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THE PROBABILITY OF COMMUTING SUBGROUPS IN ARBITRARY LATTICES OF SUBGROUPS

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Abstract. A finite group G, in which two randomly chosen subgroups H and K commute, has been classified by Iwasawa in 1941. It is possible to define a probabilistic notion, which "measures the distance" of G from the groups of Iwasawa. Here we introduce the generalized subgroup commutativity degree $gsd(G)$ for two arbitrary sublattices $\mathrm{S}(G)$ and $\mathrm{T}(G)$ of the lattice of subgroups $\mathrm{L}(G)$ of G. Upper and lower bounds for $gsd(G)$ are shown and we study the behaviour of $gsd(G)$ with respect to subgroups and quotients, showing new numerical restrictions.

1. The new idea and the main results

We deal with finite groups only. The *subgroup commutativity degree* of a group G is defined by

$$sd(G) = \frac{|\{(X,Y) \in \mathrm{L}(G) \times \mathrm{L}(G) \mid XY = YX\}|}{|\mathrm{L}(G)|\ |\mathrm{L}(G)|} \tag{1.1}$$

in [15, Page 2509] and represents the probability of commuting subgroups in the subgroups lattice $\mathrm{L}(G)$ of G. Iwasawa [7] gave the first classifications of groups rich in commuting subgroups (see also [5, 13]). The subgroup commutativity degree extends the probability of commuting elements in G (known as *commutativity degree* of G, see [1, 2, 3, 6]) to the context of lattice theory. Given two sublattices $\mathrm{S}(G)$ and $\mathrm{T}(G)$ of $\mathrm{L}(G)$, we can rephrase (1.1) via the characteristic function

$$\chi : (H,K) \in \mathrm{S}(G) \times \mathrm{T}(G) \longmapsto \chi(H,K) = \begin{cases} 1, & \text{if } HK = KH, \\ 0, & \text{if } HK \neq KH, \end{cases} \tag{1.2}$$

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noting that $\chi(H,K)=1$ if and only if $HK \in \mathrm{L}(G)$. In particular, if $\mathrm{S}(G)=\mathrm{T}(G)=\mathrm{L}(G)$, then it has been shown in [15, Page 2511] that (1.1) becomes exactly

$$sd(G)=\left(\frac{1}{|\mathrm{L}(G)|\cdot|\mathrm{L}(G)|}\right)\sum_{(H,K)\in \mathrm{L}(G)\times \mathrm{L}(G)}\chi(H,K). \tag{1.3}$$

Therefore (1.3) can be taken as an equivalent definition of the subgroup commutativity degree and it allows us to introduce the following new notion, which generalizes those in [8, 9, 10, 11, 15].

Definition 1.1. *For a group G and two sublattices $\mathrm{S}(G)$ and $\mathrm{T}(G)$ of $\mathrm{L}(G)$, the number*

$$gsd(G)=\left(\frac{1}{|\mathrm{S}(G)|\cdot|\mathrm{T}(G)|}\right)\sum_{(H,K)\in \mathrm{S}(G)\times \mathrm{T}(G)}\chi(H,K)$$

is the generalized subgroup commutativity degree of G.

Of course, $gsd(G)=sd(G)$ when $\mathrm{S}(G)=\mathrm{T}(G)=\mathrm{L}(G)$, so it turns out to generalize in particular the invariants in [11, 12, 15, 16, 17]. Now consider a normal subgroup N of G and introduce

$$\alpha(\mathrm{S}(G/N),\mathrm{S}(N))=\frac{1}{|\mathrm{S}(G)|^2}\cdot\Big((|\mathrm{S}(N)|+|\mathrm{S}(G/N)|-1)^2+(gsd(N)-1)\cdot|\mathrm{S}(N)|^2$$
$$+(gsd(G/N)-1)\cdot|\mathrm{S}(G/N)|^2\Big), \tag{1.4}$$

which is only depending on $\mathrm{S}(G/N)$ and $\mathrm{S}(N)$. Similarly we may consider the sets

$$A_1=\{X\in \mathrm{S}(G)\mid N\subseteq X\}, A_2=\{X\in \mathrm{S}(G)\mid X\subset N\} \tag{1.5}$$

$$B_1=\{X\in \mathrm{T}(G)\mid N\subseteq X\}, \text{and } B_2=\{X\in \mathrm{T}(G)\mid X\subset N\} \tag{1.6}$$

and introduce the quantities

$$gsd_1(G)=\frac{1}{|A_1\cup A_2|^2}\sum_{(X,Y)\in(A_1\cup A_2)^2}\chi(X,Y) \tag{1.7}$$

depending on $\mathrm{S}(G)$ and N;

$$gsd_2(G)=\frac{1}{|B_1\cup B_2|^2}\sum_{(X,Y)\in(B_1\cup B_2)^2}\chi(X,Y). \tag{1.8}$$

depending on $\mathrm{T}(G)$ and N;

$$gsd_3(G)=\frac{1}{|A_1|\cdot|B_1|}\sum_{(X,Y)\in A_1\times B_1}\chi(X,Y) \tag{1.9}$$

depending on $\mathrm{S}(G)$, $\mathrm{T}(G)$ and N. Then one can consider the quantity

$$\beta(\mathrm{S}(G),\mathrm{T}(G),N)=\frac{1}{|\mathrm{S}(G)|\;|\mathrm{T}(G)|}\left(|A_1|\;|B_1|\;gsd_3(G)+|B_1-A_1|+|A_1-B_1|\right) \tag{1.10}$$

depending on $\mathrm{S}(G)$, $\mathrm{T}(G)$ and N. Our first main result deals with new bounds for the generalized subgroup commutativity degree in terms of (1.1), (1.7), (1.8), (1.9).

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Theorem 1.2. *Assume that two given sublattices* $\mathrm{S}(G)$ *and* $\mathrm{T}(G)$ *of* $\mathrm{L}(G)$ *satisfy* $\mathrm{S}(G)\cap\mathrm{T}(G) \subseteq \mathrm{N}(G)$, *where* $\mathrm{N}(G)$ *is the sublattice of the normal subgroups of* G.

(i). *If* $\mathrm{S}(G) = \mathrm{T}(G)$, *then* $gsd_1(G) = gsd_2(G)$ *and*

$$|\mathrm{L}(G)|^2\ sd(G) \geq |\mathrm{S}(G)|^2\ gsd(G) \geq |A_1 \cup A_2|^2\ gsd_1(G).$$

(ii). *If* $\mathrm{S}(G) \neq \mathrm{T}(G)$ *and* $A_1 \times B_1 \subseteq (\mathrm{S}(G) - (\mathrm{S}(G)\cap\mathrm{T}(G))) \times (\mathrm{T}(G) - (\mathrm{S}(G)\cap\mathrm{T}(G)))$, *then*

$$|\mathrm{L}(G)|^2\ sd(G) \geq |\mathrm{S}(G)|\ |\mathrm{T}(G)|\ gsd(G) \geq |A_1|\ |B_1|\ gsd_3(G).$$

Our second main result deals with semidirect products $N \rtimes H$ with normal factor N.

Theorem 1.3. *If* $G = N \rtimes H$ *and* $N \in \mathrm{S}(G)\cap\mathrm{T}(G)$, *then*

$$gsd(G) \geq \max\{\alpha(\mathrm{S}(G/N), \mathrm{S}(N)), \beta(\mathrm{S}(G), \mathrm{T}(G), N)\}.$$

The real issue, which we leave open, is the possibility to have an approach in terms of characters and representation theory for the generalized subgroup commutativity degree. It has been shown in [12, Theorem 3.2] that the "strong subgroup commutativity degree" (defined as (1.1), but replacing the condition "$HK = KH$" by the condition "$[H,K] = 1$") can be regarded as a generalized $\mathbb{Q}$-character of G. Therefore:

Question 1.4. *Is it possible to formulate Definition 1.1 in terms of generalized characters of* G *?*

Because if this is possible, then one can find important relations with the theory of the so-called *T-systems* in [4] and with corresponding problems on probabilities on words in [14].

2. Properties of measures and natural bounds

A computational advantage may be found in the calculation of $gsd(G_1 \times G_2)$, where G_1 and G_2 are two given groups.

Corollary 2.1. *Let* G_1 *and* G_2 *be groups of coprime orders. Then*

$$gsd(G_1 \times G_2) = gsd(G_1) \cdot gsd(G_2).$$

Proof. Using [13, Theorem 1.6.9] from $\gcd(G_1, G_2) = 1$, we have $\mathrm{L}(G_1)\cap\mathrm{L}(G_2) = \{1\}$, so $\mathrm{S}(G_1)\cap\mathrm{S}(G_2) = \{1\}$ and $\mathrm{T}(G_1)\cap\mathrm{T}(G_2) = \{1\}$. Then $\mathrm{S}(G_1\times G_2) = \mathrm{S}(G_1)\times\mathrm{S}(G_2)$ and $\mathrm{T}(G_1\times G_2) = \mathrm{T}(G_1)\times\mathrm{T}(G_2)$. Therefore

$$|\mathrm{S}(G_1\times G_2)|\cdot|\mathrm{T}(G_1\times G_2)|\cdot gsd(G_1\times G_2)$$
$$= \sum_{\substack{(Y_1,Y_2)\in\mathrm{T}(G_1\times G_2)\\(X_1,X_2)\in\mathrm{S}(G_1\times G_2)}} \chi((X_1,X_2),(Y_1,Y_2)) = \sum_{\substack{(X_2,Y_2)\in\mathrm{S}(G_2)\times\mathrm{T}(G_2)\\(X_1,Y_1)\in\mathrm{S}(G_1)\times\mathrm{T}(G_1)}} \chi(X_1\times Y_1, X_2\times Y_2)$$
$$= \left(\sum_{(X_1,Y_1)\in\mathrm{S}(G_1)\times\mathrm{T}(G_1)} \chi(X_1,Y_1)\right)\cdot\left(\sum_{(X_2,Y_2)\in\mathrm{S}(G_2)\times\mathrm{T}(G_2)} \chi(X_2,Y_2)\right)$$

$$= (|\mathrm{S}(G_1)| \cdot |\mathrm{T}(G_1)| \cdot gsd(G_1)) \ \cdot \ (|\mathrm{S}(G_2)| \cdot |\mathrm{T}(G_2)| \cdot gsd(G_2))$$

□

If G_1, G_2,...,G_n are groups such that $gcd(|G_i|, |G_j|) = 1$ for all $i, j \in \{1, \ldots, n\}$, then Corollary 2.1 may be generalized to

$$gsd(G_1 \times G_2 \times \cdot \times G_2) = gsd(G_1) \cdot gsd(G_2) \cdots gsd(G_n). \tag{2.1}$$

The proof is omitted because it is by analogy with that of Corollary 2.1.

A classical situation, where we can apply (2.1), is when G is abelian. Recall that an abelian group G of order $n = p_1^{n_1} p_2^{n_2} \cdots p_m^{n_m}$, where $p_1, p_2, \ldots, p_m$ are distinct primes and $n_1, n_2, \ldots, n_m$ are positive integers, has a canonical decomposition of the form $G = G_1 \times G_2 \times \cdots \times G_m$, where $G_1, G_2, \ldots, G_m$ are called p_i- primary components. It is well known from [13, Theorem 1.6.9] that a nilpotent group G has $\mathrm{L}(G) = \mathrm{L}(G_1) \times \mathrm{L}(G_2) \times \ \cdots \times \mathrm{L}(G_m)$ and we have $|\mathrm{L}(G)| = |\mathrm{L}(G_1)| \ |\mathrm{L}(G_2)| \ \cdots \ |\mathrm{L}(G_m)|$. The following consequence of Corollary 2.1 reduces the study of $gsd(G)$ for a nilpotent group G to p-groups.

Corollary 2.2. *If G is nilpotent and G_i a p_i- primary component, then $gsd(G) = \prod_{i=1}^m gsd(G_i)$.*

Introducing the symbol $\mathrm{S}^{\perp}(G)$ for the sublattice of $\mathrm{L}(G)$ containing all subgroups X of G which are permutable with all $S \in \mathrm{S}(G)$, the following result is straightforward, so we omit its proof.

Corollary 2.3. *In a group G we have $gsd(G) = 1$ if and only if $\mathrm{S}(G) \subseteq \mathrm{T}^{\perp}(G)$ or $\mathrm{T}(G) \subseteq \mathrm{S}^{\perp}(G)$.*

We show that the generalized subgroup commutativity degree of G is naturally upper bounded by the subgroup commutativity degree of G.

Lemma 2.4. *In a group G we have*

$$\frac{|\mathrm{S}(G)| \cdot |\mathrm{T}(G)|}{|\mathrm{L}(G)|^2} \cdot gsd(G) \leq \ sd(G)$$

and the bound is achieved, if $\mathrm{S}(G) = \mathrm{T}(G) = \mathrm{L}(G)$. Viceversa, if the previous bound is exact, then

$$\sum_{(X,Y) \in \mathrm{S}(G) \times \mathrm{T}(G)} \chi(X,Y) \geq \sum_{(X,Y) \in \mathrm{L}(G) \times \mathrm{L}(G)} \chi(X,Y).$$

Proof. Since $\mathrm{S}(G) \times \mathrm{T}(G) \subseteq \mathrm{L}(G)^2$, we have

$$\{(X,Y) \in \mathrm{S}(G) \times \mathrm{T}(G) \mid XY = YX\} \subseteq \ \{(X,Y) \in \mathrm{L}(G)^2 \mid XY = YX\}.$$

Then $|\mathrm{S}(G)| \ |\mathrm{T}(G)| \ gsd(G) = |\{(X,Y) \in \mathrm{S}(G) \times \mathrm{T}(G) \mid XY = YX\}| \leq |\{(X,Y) \in \mathrm{L}(G)^2 \mid XY = YX\} = |\mathrm{L}(G)|^2 \ sd(G)$ therefore the bound follows and is clearly achieved when $\mathrm{S}(G) = \mathrm{T}(G) = \mathrm{L}(G)$. On the other hand, if the bound is exact, then

$$|\{(X,Y) \in \mathrm{S}(G) \times \mathrm{T}(G) \mid XY = YX\}| = |\{(X,Y) \in \mathrm{L}(G) \times \mathrm{L}(G) \mid XY = YX\}|,$$

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where the condition $\mathrm{S}(G) \times \mathrm{T}(G) \subseteq \mathrm{L}(G)^2$ shows that

$$|\{(X,Y) \in \mathrm{S}(G) \times \mathrm{T}(G) \mid XY = YX\}| \leq |\{(X,Y) \in \mathrm{L}(G) \times \mathrm{L}(G) \mid XY = YX\}|$$

is always satisfied. The result follows. □

Another basic property is to relate $gsd(G)$ with quotients and subgroups of G.

Lemma 2.5. *If H is a subgroup of G, then*

$$gsd(G) \geq \frac{|\mathrm{S}(H)| \cdot |\mathrm{T}(H)|}{|\mathrm{S}(G)| \cdot |\mathrm{T}(G)|} \cdot gsd(H).$$

Proof. Since $\mathrm{S}(H) \times \mathrm{T}(H) \subseteq \mathrm{S}(G) \times \mathrm{T}(G)$ and

$$\mathrm{S}(G) \times \mathrm{T}(G) = \Big(\mathrm{S}(H) \times \mathrm{T}(H)\Big) \cup \Big((\mathrm{S}(G) - \mathrm{S}(H)) \times (\mathrm{T}(G) - \mathrm{T}(H))\Big),$$

we have that

$$|\mathrm{S}(G)| \cdot |\mathrm{T}(G)| \cdot gsd(G) = \sum_{(X,Y) \in \mathrm{S}(G) \times \mathrm{T}(G)} \chi(X,Y)$$

$$= \sum_{(X,Y) \in \mathrm{S}(H) \times \mathrm{T}(H)} \chi(X,Y) + \sum_{(X,Y) \in (\mathrm{S}(G) - \mathrm{S}(H)) \times (\mathrm{T}(G) - \mathrm{T}(H))} \chi(X,Y)$$

$$\geq \sum_{(X,Y) \in \mathrm{S}(H) \times \mathrm{T}(H)} \chi(X,Y) = |\mathrm{S}(H)| \cdot |\mathrm{T}(H)| \cdot gsd(H).$$

□

In particular we have the following result for semidirect products.

Lemma 2.6. *If $G = N \rtimes H$, then*

$$gsd(G) \geq \frac{|\mathrm{S}(G/N)| \cdot |\mathrm{T}(G/N)|}{|\mathrm{S}(G)| \cdot |\mathrm{T}(G)|} \cdot gsd(G/N).$$

Proof. Note that $H \simeq G/N$ and apply Lemma 2.5. □

Most of the results which we have seen in this section will be applied to the proof of Theorem 1.3. In particular, the above lemma will play an important role.

3. The main results and their proofs

We lower bound $sd(G)$ and $gsd(G)$ in terms of (1.7), (1.8), and (1.9).

Proof of Theorem 1.2. Case (i). We note that $(A_1 \cup A_2) \times (B_1 \cup B_2) \subseteq \mathrm{S}(G) \times \mathrm{T}(G)$, but $\mathrm{S}(G) = \mathrm{T}(G)$ implies $A_1 = B_1$, $A_2 = B_2$ so

$$\sum_{(X,Y) \in \mathrm{S}(G) \times \mathrm{S}(G)} \chi(X,Y) \geq \sum_{(X,Y) \in (A_1 \cup A_2) \times (B_1 \cup B_2)} \chi(X,Y) \tag{3.1}$$

$$= \sum_{(X,Y) \in (A_1 \cup A_2) \times (A_1 \cup A_2)} \chi(X,Y) = gsd_1(G) \ \cdot \ |A_1 \cup A_2|^2,$$

on the other hand

$$gsd_1(G) \cdot |A_1 \cup A_2|^2 = \sum_{(X,Y)\in(B_1\cup B_2)\times(B_1\cup B_2)} \chi(X,Y) \tag{3.2}$$

and $gsd_1(G) = gsd_2(G)$ follows. Note that (3.1) gives

$$|\mathrm{S}(G)| \cdot |\mathrm{S}(G)| \cdot gsd(G) \geq gsd_1(G) \cdot |A_1 \cup A_2|^2 \tag{3.3}$$

and now the result follows from Lemma 2.4.

Case (ii). We begin to write

$$\mathrm{S}(G) = \big(\mathrm{S}(G) - (\mathrm{S}(G)\cap \mathrm{T}(G))\big) \cup (\mathrm{S}(G)\cap\mathrm{T}(G))$$

and, since the same is true also for $\mathrm{T}(G)$, we get

$$\begin{aligned}&\Big(\mathrm{S}(G) - (\mathrm{S}(G)\cap\mathrm{T}(G)) \cup (\mathrm{S}(G)\cap\mathrm{T}(G))\Big) \times \Big((\mathrm{T}(G) - (\mathrm{S}(G)\cap\mathrm{T}(G)) \cup (\mathrm{S}(G)\cap\mathrm{T}(G))\Big)\\ &= \Big((\mathrm{S}(G) - (\mathrm{S}(G)\cap\mathrm{T}(G))) \times (\mathrm{T}(G) - (\mathrm{S}(G)\cap\mathrm{T}(G)))\Big)\\ &\cup\Big((\mathrm{S}(G) - (\mathrm{S}(G)\cap\mathrm{T}(G))) \times (\mathrm{S}(G)\cap\mathrm{T}(G))\Big) \cup \Big((\mathrm{S}(G)\cap\mathrm{T}(G)) \times (\mathrm{T}(G) - (\mathrm{S}(G)\cap\mathrm{T}(G)))\Big)\\ &\cup\Big((\mathrm{S}(G)\cap\mathrm{T}(G)) \times (\mathrm{S}(G)\cap\mathrm{T}(G))\Big) = \mathrm{S}(G)\times\mathrm{T}(G).\end{aligned} \tag{3.4}$$

Therefore

$$\begin{aligned}\sum_{(X,Y)\in\mathrm{S}(G)\times\mathrm{T}(G)} \chi(X,Y) &= \sum_{\substack{X\in\mathrm{S}(G)-(\mathrm{S}(G)\cap\mathrm{T}(G))\\ Y\in T(G)-(\mathrm{S}(G)\cap\mathrm{T}(G))}} \chi(X,Y) +\\ &\sum_{\substack{X\in\mathrm{S}(G)-(\mathrm{S}(G)\cap\mathrm{T}(G))\\ Y\in\mathrm{S}(G)\cap\mathrm{T}(G)}} \chi(X,Y) + \sum_{\substack{Y\in\mathrm{T}(G)-(\mathrm{S}(G)\cap\mathrm{T}(G))\\ X\in\mathrm{S}(G)\cap\mathrm{T}(G)}} \chi(X,Y) + \sum_{\substack{X\in\mathrm{S}(G)\cap\mathrm{T}(G)\\ Y\in\mathrm{S}(G)\cap\mathrm{T}(G)}} \chi(X,Y)\\ &= \Big(\sum_{\substack{X\in\mathrm{S}(G)-(\mathrm{S}(G)\cap\mathrm{T}(G))\\ Y\in T(G)-(\mathrm{S}(G)\cap\mathrm{T}(G))}} \chi(X,Y)\Big) + |\mathrm{S}(G) - (\mathrm{S}(G)\cap\mathrm{T}(G))|\ |\mathrm{S}(G)\cap\mathrm{T}(G)| +\\ &+|\mathrm{T}(G) - (\mathrm{S}(G)\cap\mathrm{T}(G))|\ |\mathrm{S}(G)\cap\mathrm{T}(G)| + |\mathrm{S}(G)\cap\mathrm{T}(G)|^2\end{aligned} \tag{3.5}$$

$$= \Big(\sum_{\substack{X\in\mathrm{S}(G)-(\mathrm{S}(G)\cap\mathrm{T}(G))\\ Y\in T(G)-(\mathrm{S}(G)\cap\mathrm{T}(G))}} \chi(X,Y)\Big) + |\mathrm{S}(G)\cap\mathrm{T}(G)| \cdot \Big(|\mathrm{S}(G)| + |\mathrm{T}(G)| - |\mathrm{S}(G)\cap\mathrm{T}(G)|\Big) \tag{3.6}$$

$$\geq \sum_{\substack{X\in\mathrm{S}(G)-(\mathrm{S}(G)\cap\mathrm{T}(G))\\ Y\in T(G)-(\mathrm{S}(G)\cap\mathrm{T}(G))}} \chi(X,Y) \tag{3.7}$$

Therefore by our assumption,

$$\sum_{(X,Y)\in\mathrm{S}(G)\times\mathrm{T}(G)} \chi(X,Y) \geq \sum_{\substack{X\in\mathrm{S}(G)-(\mathrm{S}(G)\cap\mathrm{T}(G))\\ Y\in T(G)-(\mathrm{S}(G)\cap\mathrm{T}(G))}} \chi(X,Y) \geq \sum_{\substack{X\in A_1\\ Y\in B_1}} \chi(X,Y)$$

Hence the result follows. □

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We can not remove the additional condition of Theorem 1.2 (ii) because of the following example.

Example 3.1. The dihedral group of order 8 is $D_8 = \langle a, b \mid a^2 = b^4 = 1, a^{-1}ba = b^{-1}\rangle$ and has

$$\mathrm{L}(D_8) = \{\{1\}, \langle b\rangle, \langle b^2\rangle, \langle a\rangle, \langle ba\rangle, \langle b^2a\rangle, \langle b^3a\rangle, \{1, b^2, a, b^2a\}, \{1, b^2, ba, b^3a\}, D_8\}.$$

Denoting $B = \langle b\rangle$, $Z(D_8) = \langle b^2\rangle$, $M_1 = \{1, b^2, a, b^2a\}$, $M_2 = \{1, b^2, ba, b^3a\}$, we have $\mathrm{N}(D_8) = \{D_8, \{1\}, B, Z(D_8), M_1, M_2\}$. Notice that $H = \langle b^2a\rangle$ and $K = \langle a\rangle$ are contained in M_1, while $U = \langle ba\rangle$ and $V = \langle b^3a\rangle$ in M_2. Suppose to have the following two sublattices of $\mathrm{L}(D_8)$; $\mathrm{S}(G) = \{\{1\}, H, K, M_1, Z(D_8), B, D_8\}$ and $\mathrm{T}(G) = \{\{1\}, U, V, M_2, Z(D_8), B, D_8\}$, and choose now $A_1 = \{Z(D_8), M_1, B, D_8\}$ and $B_1 = \{Z(D_8), M_2, B, D_8\}$. Then $\mathrm{S}(G) \cap \mathrm{T}(G) = \{Z(D_8), B, D_8\}$, $\mathrm{S}(G) - (\mathrm{S}(G) \cap \mathrm{T}(G)) = \{\{1\}, H, K, M_1\}$, $\mathrm{T}(G) - (\mathrm{S}(G) \cap \mathrm{T}(G)) = \{\{1\}, U, V, M_2\}$ and of course $Z(D_8) \times Z(D_8) \in A_1 \times B_1$ but $Z(D_8) \times Z(D_8) \notin (\mathrm{S}(G) - (\mathrm{S}(G) \cap \mathrm{T}(G))) \times (\mathrm{T}(G) - (\mathrm{S}(G) \cap \mathrm{T}(G)))$.

Now we begin with a first case of the proof of Theorem 1.3.

Lemma 3.2. *If a group G has a normal subgroup $N \in \mathrm{S}(G)$ and $\mathrm{S}(G) = \mathrm{T}(G)$, then*

$$gsd(G) \geq \alpha(\mathrm{S}(G/N), \mathrm{S}(N)).$$

Proof. Since $N \in \mathrm{S}(G)$ and $\mathrm{S}(G) = \mathrm{T}(G)$, [15, Proposition 2.4] gives a method to calculate $gsd_1(G)$:

$$|A_1 \cup A_2|^2 \; gsd_1(G) = \sum_{(X,Y)\in(A_1\cup A_2)^2} \chi(X, Y) = \sum_{X,Y\in A_1\cup A_2} \chi(X, Y) \tag{3.8}$$

$$= \sum_{X,Y\in A_1} \chi(X, Y) + \sum_{X,Y\in A_2} \chi(X, Y) + 2 \sum_{X\in A_1} \sum_{Y\in A_2} \chi(X, Y),$$

and so we evaluate the three terms separately:

$$\sum_{X,Y\in A_1} \chi(X, Y) = \sum_{(X,Y)\in A_1\times A_1} \chi(X, Y) = gsd(G/N) \cdot |\mathrm{S}(G/N)|^2; \tag{3.9}$$

$$\sum_{X,Y\in A_2} \chi(X, Y) = \sum_{X,Y\in A_2\cup\{N\}} \chi(X, Y) - 2 \sum_{X\in A_2\cup\{N\}} \chi(X, N) + 1 \tag{3.10}$$

$$= gsd(N) \cdot |\mathrm{S}(N)|^2 - 2|\mathrm{S}(N)| + 1;$$

$$2 \sum_{X\in A_1} \sum_{Y\in A_2} \chi(X, Y) = 2 \; |A_1| \cdot |A_2| = 2 \; |\mathrm{S}(G/N)| \cdot (|\mathrm{S}(N)| - 1). \tag{3.11}$$

Therefore we may apply Theorem 1.2 (i), and we get the result. □

Now we focus on the case of a semidirect product.

Lemma 3.3. *Assume $G = N \rtimes H$ and $N \in \mathrm{S}(G) \cap \mathrm{T}(G)$. If $\mathrm{S}(G) \neq \mathrm{T}(G)$, then*

$$gsd(G) \geq \beta(\mathrm{S}(G), \mathrm{T}(G), N).$$

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Proof. Since $\mathrm{S}(G) \neq \mathrm{T}(G)$, the argument of Lemma 3.2 gives problems due to the application of Theorem 1.2 (i) in its final part. On the other hand,

$$gsd(G/N)\ |\mathrm{S}(G/N)|\ |\mathrm{T}(G/N)| = \sum_{X,Y\in A_1\cup B_1} \chi(X,Y) \tag{3.12}$$

$$= \sum_{\substack{X\in A_1\\ Y\in B_1}} \chi(X,Y) + \sum_{\substack{X\in B_1-A_1\\ Y\in B_1}} \chi(X,Y) + \sum_{\substack{X\in A_1\cup B_1\\ Y\in A_1-B_1}} \chi(X,Y)$$

and the fact that $N \in A_1 \cup B_1$ and $N \in B_1$ imply

$$\sum_{\substack{X\in B_1-A_1\\ Y\in B_1}} \chi(X,Y) + \sum_{\substack{X\in A_1\cup B_1\\ Y\in A_1-B_1}} \chi(X,Y) \geq |B_1 - A_1| + |A_1 - B_1|$$

and we may conclude

$$gsd(G/N)\ |\mathrm{S}(G/N)|\ |\mathrm{T}(G/N)| \geq |A_1|\ |B_1|\ gsd_3(G) + |B_1 - A_1| + |A_1 - B_1|.$$

Now the result follows from Lemma 2.6. □

Note that the proof of Lemma 3.3 shows a lower bound of independent interest.

Corollary 3.4. *If $G = N \rtimes H$ and $N \in \mathrm{S}(G) \cap \mathrm{T}(G)$ with $\mathrm{S}(G) \neq \mathrm{T}(G)$, then*

$$|\mathrm{S}(G/N)|\ |\mathrm{T}(G/N)|\ gsd(G/N) \geq\ gsd_3(G)\ |A_1|\ |B_1|.$$

We collect the result we obtained and our second main result follows.

Proof of Theorem 1.3. It follows from Lemmas 3.2 and 3.3. □

4. Applications

There are some interesting specializations of Theorem 1.3.

Corollary 4.1. *Assume that $G = N \rtimes H$ has $N \in \mathrm{S}(G) \cap \mathrm{T}(G)$. If $gsd(N) = gsd(G/N) = 1$, then*

$$gsd(G) \geq \max\left\{ \left(\frac{|\mathrm{S}(N)| + |\mathrm{S}(G/N)| - 1}{|\mathrm{S}(G)|} \right)^2, \beta(\mathrm{S}(G), \mathrm{T}(G), N) \right\}.$$

Proof. Application of Theorem 1.3. □

A classical situation, in which Corollary 4.1 is applicable, is when $G = N \rtimes H$ is metabelian. Here if $N = G'$ and $\mathrm{S}(G) = \mathrm{T}(G) = \mathrm{L}(G)$, then we get exactly [15, Corollary 2.5].

Corollary 4.2. *Assume $G = N \rtimes H$ and $N \in \mathrm{S}(G) \cap \mathrm{T}(G)$. If N is of prime index, then*

$$gsd(G) \geq \max\left\{ \frac{gsd(N) \cdot |\mathrm{S}(N)|^2 + 2|\mathrm{S}(N)| + 1}{|\mathrm{S}(G)|^2}, \beta(\mathrm{S}(G), \mathrm{T}(G), N) \right\}.$$

Proof. Since N is of prime index, $|\mathrm{S}(G/N)| = |\mathrm{T}(G/N)| = 2$ and $gsd_1(G/N) = gsd_2(G/N) = 1$. The result follows from Theorem 1.3. □

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It is useful to compare Corollary 4.2 with [15, Corollary 2.6]. We now offer an example in which the conditions of Theorem 1.3 are satisfied.

Example 4.3. The symmetric group $G = S_3$ has a unique minimal normal subgroup $N = A_3$ and this is atomic, that is, it covers the identity element in $\mathrm{L}(S_3)$ (see [13]). Here any choice of $\mathrm{S}(G)$ and $\mathrm{T}(G)$ satisfies the assumptions of Theorem 1.3 with $N \in \mathrm{S}(G) \cap \mathrm{T}(G)$. On the other hand, if we consider $G = S_4$, then there is again a normal subgroup $N = A_4$, but it is well known that A_4 is not atomic in $\mathrm{L}(S_4)$, so for $G = S_4$ an appropriate choice of N, depending on a corresponding choice for $\mathrm{S}(G)$ and $\mathrm{T}(G)$ must be taken into account (because in general the condition $N \in \mathrm{S}(G) \cap \mathrm{T}(G)$ might be false). More generally for an odd prime p and $r \geq 1$,

$$G = \langle x, a_1, a_2, \ldots, a_r \mid x^2 = a_1^p = a_2^p = \cdots = a_r^p = 1, x^{-1}a_ix = a_i^{-1}, [a_i, a_j] = 1, \ \forall\ i, j \in \{1, 2, \ldots, r\}\rangle$$

is of order $2p^r$ may be written in the form $G = N \rtimes H$, where N is an elementary abelian p-subgroup of rank r and $H = \mathbb{Z}_2 = \langle x \rangle$ is of order two acting on N by inversion. Here N turns out to be atomic in $\mathrm{L}(G)$. Here G satisfies the assumptions of Theorem 1.3, when $\mathrm{S}(G)$ and $\mathrm{T}(G)$ are chosen in such a way that $N \in \mathrm{S}(G) \cap \mathrm{T}(G)$.

The presence of atomic normal subgroups implies the following result.

Corollary 4.4. *If N is an atomic normal subgroup of $G = N \rtimes H$, $N \in \mathrm{S}(G)$ and $\mathrm{S}(G) = \mathrm{T}(G)$, then $gsd(G)$ is lower bounded by a function depending only on G/N, namely*

$$gsd(G) \geq \frac{1}{|\mathrm{S}(G)|^2} \cdot \Big(gsd_1(G/N).|\mathrm{S}(G/N)|^2 + 2|\mathrm{S}(G/N)| + 1\Big).$$

Proof. Assume $\mathrm{S}(G) = \mathrm{T}(G)$ and $N \in \mathrm{S}(G)$. We can calculate $gsd_1(G)$ by the argument in Lemma 3.2. Since N is atomic, $|\mathrm{S}(N)| = 2$ and $gsd_1(N) = 1$, and so we may apply Theorem 1.2 (i) and Lemma 3.2, getting the result. □

We shall compute $\alpha(\mathrm{S}(G/N), \mathrm{S}(N))$ and $\beta(\mathrm{S}(G), \mathrm{T}(G), N)$ explicitly if G has an abelian normal subgroup $N = \mathbb{Z}_{p^{\alpha_1}} \times \mathbb{Z}_{p^{\alpha_2}}$ with $1 \leq \alpha_1 \leq \alpha_2$ and prime p. This will involve a polynomial function which has been studied in [17].

Corollary 4.5. *Suppose a group $G = N \rtimes H$ has an abelian subgroup $N = \mathbb{Z}_{p^{\alpha_1}} \times \mathbb{Z}_{p^{\alpha_2}}$ with $1 \leq \alpha_1 \leq \alpha_2$ and p prime. If $N \in \mathrm{S}(G)$ and $\mathrm{S}(G) = \mathrm{T}(G)$ with $H \simeq G/N$ of prime order, then*

$$gsd(G) \geq \frac{1}{(p-1)^4 \cdot |\mathrm{S}(G)|^2} \cdot [(\alpha_2 - \alpha_1 + 1)p^{\alpha_1+2} - (\alpha_2 - \alpha_1 - 1)p^{\alpha_1+1} - (\alpha_2 + \alpha_1 + 3)p + (\alpha_2 + \alpha_1 + 2)]^2.$$

Proof. Assume $\mathrm{S}(G) = \mathrm{T}(G)$ and $N = \mathbb{Z}_{p^{\alpha_1}} \times \mathbb{Z}_{p^{\alpha_2}} \in \mathrm{S}(G)$. From Lemma 3.2,

$$gsd(G) \geq \frac{1}{|\mathrm{S}(G)|^2} \cdot \Big((|\mathrm{S}(N)| + |\mathrm{S}(G/N)| - 1)^2 + (gsd(N) - 1) \cdot |\mathrm{S}(N)|^2 + (gsd(G/N) - 1) \cdot |\mathrm{S}(G/N)|^2\Big).$$

Since $gsd(G/N) = gsd(N) = 1$ and $|\mathrm{S}(G/N)| = 2$, we obtain

$$gsd(G) \geq \Big(\frac{|\mathrm{S}(N)|+1}{|\mathrm{S}(G)|}\Big)^2.$$

Now [17, Theorem 3.3] implies that

$$|\mathrm{S}(N)| = \frac{1}{(p-1)^2} \cdot [(\alpha_2-\alpha_1+1)p^{\alpha_1+2} - (\alpha_2-\alpha_1-1)p^{\alpha_1+1} - (\alpha_2+\alpha_1+3)p + (\alpha_2+\alpha_1+1)]$$

and then,

$$\Big(\frac{|\mathrm{S}(N)|+1}{|\mathrm{S}(G)|}\Big)^2 = \frac{1}{(p-1)^4 \cdot |\mathrm{S}(G)|^2} \cdot [(\alpha_2-\alpha_1+1)p^{\alpha_1+2} - (\alpha_2-\alpha_1-1)p^{\alpha_1+1} - (\alpha_2+\alpha_1+3)p + (\alpha_2+\alpha_1+1)+1]^2.$$

□

Corollary 4.5 improves [11, Lemma 2.6], where specific choices of the sublattices are involved. Another generalization is reported separately for the subgroup commutativity degree.

Corollary 4.6. *In the same assumptions of Corollary 4.5,*

$$sd(G) \geq \frac{1}{(p-1)^4 \cdot |\mathrm{L}(G)|^2} \cdot [(\alpha_2-\alpha_1+1)p^{\alpha_1+2} - (\alpha_2-\alpha_1-1)p^{\alpha_1+1} - (\alpha_2+\alpha_1+3)p + (\alpha_2+\alpha_1+2)]^2.$$

Proof. See Lemma 2.4 and Corollary 4.5. □

Note that Corollary 4.6 improves the bound in [11, Theorem 2.8].

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