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SUBGROUPS OF ARBITRARY EVEN ORDINARY DEPTH

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ABSTRACT. We show that for each positive integer n, there exist a group G and a subgroup H such that the ordinary depth $d(H, G)$ is $2n$. This solves the open problem posed by Lars Kadison whether even ordinary depth larger than 6 can occur.

1. Introduction

The notion of depth was originally defined for von-Neumann algebras, see [9]. Later it was also defined for Hopf algebras, see [20]. For some recent results in this direction, see [11, 17, 10]. In [19] and later in [4], the depth of semisimple algebra inclusions was studied, by Burciu, Kadison and Külshammer. First results concerned the depth 2 case. Later these were generalized for arbitrary n. In the case of group algebra inclusion $\mathbb{C}H \subseteq \mathbb{C}G$ it was shown that the depth is at most 2 if and only if H is normal in G, see [19]. For similar results on group algebras over commutative rings, see [3].

Let F be a field. We say that the *depth of the group algebra inclusion* $FH \subseteq FG$ is $2n$, for a positive integer n, if $FG \otimes_{FH} \cdots \otimes_{FH} FG$ ($n+1$-times FG) is isomorphic to a direct summand of $\oplus_{i=1}^{a} FG \otimes_{FH} \cdots \otimes_{FH} FG$ (n times FG) as $FG - FH$-bimodules (or equivalently as $FH - FG$-bimodules) for some positive integer a.

Furthermore, FH is said to have depth $2n+1$, for a positive integer n, in FG if the same assertion holds for $FH - FH$-bimodules, for some positive integer a. Finally FH has depth 1 in FG if FG is isomorphic to a direct summand of $\oplus_{i=1}^{a} FH$ as $FH - FH$ bimodules, for some positive integer a.

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The *ordinary depth* $d(H,G)$ of a subgroup H in a finite group G is defined as the minimal depth of the group algebra inclusion $\mathbb{C}H \subseteq \mathbb{C}G$. This is well defined. It is shown in [1, Remark 4.5] by Boltje, Danz and Külshammer that the depth of group algebra inclusion does not depend on the field, only on the characteristic. If the characteristic is prime then we get the notion of modular depth. From now on we will always consider group algebras over $\mathbb{C}$.

The ordinary depth can be obtained from the so called *inclusion or Frobenius matrix* M. If $\chi_1,\ldots,\chi_s$ are all of the irreducible characters of G and $\psi_1,\ldots,\psi_r$ are all of the irreducible characters of H, then $m_{i,j} := (\psi_i^G, \chi_j)$. Let $M = (m_{i,j})$. Some kinds of powers of M are defined by $M^{(1)} := M$, $M^{(2l)} := M^{(2l-1)}M^T$, $M^{(2l+1)} := M^{(2l)}M$, for positive integers l, and $M^{(0)}$ is the $r \times r$ unit matrix. The ordinary depth $d(H,G)$ can be obtained as the smallest positive integer n such that $M^{(n+1)} \leq aM^{(n-1)}$ for some positive integer a, where the inequality of matrices means that this inequality holds componentwise.

The results on characters in [4] help to determine $d(H,G)$. Two irreducible characters $\alpha, \beta \in \mathrm{Irr}(H)$ are called *related*, $\alpha \sim_G \beta$, if they are constituents of χ_H, for some $\chi \in \mathrm{Irr}(G)$. The *distance* $d_G(\alpha,\beta) = m$ is the smallest integer m such that there is a chain of irreducible characters of H such that $\alpha = \psi_0 \sim_G \psi_1 \cdots \sim_G \psi_m = \beta$. If there is no such chain then $d_G(\alpha,\beta) = -\infty$ and if $\alpha = \beta$ then the distance is zero. If X is the set of irreducible constituents of χ_H then we set $m(\chi) := \max\{\min\{d_G(\alpha,\psi); \psi \in X\}; \alpha \in \mathrm{Irr}(H)\}$. We will use the following result from [4].

Theorem 1.1. [4, Theorem 3.6, Theorem 3.10]

Let H be a subgroup of a finite group G.

(i) *Let $m \geq 1$. Then H has ordinary depth $\leq 2m+1$ in G if and only if the distance between two irreducible characters of H is at most m.*

(ii) *Let $m \geq 2$. Then H has ordinary depth $\leq 2m$ in G if and only if $m(\chi) \leq m-1$ for all $\chi \in \mathrm{Irr}(G)$.*

Thus we have the following.

Corollary 1.2. *Let H be a subgroup of a finite group G. The* ordinary depth *$d(H,G)$ is the minimal possible positive integer which can be determined from the upper bounds (i) and (ii) of Theorem 1.1 and from*

(iii) *$d(H,G) \leq 2$ if and only if H is normal in G, see* [19, Corollary 3.2],

(iv) *$d(H,G) = 1$ if and only if $G = HC_G(x)$ for all $x \in H$, see* [2, Theorem 1.7].

We will also use the following result from [4].

Theorem 1.3. [4, Theorem 6.9] *Suppose that H is a subgroup of a finite group G and $N = \mathrm{Core}_G(H)$ is the intersection of m conjugates of H. Then $d(H,G) \leq 2m$. If additionally $N \leq Z(G)$ holds then $d(H,G) \leq 2m-1$.*

In recent publications, several authors determined the ordinary depth of subgroups in some special series of groups, e.g. $PSL(2,q)$, Suzuki groups, Ree groups, symmetric and alternating groups, see

[4], [6], [7], [14], [15]. In [5], twisted group algebra inclusions for symmetric and alternating groups are studied.

It is known that odd ordinary depth of a subgroup in a finite group can be arbitrarily large: It is shown in [4] that the (minimal) ordinary depth of the symmetric group S_n in S_{n+1} is $2n-1$.

Lars Kadison posed the following open problem on his homepage, see [18]: Are there subgroups of (minimal) ordinary depth $2n$ where $n > 3$?

If one looks at the results of the above papers or the calculations presented in [13], one has the impression that in most cases the depth of subgroups is odd. However still one can find examples of arbitrarily large even depth. In our examples wreath products will play an important role. In this short note we will always consider ordinary depth, so in the following depth will always mean ordinary depth.

The main result of this paper is the following.

Theorem 1.4. *There exists a series of groups and subgroups (G_n, H_n) such that $d(H_n, G_n) = 2n$ for every positive integer n.*

2. **Constructing examples**

An example of a subgroup H of depth 6 in a group G is mentioned in [4] as found with GAP [8]: One takes $G = AGL(2,3)$ and $H = N_G(P)$, where $P \in Syl_3(G)$. Note that $|G| = 432$ and $|H| = 108$.

The smallest examples of depth 6 are G of structure $C_2 \times C_4^2 \rtimes C_3$ and $H \cong C_4^2$, and G of structure $C_2 \times C_2^4 \rtimes C_3$ and $H \cong C_2^4$, see [13]. The groups G can be found in the Small groups library of GAP [8] as SmallGroup(96, 68) and SmallGroup(96, 229), respectively.

More examples of depth 6 were found with GAP [8] among maximal subgroups of some alternating groups, see [13]: $d(2^4 : (S_3 \times S_3), A_8) = 6$ and $d(S_7, A_9) = 6$.

The following examples of subgroups of depth 8 had been constructed earlier by the third author with the help of the GAP system [8], see [13]: $d(A_{15} \cap (S_{12} \times S_3), A_{15}) = 8$, $d(2^6 : U_4(2), O_8^-(2)) = 8$, and $d(G \cap (A_8 \times A_8), G) = 8$, for $G = ((C_2 \wr C_2) \wr C_2) \wr C_2$.

It was shown already in [4] that $d(D_8, S_4) = 4$ holds. The first author found with GAP that $d(D_8 \times S_4, S_4 \wr C_2) = 8$. Continuing this process, we obtained that $d((D_8 \times S_4) \times (S_4 \wr C_2), (S_4 \wr C_2) \wr C_2) = 16$. In general, we can define

- $G_0 := S_4$, $H_0 := D_8$,
- $G_n := G_{n-1} \wr C_2$, $H_n := H_{n-1} \times G_{n-1} < G_{n-1} \times G_{n-1} < G_n$,

and get $d(H_n, G_n) = 2^{n+2}$.

The idea of the proof is to use Theorem 1.3 to prove that $d(H_n, G_n) \leq 2^{n+2}$. Then we show that the depth cannot be at most $2^{n+2} - 1 = 2(2^{n+1} - 1) + 1$, since by Corollary 1.2 then the distance of any two characters of H_n would be at most $2^{n+1} - 1$, however there are irreducible characters of H_n of distance exactly 2^{n+1}. The proof is a rather complicated induction, see [16].

We wanted to simplify the construction. Our aim was also to construct as depth more even numbers. We can generalize the first two steps of the former construction in another way as follows:

- $d(D_8, S_4) = 4$,
- $d(D_8 \times S_4, S_4 \wr C_2) = 8$,
- $d(D_8 \times S_4 \times S_4, S_4 \wr C_3) = 12$.

In general, we take

- $G_1 := S_4$, $H_1 := D_8$,
- $G_n := G_1 \wr C_n$, $H_n := H_1 \times G_1^{n-1} < G_1^n < G_n$.

Then we have that $d(H_n, G_n) = 4n$. The proof is again using Theorem 1.3 to prove that $d(H_n, G_n) \leq 4n$. If $d(H_n, G_n) \leq 4n - 1 = 2(2n-1)+1$, then by Corollary 1.2 any two irreducible characters of H_n have distance at most $2n - 1$. However, one can show that there exist irreducible characters of H_n of distance $2n$.

If we want to get every even number then we can use a modified construction. We take the Klein four group $V_4 \lhd S_4$ instead of D_8 and get:

- $d(V_4, S_4) = 2$,
- $d(V_4 \times S_4, S_4 \wr C_2) = 4$,
- $d(V_4 \times S_4 \times S_4, S_4 \wr C_3) = 6$.

In general, we have a series of groups and subgroups such that $d(H_n, G_n) = 2n$ holds. The idea of the proof will be the same as before, for the inequality we will use again Theorem 1.3, and to prove that it cannot be a strict inequality, we find two irreducible characters of distance n in H_n. For that, we consider suitable characters of the base group of the wreath product and define a Cartesian product of graphs that encodes the relation $\sim$.

3. Proof of Theorem 1.4

Let G be the symmetric group on four points, and N be its normal Klein four subgroup. Set $G_1 = G$, $H_1 = N$. Then $d(H_1, G_1) = 2$, by Corollary 1.2. Define for $n \geq 2$

$$\begin{aligned} \sigma_n &= \prod_{j=1}^{4} (j, j+4, j+8, \cdots, j+4(n-1)), \\ G_n &= \langle G, \sigma_n \rangle, \\ H_n &= \langle N, G^{\sigma_n}, G^{\sigma_n^2}, \ldots, G^{\sigma_n^{n-1}} \rangle. \end{aligned}$$

Let C_n denote the cyclic group of order n. Then $H_n < G_n \cong G \wr C_n$ and

$$H_n \cong N \times G^{n-1} \leq G^n < G \wr C_n.$$

Let $N_n = \mathrm{Core}_{G_n}(H_n)$, the largest normal subgroup of G_n that is contained in H_n. Then $N_1 = N$, and

$$N_n = \langle N, N^{\sigma_n}, \ldots, N^{\sigma_n^{n-1}} \rangle = \bigcap_{i=0}^{n-1} H_n^{\sigma_n^i}$$

is an intersection of n conjugates of H_n, and Theorem 1.3 yields $d(H_n, G_n) \leq 2n$. Set

$$K_n = \langle G, G^{\sigma_n}, \ldots, G^{\sigma_n^{n-1}} \rangle \leq G_n.$$

Then $H_n \leq K_n \cong G^n$.

The character tables of N and G are as follows, where the columns are indexed by the conjugacy classes of the elements $g_1 = ()$, $g_2 = (1,3)(2,4)$, $g_3 = (1,2)(3,4)$, $g_3' = (1,2,3)$, $g_4 = (1,4)(2,3)$, $g_4' = (1,3)$, $g_5 = (1,2,3,4)$.

	g_1	g_2	g_3	g_4
ν_1	1	1	1	1
ν_2	1	1	−1	−1
ν_3	1	−1	1	−1
ν_4	1	−1	−1	1

	g_1	g_2	g_3'	g_4'	g_5
χ_1	1	1	1	1	1
χ_2	1	1	1	−1	−1
χ_3	2	2	−1	0	0
χ_4	3	−1	0	1	−1
χ_5	3	−1	0	−1	1

$$\begin{aligned} \chi_1|_N &= \nu_1 \\ \chi_2|_N &= \nu_1 \\ \chi_3|_N &= 2\nu_1 \\ \chi_4|_N &= \nu_2 + \nu_3 + \nu_4 \\ \chi_5|_N &= \nu_2 + \nu_3 + \nu_4 \end{aligned}$$

Set

$$X_n = \{\chi_{i_1} \times \chi_{i_2} \times \cdots \times \chi_{i_n} \in \mathrm{Irr}(K_n); i_1 \in \{4,5\}, i_j \in \{1,2,3\} \text{ for } 2 \leq j \leq n\}$$

and

$$Y_n = \left\{\chi^{G_n}; \chi \in X_n\right\}.$$

Let Γ_1 be the undirected graph with vertex set $\{4,5\}$ and edge set $\{\{4,5\}\}$, Γ_0 be the undirected graph with vertex set $\{1,2,3\}$ and edge set $\{\{1,3\},\{2,3\},\{1,2\}\}$. For $n \geq 2$, let Γ_n be the Cartesian product of Γ_1 and $n-1$ copies of Γ_0, that is, Γ_n has vertex set

$$\{(i_1, i_2, \ldots, i_n); i_1 \in \{4,5\}, i_j \in \{1,2,3\} \text{ for } 2 \leq j \leq n\},$$

and there is an edge between $(i_1, i_2, \ldots, i_n)$ and $(i_1', i_2', \ldots, i_n')$ if and only if there is a (unique) j such that $i_k = i_k'$ for $k \neq j$ and $i_j \neq i_j'$ and either $\{i_j, i_j'\} = \{4,5\}$ or $\{i_j, i_j'\} \subset \{1,2,3\}$.

Lemma 3.1.

(i) *$Y_n \subseteq \mathrm{Irr}(G_n)$, and mapping χ to χ^{G_n} defines a bijection from X_n to Y_n.*

(ii) *For $\psi \in Y_n$ and $\psi' \in \mathrm{Irr}(G_n)$, if $\psi|_{H_n}$ and $\psi'|_{H_n}$ have a common constituent then $\psi' \in Y_n$.*

(iii) *Let $\psi = \chi^{G_n}$, $\psi' = (\chi')^{G_n}$ for $\chi, \chi' \in X_n$, with $\psi \neq \psi'$. Then $\psi|_{H_n}$ and $\psi'|_{H_n}$ have a common constituent if and only if there is an edge between $(i_1, i_2, \ldots, i_n)$ and $(i_1', i_2', \ldots, i_n')$ in Γ_n, where $\chi = \chi_{i_1} \times \chi_{i_2} \times \cdots \times \chi_{i_n}$ and $\chi' = \chi_{i_1'} \times \chi_{i_2'} \times \cdots \times \chi_{i_n'}$.*

(iv) *The distance of the vertices $(4,1,1,\ldots,1)$ and $(4,2,2,\ldots,2)$ of Γ_n is $n-1$.*

(v) *The distance $d_{G_n}(\alpha_n, \omega_n)$ of the characters $\alpha_n := \nu_2 \times \chi_1 \times \cdots \times \chi_1$ and $\omega_n := \nu_2 \times \chi_2 \times \cdots \times \chi_2$ of H_n is n.*

Proof. Let $\psi = \chi^{G_n}$, where $\chi = \chi_{i_1} \times \chi_{i_2} \times \cdots \times \chi_{i_n} \in X_n$, that is, χ_{i_1} is faithful and the other χ_{i_j} are not.

For part (i), χ has inertia subgroup K_n inside G_n. Hence by Clifford's Theorem, see [12, Theorem 6.11], χ^{G_n} is irreducible. The irreducible constituents of the restriction $\psi|_{K_n}$ are the n conjugates of χ by σ_n, i. e., those characters where the n components of χ are cyclically permuted. Thus each constituent has exactly one faithful component. Hence χ is the only constituent of $\psi|_{K_n}$ that lies in X_n. Thus we get an inverse to the map $\chi \mapsto \chi^{G_n}$.

For part (ii), consider the restriction of the constituents of $\psi|_{K_n}$ to H_n. We get irreducible constituents where the first component is a nontrivial character of N and all other components are non-faithful characters of G, and irreducible constituents where the first component is the trivial character of N and exactly one other component is faithful. Let $\psi' \in \mathrm{Irr}(G_n)$ have the property that $\psi'|_{H_n}$ and $\psi|_{H_n}$ have a common irreducible constituent, which means that $0 \neq (\psi|_{H_n}, \psi'|_{H_n}) = ((\psi|_{H_n})^{G_n}, \psi')$. If this constituent is of the first kind then inducing it to K_n yields a character with first component $\chi_4 + \chi_5$ and all other components non-faithful. If the common constituent is of the second kind then inducing it to K_n yields a character with first component $\chi_1 + \chi_2 + 2\chi_3$ and exactly one other component faithful. (Here we used that $(\mu \times \theta_2 \times \cdots \times \theta_n)^{K_n} = (\mu^G \times \theta_2 \times \cdots \times \theta_n)$, where $\theta_i \in \mathrm{Irr}(G)$, for $i = 2 \cdots n$, $\mu \in \mathrm{Irr}(N)$.)

In both cases, the irreducible constituents are cyclic shifts of characters in X_n, thus inducing further from K_n to G_n yields characters all whose irreducible constituents lie in Y_n. Now note that ψ' is one of them.

For part (iii), note that there is an edge between $(i_1, i_2, \ldots, i_n)$ and $(i'_1, i'_2, \ldots, i'_n)$ in Γ_n if and only if $\chi := \chi_{i_1} \times \chi_{i_2} \times \cdots \times \chi_{i_n}$ and $\chi' := \chi_{i'_1} \times \chi_{i'_2} \times \cdots \times \chi_{i'_n}$ differ in exactly one component χ_{i_j}, $\chi_{i'_j}$, such that $\chi_{i_j}|_N$ and $\chi_{i'_j}|_N$ have a common constituent. Let $\psi := (\chi_{i_1} \times \chi_{i_2} \times \cdots \times \chi_{i_n})^{G_n}$, and $\psi' := (\chi_{i'_1} \times \chi_{i'_2} \times \cdots \times \chi_{i'_n})^{G_n}$. Then $\psi|_{K_n}$ contains as a constituent $\chi_{i_1} \times \chi_{i_2} \times \cdots \times \chi_{i_n}$ and all its cyclic shifts, ψ'_{K_n} contains as a constituent $\chi_{i'_1} \times \chi_{i'_2} \times \cdots \times \chi_{i'_n}$ and all its cyclic shifts. When restricted further to H_n the scalar product can be nonzero if and only if some of cyclic shifts of χ and some of cyclic shifts of χ' have in the first component a restriction that have a common component and all other components are equal. But then they must be shifted in the same way, since otherwise the faithful components were in different place. Thus $\chi_{i_1} \times \chi_{i_2} \times \cdots \times \chi_{i_n}$ and $\chi_{i'_1} \times \chi_{i'_2} \times \cdots \times \chi_{i'_n}$ differ in exactly one component χ_{i_j}, $\chi_{i'_j}$, such that $\chi_{i_j}|_N$ and $\chi_{i'_j}|_N$ have a common constituent.

For part (iv), observe that any shortest path from $(4, 1, \ldots, 1)$ to $(4, 2, \ldots, 2)$ in Γ_n replaces in each step exactly one 1 by a 2.

For part (v), fix n and let $\alpha_n \sim_{G_n} \psi_1 \sim_{G_n} \psi_2 \sim_{G_n} \cdots \sim_{G_n} \psi_m \sim_{G_n} \omega_n$ be a shortest path of related characters in $\mathrm{Irr}(H_n)$, of length $m+1$. This means that there are irreducible characters $\Phi_1, \Phi_2, \ldots, \Phi_{m+1}$ of G_n such that α_n and ψ_1 are constituents of $\Phi_1|_{H_n}$, ψ_i and ψ_{i+1} are constituents of $\Phi_{i+1}|_{H_n}$, for $1 \leq i \leq m-1$, and ψ_m and ω_n are constituents of $\Phi_{m+1}|_{H_n}$. By Frobenius reciprocity we have that $(\alpha_n^{G_n}, \Phi_1) \neq 0$. Since $\alpha_n^{K_n} = (\chi_4 + \chi_5) \times \chi_1 \times \cdots \times \chi_1$ is a sum of characters in X_n, we know that $\Phi_1 \in Y_n$, and part (ii) implies that $\Phi_i \in Y_n$ for all $i \in \{1, 2, \ldots, m+1\}$. Let Θ_i be the unique character in X_n with the property $\Phi_i = \Theta_i^{G_n}$, for $1 \leq i \leq m+1$. By part (iii), Θ_i and Θ_{i+1} differ in at most one component. Now Θ_1 has $n-1$ components χ_1, and Θ_{m+1} has $n-1$ components χ_2, thus $m \geq n-1$ holds. Conversely, any path of length $n-1$ between $(4,1,1,\ldots,1)$ and $(4,2,2,\ldots,2)$ in Γ_n yields a path of related characters from α_n to ω_n, of length n, hence $m+1 = n$. □

In order to prove that $d(H_n, G_n) = 2n$, it remains to show that $d(H_n, G_n) \geq 2n$ holds. If $d(H_n, G_n) \leq 2n-1 = 2(n-1)+1$, then by Corollary 1.2 we have that every two irreducible characters of H_n have distance at most $n-1$. However, the characters α_n and ω_n constructed in Lemma 3.1 have distance n, which is a contradiction. So we are done.

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