we can suppose $i = 2$. Then we can write

$$S = (S_1 \cup S_2 \cup \cdots \cup S_j) \cup S_{j+1} \cup \cdots \cup S_t,$$

with $\langle S_1 \cup S_2 \cup \cdots \cup S_j \rangle$ nilpotent and $\langle S_1 \cup S_2 \cup \cdots \cup S_j \cup S_l \rangle$ non nilpotent for any l, $j < l \leq t$ (eventually $j = t$). Write $X = S_1 \cup S_2 \cup \cdots \cup S_j$, $Y = \langle a, X \rangle$. We have $a^\alpha g_1, a^\beta g_1 \in S_1$ for suitable $\alpha > \beta$, hence $a^{\alpha-\beta} \in \langle X \rangle$ and from $\langle X \rangle$ nilpotent it follows Y nilpotent by Proposition 2.1. Then $g_l \notin Y$, hence $X^2 \cap S_1S_l = \emptyset$, for every l, $j < l \leq t$. Obviously $S_1S_h \cap S_1S_v = \emptyset$, for any $h \neq v$, with $h, v > 1$. Then we have $|S^2| \geq |X^2| + |S_1S_{j+1}| + \cdots + |S_1S_t|$. Now the set X is not contained in a coset of a cyclic subgroup $\langle b \rangle g$. In fact suppose that $X \subseteq \langle b \rangle g$, for some $b, g \in G$. From $S_1 \subseteq X \subseteq \langle b \rangle g$, $|S_1| > 1$, we get $a^\gamma g_1 = b^\delta g$, $a^\mu g_1 = b^\nu g$, for some integers $\gamma \neq \mu$, then we have $1 \neq a^{\gamma-\mu} = b^{\delta-\nu}$, therefore $\langle a, b \rangle$ is abelian by Proposition 2.1, and $\langle a, b \rangle = \langle c \rangle$, for some $c \in G$. Then $\langle a \rangle \subseteq \langle c \rangle, \langle b \rangle \subseteq \langle c \rangle$ implies $S \subseteq \langle c \rangle g \cup \langle c \rangle g_{j+1} \cup \cdots \cup \langle c \rangle g_t$, a union of at most $t-1$ cosets of a cyclic subgroup, contradicting the minimality of t. Therefore, by Theorem1.8, $|X^2| \geq 3|X| - 3$ and we have $|S^2| \geq 3|X| - 3 + s_{j+1} + s_1 - 1 + \cdots + s_t + s_1 - 1 \geq 2|S| + |X| - t \geq 2|S| + s_1 - t \geq 2|S| + |S|^{\frac{1}{2}} + \frac{1}{2}|S|^{\frac{1}{2}}$, since $s_1 \geq 2|S|^{\frac{1}{2}}$ and $t \leq \frac{\sqrt{|S|}}{2}$.

Now suppose that $\langle S_1 \cup S_i \rangle$ is not nilpotent, for every $i > 1$. Without loss of generality we can suppose $s_2 \geq s_3 \geq \cdots \geq s_t$.

If $s_2 < |S|^{\frac{1}{2}}$, then $s_2 + \cdots + s_t < |S|^{\frac{1}{2}}(t-1) \leq |S|^{\frac{1}{2}}(\frac{\sqrt{|S|}}{2} - 1) = \frac{|S|}{2} - |S|^{\frac{1}{2}} < \frac{|S|}{2}$, hence $s_1 > \frac{|S|}{2}$. But we have $|S^2| \geq 2s_1 - 1 + s_1 + s_2 - 1 + \cdots + s_1 + s_t - 1$ with $t \geq 4$. Hence $|S^2| \geq 5s_1 - 1 \geq 4s_1 + s_1 - 1 \geq 4\frac{|S|}{2} + s_1 - 1 \geq 2|S| + |S|^{\frac{1}{2}} + 1$, as required.

Then we can suppose

$$s_2 \geq |S|^{\frac{1}{2}}.$$

If $[a, g_2] \neq 1$, then $|S_2^2| = |S_2|^2$, arguing as in Lemma 3.2, moreover we have $S_1^2 \cap S_1S_2 = S_1S_2 \cap S_2^2 = \emptyset$. We have also $S_1^2 \cap S_2^2 = \emptyset$, otherwise from $a^\alpha g_2 a^\beta g_2 \in \langle a, g_1 \rangle$ we obtain $(a^\beta g_2)^2 \in \langle a, g_1 \rangle$ and $g_2 \in C_G(\langle a, g_1 \rangle)$, a contradiction since $\langle S_1 \cup S_2 \rangle$ is not nilpotent. Therefore $S_1^2 \cap S_1S_2 = S_1S_2 \cap S_2^2 = S_1^2 \cap S_2^2 = \emptyset$ and we have $|S^2| \geq |(S_1 \cup S_2)^2| \geq 2s_1 - 1 + s_1 + s_2 - 1 + s_2^2 \geq 2s_1 - 1 + s_1 + s_2 - 1 + s_2 s_2 \geq 2s_1 - 1 + s_1 + s_2 - 1 + 2s_2 + \cdots + 2s_t$, since $s_2 \geq s_3, \ldots, s_t$ and $s_2 \geq |S|^{\frac{1}{2}} \geq 2t$. Therefore $|S^2| \geq 2|S| - 1 + s_1 + s_2 - 1 \geq 2|S| + |S|^{\frac{1}{2}} + 1$, as required.

Finally assume $[a, g_2] = 1$. Suppose, without loss of generality, $[a, g_1] = [a, g_2] = \cdots = [a, g_i] = 1$ and $[a, g_{i+1}], \ldots, [a, g_t] \neq 1$. Write $T = S_1 \cup S_2 \cup \cdots \cup S_i$. Then $\langle T \rangle \subseteq C_G(a)$, hence $\emptyset = S_1S_{i+1} \cap T^2 = \cdots = S_1S_t \cap T^2$. Hence $|S^2| \geq |T^2| + s_1 + s_{i+1} - 1 + \cdots + s_1 + s_t - 1$. Also we have $\langle a \rangle \subseteq Z(\langle T \rangle)$, and $\langle T \rangle / Z(\langle T \rangle)$ is a torsion-free m-Engel group, by Proposition 2.1. We have $g_2 Z(\langle T \rangle) g_1 Z(\langle T \rangle) \neq g_1 Z(\langle T \rangle) g_2 Z(\langle T \rangle)$, since $\langle S_1 \cup S_2 \rangle$ is not nilpotent, therefore, by Proposition 2.4, there exists v, $2 \leq v \leq i$, such that, modulo $Z(\langle T \rangle)$, $g_v g_1 \notin g_1\{g_1, g_2, \ldots, g_i\}$. It follows that $S_v S_1 \cap (S_1^2 \cup S_1S_2 \cup \cdots \cup S_1S_i) = \emptyset$ and we have $|T^2| \geq |S_1^2| + |S_1S_2| + \cdots + |S_1S_i| + |S_vS_1| \geq 2s_1 - 1 + s_1 + s_2 - 1 + \cdots + s_1 + s_i - 1 + s_v + s_1 - 1$ and we have $|S^2| \geq 2|S| + s_1 + s_v - t - 1 \geq 2|S| + |S|^{\frac{1}{2}} + 1$. The lemma is proved. □

4. **Proofs**

In this section we prove Theorem A, Theorem B and Theorem C.

We start with the following definitions.

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Let $n \geq 1$ be an integer and S a finite subset of a torsion-free group G. The n-isoperimetric number $k_n(S)$ of S is the minimum of $|XS| - |X|$ where X is a subset of G with $|X| \geq n$. A subset V is an n-fragment if $|V| \geq n$ and $|VS| - |V| = k_n(S)$. An n-atom for S is an n-fragment of minimal cardinality. Obviously if V is an n-atom for S, then xV is also an n-atom for S, thus there exists an n atom containing 1. In the paper [29] Y. O. Hamidoune conjectured that any n-atom in a torsion-free group has cardinality n. The conjecture is still open, but in [7] it is proved that unique product groups satisfy the conjecture. We use this result in the proof of Theorem A. Notice that it is the only point where we use the unique product hypothesis, therefore Theorem A holds for any torsion-free m-Engel group if Hamidoune's conjecture holds.

Theorem A. *Let G be a unique product m-Engel group, S a finite subset of G, $|S| \geq 6^2$. If $|S^2| \leq 2|S| + n$, where $0 \leq n \leq \frac{1}{2}|S|^{\frac{1}{2}} - 1$, then $\langle S\rangle$ is abelian and S is a subset of a progression of length $|S| + n + 1$.*

Proof. It suffices to prove that $\langle S\rangle$ is abelian. Suppose the contrary. Then we show that $|S^2| \geq 2|S| + \frac{1}{2}|S|^{\frac{1}{2}}$, a contradiction. Let k be the maximal positive integer less or equal to $\frac{1}{2}|S|^{\frac{1}{2}}$. Then $k + 1 > 3$ and $k \geq 3$. Let A be a k-atom for S with $1 \in A$. Then $|A| = k$. We have by definition $|S^2| - |S| \geq |AS| - |A|$. If $\langle A\rangle$ is not cyclic, choose $u, v \in A \setminus \{1\}$ such that $\langle u, v\rangle$ is not cyclic and write $A_0 = \{1, u, v\}$. Then we have $|A_0S| \geq |S| + |S|^{\frac{1}{2}}$ by Lemma 3.3. Therefore: $|S^2| - |S| \geq |AS| - |A| \geq |A_0S| - |A| \geq |S| + |S|^{\frac{1}{2}} - |A| = |S| + |S|^{\frac{1}{2}} - k \geq |S| + |S|^{\frac{1}{2}} - \frac{1}{2}|S|^{\frac{1}{2}}$ and $|S^2| \geq 2|S| + \frac{1}{2}|S|^{\frac{1}{2}}$.

Finally, suppose $\langle A\rangle = \langle a\rangle$ cyclic. If S intersects one right coset of $\langle a\rangle$ then, by Lemma 3.2, $|S^2| = |S|^2 \geq 2|S| + |S| \geq 2|S| + \frac{1}{2}|S|^{\frac{1}{2}}$, if S intersects two right cosets of $\langle a\rangle$ then, by Lemma 3.2, $|S^2| \geq 3|S| - 3 \geq 2|S| + |S| - 3 \geq 2|S| + \frac{1}{2}|S|^{\frac{1}{2}}$, finally if S intersects three right cosets of $\langle a\rangle$ then $|S^2| \geq 2|S| + \frac{1}{2}|S|^{\frac{1}{2}}$. Hence we can assume that S intersects at least four right cosets of $\langle A\rangle$. Write S_1, S_2, S_3 three of these intersections and $S_4 = S \setminus (S_1 \cup S_2 \cup S_3)$. We have:
$|S^2| - |S| \geq |AS| - |A| \geq |AS_1| + |AS_2| + |AS_3| + |AS_4| - |A| \geq |A| + |S_1| - 1 + |A| + |S_2| - 1 + |A| + |S_3| - 1 + |A| + |S_4| - 1 - |A| = 3|A| + |S| - 4 = |S| + 3k - 4 = |S| + k + 1 + 2k - 5 > |S| + \frac{1}{2}|S|^{\frac{1}{2}}$ since $k + 1 > \frac{1}{2}|S|^{\frac{1}{2}}$ and $2k - 5 \geq 1$ since $k \geq 3$. Then $|S^2| \geq 2S| + \frac{1}{2}|S|^{\frac{1}{2}}$ also in this case. □

Theorem B. *Let G be a rigth-ordered m-Engel group, S a finite subset of G, $|S| \geq 3^2$. If $|S^2| \leq 2|S| + n$, where $0 \leq n \leq |S|^{\frac{1}{2}} - 4$, then $\langle S\rangle$ is abelian and S is a subset of a progression of length $|S| + n + 1$.*

Proof. It suffices to prove that $\langle S\rangle$ is abelian. Suppose the contrary. Write $S = \{x_1, x_2, \dots, x_k\}$, and suppose $x_1 < x_2 < \dots < x_k$. First suppose $|\{x_1, x_2, x_3\}S| \geq |S| + |S|^{\frac{1}{2}}$. Write $a = max(\{x_1, x_2, x_3\}S)$. From $x_1x_i < x_2x_i < x_3x_i$, for every $x_i \in S$, it follows $a = x_3x_t$ for some $t \in \{1, 2, \dots, k\}$. Then we have : $x_3x_t < x_4x_t < \dots < x_kx_t$, and $x_4x_t, \dots, x_kx_t \notin \{x_1, x_2, x_3\}S$. Therefore $|S^2| \geq |S| + |S|^{\frac{1}{2}} + |S| - 3 = 2|S| + |S|^{\frac{1}{2}} - 3$, a contradiction. Thus we have $|\{x_1, x_2, x_3\}S| \leq |S| + |S|^{\frac{1}{2}} - 1$. Write $c = x_1^{-1}x_2$, $d = x_1^{-1}x_3$, then from $|\{x_1, x_2, x_3\}S| = |x_1\{1, c, d\}S|$ we obtain that also $|\{1, c, d\}S| \leq |S| + |S|^{\frac{1}{2}} - 1$. Therefore, by Lemma 3.3, $\{1, c, d\}$ is contained in a cyclic subgroup $\langle a\rangle$

of G. Write $S = (S \cap \langle a \rangle g_1) \cdot \cup (S \cap \langle a \rangle g_2) \dot\cup \cdots \dot\cup (S \cap \langle a \rangle g_t)$ and $s_i = |(S \cap \langle a \rangle g_t)|$. If $t \leq \frac{\sqrt{|S|}}{2}$, then $|S^2| \geq 2|S| + |S|^{\frac{1}{2}} + 1$ by Lemma 3.4, a contradiction. Therefore $t > \frac{\sqrt{|S|}}{2}$. But in this case we have $\{1, c, d\}S = (\{1, c, d\}(S \cap \langle a \rangle g_1)) \dot\cup (\{1, c, d\}(S \cap \langle a \rangle g_2)) \dot\cup \cdots \dot\cup (\{1, c, d\}(S \cap \langle a \rangle g_t))$ and $|\{1, c, d\}S| \geq 3 + s_1 - 1 + 3 + s_2 - 1 + \cdots + 3 + s_t - 1 = |S| + 2t > |S| + |S|^{\frac{1}{2}}$, again a contradiction. □

Theorem C. *Let G be a right-ordered m-Engel group, S a finite subset of G, $|S| > 20$. If $|S^2| \leq 2|S| + 2$, then $\langle S \rangle$ is abelian and S is a subset of a geometric progression of length $|S| + 3$.*

Proof. It suffices to prove that $\langle S \rangle$ is abelian. Suppose that contrary. Write $S = \{x_1, x_2, \ldots, x_n\}$, with $x_1 < x_2 < \cdots < x_n$. If $|\{x_1, x_2\}S| \geq |S| + 5$, then write $a = max(\{x_1, x_2\}S)$. From $x_1 x_i \leq x_2 x_i$, for every $x_i \in S$, it follows $a = x_2 x_t$ for some $t \in \{1, 2, \ldots, n\}$. Then we have : $x_2 x_t < x_3 x_t < x_4 x_t < \cdots < x_n x_t$, with $x_3 x_t, \ldots, x_n x_t \notin \{x_1, x_2\}S$. Therefore $|S^2| \geq |S| + 5 + |S| - 2 = 2|S| + 3$, a contradiction. We can argue similarly if $|\{x_1, x_2, x_3\}S| \geq |S| + 6$. Hence we can suppose $|\{x_1, x_2\}S| \leq |S| + 4$ and $|\{x_1, x_2, x_3\}S| \leq |S| + 5$. Write $c = x_1^{-1} x_2$, $d = x_1^{-1} x_3$. Then $|x_1\{1, c\}S| \leq |S| + 4$, $|x_1\{1, c, d\}S| \leq |S| + 5$ imply that $|\{1, c\}S| \leq |S| + 4$, $|\{1, c, d\}S| \leq |S| + 5$. By Lemma 3.1 we have $S = (\langle c \rangle g_1 \cap S) \dot\cup (\langle c \rangle g_2 \cap S) \dot\cup \cdots \dot\cup (\langle c \rangle g_l \cap S)$ with $l \leq 4$, and $S = (\langle d \rangle y_1 \cap S) \dot\cup (\langle d \rangle y_2 \cap S) \dot\cup \cdots \dot\cup (\langle d \rangle y_v \cap S)$, with $v \leq 5$.

We show that $\langle c, d \rangle$ is cyclic.

Obviously it suffices to prove that $\langle c \rangle \cap \langle d \rangle \neq \{1\}$, by Proposition 2.1 since G is torsion-free.

There exists an index i such that $|(\langle c \rangle g_i \cap S)| > 5$, since $|S| > 20$. We have that $v \leq 5$ then there exist $c^\alpha g_i$, $c^\beta g_i$ such that $c^\alpha \neq c^\beta$ and $c^\alpha g_i$ congruent to $c^\beta g_i$ mod $\langle d \rangle$. Therefore $c^{\alpha - \beta} \in \langle d \rangle$ and $\langle c \rangle \cap \langle d \rangle \neq \{1\}$, as required.

Write $\langle c, d \rangle = \langle a \rangle$ and put $S = (\langle a \rangle z_1 \cap S) \dot\cup (\langle a \rangle z_2 \cap S) \dot\cup \cdots \dot\cup (\langle a \rangle z_t \cap S)$.

Then

$$\{1, c, d\}S = (\{1, c, d\}(\langle a \rangle z_1 \cap S)) \dot\cup (\{1, c, d\}(\langle a \rangle z_2 \cap S)) \dot\cup \cdots \dot\cup (\{1, c, d\}(\langle a \rangle z_t \cap S)).$$

If $t = 1$, then, by Lemma 3.2, $|S^2| = |S|^2 \geq 2|S| + 18$, a contradiction. If $t = 2$, then, by Lemma 3.2, we have $|S^2| \geq 3|S| - 3 \geq 2|S| + 17$, a contradiction. Therefore $t \geq 3$. Write $s_i = |(\langle a \rangle z_i \cap S)|$. Then we have $|\{1, c, d\}S| \geq (3 + s_1 - 1) + (3 + s_2 - 1) + \cdots + (3 + s_t - 1) = 2t + |S| \geq 6 + |S|$, a contradiction with $|\{1, c, d\}S| \leq |S| + 5$. □

With similar arguments we can prove Theorem D.

Theorem D. *Let G be a right-orderable m-Engel group, S a finite subset of G, $|S| > 30$. If $|S^2| \leq 2|S| + 3$, then $\langle S \rangle$ is abelian and S is a subset of a geometric progression of length $|S| + 4$.*

Proof. It suffices to prove that $\langle S \rangle$ is abelian. Suppose the contrary. Write $S = \{x_1, x_2, \ldots, x_n\}$, with $x_1 < x_2 < \cdots < x_n$. Arguing as in the proof of Theorem B we can suppose $|\{x_1, x_2\}S| \leq |S| + 5$ and $|\{x_1, x_2, x_3\}S| \leq |S| + 6$. Write $c = x_1^{-1} x_2$, $d = x_1^{-1} x_3$. Then, arguing as in the proof of

Theorem B and using Lemma 3.1 we have $S = (\langle c\rangle g_1 \cap S)\dot{\cup}(\langle c\rangle g_2 \cap S)\dot{\cup}\cdots\dot{\cup}(\langle c\rangle g_l \cap S)$ with $l \leq 5$, and $S = (\langle d\rangle y_1 \cap S)\dot{\cup}(\langle d\rangle y_1 \cap S)\dot{\cup}\cdots\dot{\cup}(\langle d\rangle y_v \cap S)$, with $v \leq 6$.

We show that $\langle c, d\rangle$ is cyclic.

Obviously it suffices to prove that $\langle c\rangle \cap \langle d\rangle \neq \{1\}$ by Proposition 2.1 since G is torsion-free.

There exists an index i such that $|(\langle c\rangle g_i \cap S)| > 7$ since $|S| > 30$ and $l \leq 5$. Since $v \leq 6$ then there exist $c^\alpha g_i$, $c^\beta g_i$ such that $c^\alpha \neq c^\beta$ and $c^\alpha g_i$ congruent to $c^\beta g_i$ mod $\langle d\rangle$. Therefore $c^{\alpha-\beta} \in \langle d\rangle$ and $\langle c\rangle \cap \langle d\rangle \neq \{1\}$, as required.

Write $\langle c, d\rangle = \langle a\rangle$ and put $S = (S \cap \langle a\rangle z_1)\dot{\cup}(\langle a\rangle z_2 \cap S)\dot{\cup}\cdots\dot{\cup}(\langle a\rangle z_t \cap S)$. Then

$$\{1, c, d\}S = (\{1, c, d\}(\langle a\rangle z_1 \cap S))\dot{\cup}(\{1, c, d\}(\langle a\rangle z_2 \cap S))\dot{\cup}\cdots\dot{\cup}(\{1, c, d\}(\langle a\rangle z_t \cap S)).$$

If $t = 1$, then, by Lemma 3.2, $|S^2| = |S|^2 \geq 6|S|$, a contradiction. If $t = 2$, then, by Lemma 3.2, $|S^2| \geq 3|S| - 3 \geq 2|S| + 27$, a contradiction. If $t = 3$, then, again by Lemma 3.2, $|S^2| \geq 2|S| + |S|^{\frac{1}{2}} \geq 2|S| + 5$, a contradiction. Finally, suppose $t \geq 4$. Write $s_i = |(\langle a\rangle z_i \cap S)|$. Then we have $|\{1, c, d\}S| \geq (3 + s_1 - 1) + (3 + s_2 - 1) + \cdots + (3 + s_t - 1) = 2t + |S| \geq 8 + |S|$, a contradiction since $|\{1, c, d\}S| \leq |S| + 7$. □

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