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INDUCED OPERATORS ON THE GENERALIZED SYMMETRY CLASSES OF TENSORS

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Abstract. Let V be a unitary space. Suppose G is a subgroup of the symmetric group of degree m and Λ is an irreducible unitary representation of G over a vector space U. Consider the generalized symmetrizer on the tensor space $U \otimes V^{\otimes m}$,

$$S_\Lambda(u \otimes v^{\otimes}) = \frac{1}{|G|} \sum_{\sigma \in G} \Lambda(\sigma) u \otimes v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(m)}$$

defined by G and Λ. The image of $U \otimes V^{\otimes m}$ under the map S_Λ is called the generalized symmetry class of tensors associated with G and Λ and is denoted by $V_\Lambda(G)$. The elements in $V_\Lambda(G)$ of the form $S_\Lambda(u \otimes v^{\otimes})$ are called generalized decomposable tensors and are denoted by $u \circledast v^{\circledast}$. For any linear operator T acting on V, there is a unique induced operator $K_\Lambda(T)$ acting on $V_\Lambda(G)$ satisfying

$$K_\Lambda(T)(u \otimes v^{\otimes}) = u \circledast Tv_1 \circledast \cdots \circledast Tv_m.$$

If $\dim U = 1$, then $K_\Lambda(T)$ reduces to $K_\lambda(T)$, induced operator on symmetry class of tensors $V_\lambda(G)$. In this paper, the basic properties of the induced operator $K_\Lambda(T)$ are studied. Also some well-known results on the classical Schur functions will be extended to the case of generalized Schur functions.

1. **Introduction**

Let V be a unitary space of dimension n and denote by $V^{\otimes m}$, the mth tensor power of V. Let U be a unitary space and End(U) be the set of all linear operators on U. Then $U \otimes V^{\otimes m}$ is a unitary

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space with the induced inner product that satisfies

$$(u \otimes x^{\otimes}, v \otimes y^{\otimes}) = (u, v) \prod_{i=1}^{m} (x_i, y_i),$$

where $u, v \in U$ and $x^{\otimes} = x_1 \otimes \cdots \otimes x_m,\ y^{\otimes} = y_1 \otimes \cdots \otimes y_m \in V^{\otimes m}$.

Let S_m be the full symmetric group of degree m and G be a subgroup of S_m. Suppose Λ is an irreducible unitary representation of G over U. The generalized symmetrizer associated with G and Λ is defined by

$$S_\Lambda = \frac{1}{|G|} \sum_{\sigma \in G} \Lambda(\sigma) \otimes P(\sigma) \in \mathrm{End}(U \otimes V^{\otimes m}),$$

where

$$P(\sigma) v_1 \otimes \cdots \otimes v_m = v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(m)}$$

is the permutation operator (see [2]).

It is well-known that S_Λ is an orthogonal projection on $U \otimes V^{\otimes m}$. The image of $U \otimes V^{\otimes m}$ under the map S_Λ is called the generalized symmetry class of tensors associated with G and Λ and is denoted by $V_\Lambda(G)$. The elements in $V_\Lambda(G)$ of the form

$$u \circledast v^{\circledast} = S_\Lambda(u \otimes v^{\otimes})$$

are called the generalized decomposable symmetrized tensors (for more details, see [6, 7]).

Let $\Gamma_{m,n}$ be the set of all sequences $\alpha = (\alpha(1), \ldots, \alpha(m))$ with $1 \le \alpha(i) \le n, 1 \le i \le m$. The group G acts on $\Gamma_{m,n}$ by

$$\alpha\sigma = (\alpha(\sigma(1)), \ldots, \alpha(\sigma(m)))\,.$$

Two sequences α and β in $\Gamma_{m,n}$ are said to be equivalent modulo G, denoted by $\alpha \sim \beta \pmod G$, if there exist $\sigma \in G$ such that $\beta = \alpha\sigma$. Let Δ be a system of representatives for the orbits such that each sequence in Δ is first in its orbit relative to the lexicographic order.

Suppose $\mathbb{F} = \{u_1, \ldots, u_r\}$ and $\mathbb{E} = \{e_1, \ldots, e_n\}$ are orthonormal bases for unitary spaces U and V, respectively. For each $\alpha \in \Gamma_{m,n}$, the subspace

$$V_\alpha^{\circledast} = \langle\, u_i \circledast e_\alpha^{\circledast} \mid 1 \le i \le r \,\rangle = \langle\, u_1 \circledast e_{\alpha\sigma}^{\circledast} \mid \sigma \in G \,\rangle$$

is called the generalized orbital subspace corresponding to α. It is well-known that

$$V_\Lambda(G) = \bigoplus_{\alpha \in \bar{\Delta}} V_\alpha^{\circledast},$$

where

$$\bar{\Delta} = \bigcup_{j=1}^{r} \bar{\Delta}_j,\ \ \bar{\Delta}_j = \{\alpha \in \Delta | u_j \circledast e_\alpha^{\circledast} \neq 0\}.$$

Notice that

$$\bar{\Delta} = \{\alpha \in \Delta | \sum_{\sigma \in G_\alpha} \lambda(\sigma) \neq 0\},$$

where λ is the corresponding character of Λ. For $\alpha \in \bar{\Delta}$, choose a lexicographically ordered set $\{\alpha_1 = \alpha, \alpha_2, \ldots, \alpha_{s_\alpha}\}$ from $\{\alpha\sigma | \sigma \in G\}$ such that

$$\{u_1 \circledast e^{\circledast}_{\alpha_1}, u_1 \circledast e^{\circledast}_{\alpha_2}, \ldots, u_1 \circledast e^{\circledast}_{\alpha_{s_\alpha}}\}$$

is a basis of $V^{\circledast}_\alpha$. The same is done for any $\alpha \in \bar{\Delta}$. Let $\{\alpha, \beta, \gamma, \ldots\}$ ordered lexicographically and define $\hat{\Delta}$ as

$$\hat{\Delta} = \{\alpha_1, \ldots, \alpha_{s_\alpha}, \beta_1, \ldots, \beta_{s_\beta}, \ldots\}$$

to be ordered as indicated. Then $\{u_1 \circledast e^{\circledast}_\alpha | \alpha \in \hat{\Delta}\}$ is a basis of $V_\Lambda(G)$. Obviously, $\bar{\Delta} = \{\alpha_1, \beta_1, \ldots\}$ is lexicographically ordered, but note that $\hat{\Delta}$ is not lexicographically ordered; it is possible that $(\alpha_2 > \beta_1)$. Such order in $\hat{\Delta}$ is called orbital order.

Denote by $\mathbb{C}_{m\times m}$, the set of all $m \times m$ complex matrices. The generalized Schur function $D_\Lambda : \mathbb{C}_{m\times m} \to \mathrm{End}(U)$ is defined by

$$D_\Lambda(A) = \sum_{\sigma \in G} \Lambda(\sigma) \prod_{i=1}^{m} a_{i\sigma(i)}$$

for $A = (a_{ij})_{m\times m} \in \mathbb{C}_{m\times m}$.

In this paper, we introduce the generalized symmetric multilinear functions and prove universal factorization property for this functions. Also, we define the induced operators on the generalized symmetry classes of tensors and study some of basic properties of this operators. Then we deduce some well-known results on the generalized Schur functions.

2. Generalized symmetric multilinear functions

Let W be a vector space. A multilinear function $\psi : U \times V^{\times m} \longrightarrow W$ is said to be symmetric with respect to G and Λ if

$$\frac{1}{|G|} \sum_{\sigma \in G} \psi(\Lambda(\sigma^{-1})u, v_{\sigma(1)}, \ldots, v_{\sigma(m)}) = \psi(u, v_1, \ldots, v_m)$$

for all $u \in U$ and $v_1, \ldots, v_m \in V$. If $\dim U = 1$, then ψ is symmetric with respect to G and λ, where λ is the corresponding character Λ (see [5]).

Lemma 2.1. *Let Λ be an irreducible unitary representation of the subgroup G of S_m. Suppose $\phi : U \times V^{\times m} \to W$ is defined as*

$$\phi(u, v_1, \ldots, v_m) = u \circledast v^{\circledast}.$$

Then ϕ is multilinear and symmetric with respect to G and Λ.

Proof. Obviously, ϕ is multilinear. We show that ϕ is symmetric with respect to G and Λ. We have

$$\begin{aligned}
\frac{1}{|G|}\sum_{\sigma\in G}\phi\left(\Lambda(\sigma^{-1})u, v_{\sigma(1)},\dots,v_{\sigma(m)}\right) &= \frac{1}{|G|}\sum_{\sigma\in G}\Lambda(\sigma^{-1})u\circledast v_{\sigma}^{\circledast}\\
&= \frac{1}{|G|}\sum_{\sigma\in G}S_{\Lambda}\left(\Lambda(\sigma^{-1})u\otimes v_{\sigma}^{\otimes}\right)\\
&= \frac{1}{|G|}\sum_{\sigma\in G}S_{\Lambda}\left(\Lambda(\sigma^{-1})u\otimes P(\sigma^{-1})v^{\otimes}\right)\\
&= S_{\Lambda}\left[\frac{1}{|G|}\sum_{\sigma\in G}\Lambda(\sigma^{-1})\otimes P(\sigma^{-1})(u\otimes v^{\otimes})\right]\\
&= S_{\Lambda}^{2}(u\otimes v^{\otimes})\\
&= S_{\Lambda}(u\otimes v^{\otimes})\\
&= \phi(u, v_1,\dots,v_m).
\end{aligned}$$

□

Theorem 2.2. *(Universal factorization property for the generalized symmetric multilinear functions) Suppose V and W are vector spaces and Λ is an irreducible unitary representation of G over U. If the multilinear function $\psi : U\times V^{\times m}\to W$ is symmetric with respect to G and Λ, then there is a unique linear function $h : V_{\Lambda}(G)\longrightarrow W$ such that $h\left(u\circledast v^{\circledast}\right)=\psi(u,v_1,\dots,v_m)$.*

Proof. According to the (ordinary) universal factorization property, there is a unique linear function $h : U\otimes V^{\otimes m}\to W$ such that $h\otimes = \psi$; that is $h(u\otimes v^{\otimes}) = \psi(u, v_1,\dots,v_m)$, for all $u\in U$ and $v_1,\dots,v_m\in V$. Therefore

$$\begin{aligned}
h(u\circledast v^{\circledast}) =& h\left[\frac{1}{|G|}\sum_{\sigma\in G}\Lambda(\sigma^{-1})u\otimes v_{\sigma}^{\otimes}\right]\\
=& \frac{1}{|G|}\sum_{\sigma\in G}\psi\left(\Lambda(\sigma^{-1})u, v_{\sigma(1)},\dots,v_{\sigma(m)}\right)\\
=& \psi(u, v_1,\dots,v_m).
\end{aligned}$$

□

3. The basic properties of the induced operator $K_{\Lambda}(T)$

Let $T\in \mathrm{End}(V)$. Then the map $\psi : U\times V^{\times m}\to V_{\Lambda}(G)$ defined by

$$\psi(u, v_1,\dots,v_m) = u\circledast Tv_1\circledast\cdots\circledast Tv_m$$

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is multilinear and symmetric respect to G and Λ, because

$$\begin{aligned}
&\frac{1}{|G|}\sum_{\sigma\in G}\psi\left(\Lambda(\sigma^{-1})u, v_{\sigma(1)},\ldots,v_{\sigma(m)}\right)\\
&=\frac{1}{|G|}\sum_{\sigma\in G}S_\Lambda\left[\Lambda(\sigma^{-1})u\otimes Tv_{\sigma(1)}\otimes\cdots\otimes Tv_{\sigma(m)}\right]\\
&=S_\Lambda\left[\frac{1}{|G|}\sum_{\sigma\in G}\left(\Lambda(\sigma^{-1})\otimes P(\sigma^{-1})\right)u\otimes Tv_1\otimes\cdots\otimes Tv_m\right]\\
&=S_\Lambda^2(u\otimes Tv_1\otimes\cdots\otimes Tv_m)\\
&=S_\Lambda(u\otimes Tv_1\otimes\cdots\otimes Tv_m)\\
&=\psi(u,v_1,\ldots,v_m).
\end{aligned}$$

Therefore according to the universal factorization property for the generalized symmetric multilinear functions, there is a unique linear operator $K_\Lambda(T)\in End(V_\Lambda(G))$, such that

$$K_\Lambda(T)(u\circledast v^{\circledast}) = u\circledast Tv_1\circledast\cdots\circledast Tv_m.$$

Such $K_\Lambda(T)$ is called *the induced operator of T on $V_\Lambda(G)$*. If $\dim U = 1$, then $K_\Lambda(T)$ coincides on $K_\lambda(T)$, where $K_\lambda(T)$ is the induced operator of T on $V_\lambda(G)$ (for more details, we refer the reader to [1, 3, 4, 5, 9]). Recently, the induced operators over symmetry classes of polynomials have been studied in [8, 10]. In this section, we verify some basic properties of the generalized induced operator $K_\Lambda(T)$.

Theorem 3.1. *Suppose $T\in End(V)$. Then $V_\Lambda(G)$ is an invariant subspace of $I\otimes T^{\otimes m}$ and $K_\Lambda(T) = I\otimes T^{\otimes m}\mid_{V_\Lambda(G)}$.*

Proof. Since $T^{\otimes m}P(\sigma) = P(\sigma)T^{\otimes m}$, so $(I\otimes T^{\otimes m})(\Lambda(\sigma)\otimes P(\sigma)) = (\Lambda(\sigma)\otimes P(\sigma))(I\otimes T^{\otimes m})$. Thus $(I\otimes T^{\otimes m})S_\Lambda = S_\Lambda(I\otimes T^{\otimes m})$, that is $V_\Lambda(G)$ is an invariant subspace of $I\otimes T^{\otimes m}$. Also we have

$$\begin{aligned}
(I\otimes T^{\otimes m})(u\circledast v^{\circledast}) =&(I\otimes T^{\otimes m})S_\Lambda(u\otimes v^{\otimes})\\
=&S_\Lambda(I\otimes T^{\otimes m})(u\otimes v^{\otimes})\\
=&S_\Lambda(u\otimes Tv_1\otimes\cdots\otimes Tv_m)\\
=&K_\Lambda(T)(u\circledast v^{\circledast}),
\end{aligned}$$

so the assertion holds. □

Theorem 3.2. *Let $S, T\in End(V)$. Then*

(i) $K_\Lambda(I_V) = I_{V_\Lambda(G)}$,
(ii) $K_\Lambda(ST) = K_\Lambda(S)K_\Lambda(T)$.

Proof. (i) It is clear.

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(ii) Using Theorem (3.1), we have

$$\begin{aligned} K_\Lambda(ST) =&(I \otimes (ST)^{\otimes m}) \mid_{V_\Lambda(G)} \\ =&(I \otimes (S^{\otimes m} T^{\otimes m})) \mid_{V_\Lambda(G)} \\ =&(I \otimes S^{\otimes m})(I \otimes T^{\otimes m}) \mid_{V_\Lambda(G)} \\ =&(I \otimes S^{\otimes m}) \mid_{V_\Lambda(G)} (I \otimes T^{\otimes m}) \mid_{V_\Lambda(G)} \\ =&K_\Lambda(S)K_\Lambda(T), \end{aligned}$$

so the assertion holds.

□

Therefore $T \to K_\Lambda(T)$ defines a representation of the general linear group $GL(V)$ on $V_\Lambda(G)$.

Theorem 3.3. *Suppose $T \in End(V)$. Then with respect to induced inner product on $V_\Lambda(G)$, we have*

(i) $K_\Lambda(T)^* = K_\Lambda(T^*)$,

(ii) *If T is normal, Hermitian, positive definite, positive semi-definite, unitary, so is $K_\Lambda(T)$.*

Proof. (i) We know that $(T^{\otimes m})^* = (T^*)^{\otimes m}$, so $(I \otimes T^{\otimes m})^* = I \otimes (T^*)^{\otimes m}$. By restricting the both sides to $V_\Lambda(G)$ we have:

$$\left((I \otimes T^{\otimes m}) \mid_{V_\Lambda(G)}\right)^* = (I \otimes T^{\otimes m})^* \mid_{V_\Lambda(G)} = I \otimes (T^*)^{\otimes m} \mid_{V_\Lambda(G)} .$$

Therefore $K_\Lambda(T)^* = K_\Lambda(T^*)$.

(ii) Suppose T is positive semi-definite. Then there is a linear operator S on V such that $T = SS^*$. So we have

$$K_\Lambda(T) = K_\Lambda(SS^*) = K_\Lambda(S)K_\Lambda(S^*) = K_\Lambda(S)K_\Lambda(S)^*$$

Then $K_\Lambda(T)$ is positive semi-definite.

The rest of the second part of theorem is similarly proved. □

Theorem 3.4. *Suppose T and S are positive semi-definite linear operators. Then*

$$K_\Lambda(T + S) \geq K_\Lambda(T) + K_\Lambda(S).$$

Proof. Notice that if T and S are positive semi-definite operators on V then

$$(T + S)^{\otimes m} \geq T^{\otimes m} + S^{\otimes m}.$$

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So for all $u \in U$ and $v \in V$, we have

$$\begin{aligned}
&\left(I \otimes (T+S)^{\otimes m}\, u \otimes v^{\otimes}, u \otimes v^{\otimes}\right) \\
=&\left(u \otimes (T+S)^{\otimes m}\, v^{\otimes},\ u \otimes v^{\otimes}\right) \\
=&(u,u)\left((T+S)^{\otimes m}\, v^{\otimes},\ v^{\otimes}\right) \\
\geq&(u,u)\left((T^{\otimes m}\, v^{\otimes},\ v^{\otimes}) + (S^{\otimes m}\, v^{\otimes},\ v^{\otimes})\right) \\
=&\left(I \otimes T^{\otimes m}\ \ u \otimes v^{\otimes}, u \otimes v^{\otimes}\right) + \left(I \otimes S^{\otimes m}\, u \otimes v^{\otimes},\ u \otimes v^{\otimes}\right).
\end{aligned}$$

Thus

$$I \otimes (T+S)^{\otimes m} \geq I \otimes T^{\otimes m} + I \otimes S^{\otimes m}.$$

Now by restricting the both sides to $V_\Lambda(G)$ we deduce

$$K_\Lambda(T+S) \geq K_\Lambda(T) + K_\Lambda(S).$$

□

Corollary 3.5. *Let T and S be positive semi-definite linear operators on V. If $T \geq S$, then*

$$K_\Lambda(T) \geq K_\Lambda(S).$$

Proof. By assumption $T - S$ is positive operator. Now, using theorems (3.3) and (3.4), we have

$$\begin{aligned}
K_\Lambda(T) &= K_\Lambda((T-S)+S) \\
&\geq K_\Lambda(T-S) + K_\Lambda(S) \\
&\geq K_\Lambda(S).
\end{aligned}$$

Therefore

$$K_\Lambda(T) \geq K_\Lambda(S).$$

□

Theorem 3.6. *If $T \in End(V)$ and $rank(T) = k$, then $rank K_\Lambda(T) = |\Gamma_{m,k} \cap \hat{\Delta}|$.*

Proof. Suppose rank $(T) = k$. Then there is a basis $\{v_1, \ldots, v_n\}$ of V such that $Tv_1, \ldots, Tv_k$ are linearly independent and $Tv_{k+1} = \cdots = Tv_n = 0$. Let $Tv_i = e_i$, $1 \leq i \leq k$, and extend them to a basis $E = \{e_1, \ldots, e_k, e_{k+1}, \ldots, e_n\}$ of V. Now we consider the basis $E_{\circledast} = \{u_1 \circledast e^{\circledast}_\alpha | \alpha \in \hat{\Delta}\}$ of $V_\Lambda(G)$ (see [6]). If $\alpha \in \Gamma_{m,k}$ then

$$K_\Lambda(T)(u_1 \circledast v^{\circledast}_\alpha) = u_1 \circledast Tv_{\alpha(1)} \circledast \cdots \circledast Tv_{\alpha(m)} = u_1 \circledast e^{\circledast}_\alpha.$$

So

$$\{K_\Lambda(T)u_1 \circledast v^{\circledast}_\alpha | \alpha \in \hat{\Delta} \cap \Gamma_{m,k}\}$$

is a subset of the basis $E_{\circledast}$ for $V_\Lambda(G)$. Thus it is a linearly independent set. When $\alpha \notin \hat{\Delta} \cap \Gamma_{m,k}$ there is some i such that $\alpha(i) > k$. Then $Tv_{\alpha(i)} = 0$ thus $K_\Lambda(T)u_1 \circledast v^{\circledast}_\alpha = 0$. Therefore rank $K_\Lambda(T) = |\Gamma_{m,k} \cap \hat{\Delta}|$. □

Theorem 3.7. *Suppose $T \in End(V)$. If $V_\Lambda(G) \neq 0$, then T is invertible if and only if $K_\Lambda(T)$ is invertible.*

Proof. If T is invertible, then

$$I = K_\Lambda(I) = K_\Lambda(TT^{-1}) = K_\Lambda(T)K_\Lambda(T^{-1}).$$

Thus $K_\Lambda(T)$ is invertible and $K_\Lambda(T^{-1}) = K_\Lambda(T)^{-1}$.

Suppose T is not invertible and $Te_1 = 0$. Then there is $\alpha \in \bar{\Delta}$ such that $1 \in Im\,\alpha$. Since $\bar{\Delta} = \bigcup_{j=1}^{r} \bar{\Delta}_j$ so there exists $1 \le j \le r$ such that $\alpha \in \bar{\Delta}_j$. Then $u_j \circledast e_\alpha^\circledast \neq 0$. But

$$K_\Lambda(T)(u_j \circledast e_\alpha^\circledast) = u_j \circledast Te_{\alpha(1)} \circledast \cdots \circledast Te_{\alpha(m)} \;=\; 0,$$

this shows that $K_\Lambda(T)$ is not invertible. □

Theorem 3.8. *Let Λ be an irreducible unitary representation of G. Let V be a vector space of dimension n. Suppose $T \in End(V)$ has eigenvalues $\lambda_1, \ldots, \lambda_n$. Then the eigenvalues of $K_\Lambda(T)$ are*

$$\lambda_\alpha = \prod_{t=1}^{m} \lambda_{\alpha(t)}, \alpha \in \hat{\Delta}\ .$$

Proof. Similar to the proof of [5, Theorem 7.49]. □

Theorem 3.9. *Suppose $T \in End(V)$ and $K_\Lambda(T)$ is the induced operator determined by G and Λ. Then*

$$det\ (K_\Lambda(T)) = (det\ (T))^{\frac{m}{n}|\hat{\Delta}|}.$$

Proof. Denote the eigenvalues of T by $\lambda_1, \ldots, \lambda_n$ (multiplicities included). Then, by Theorem (3.8),

$$\begin{aligned} det\ \ K_\Lambda(T) &= \prod_{\omega\in\hat{\Delta}} \prod_{i=1}^{m} \lambda_{\omega(i)} \\ &= \prod_{\omega\in\hat{\Delta}} \prod_{t=1}^{n} \lambda_t^{m_t(\omega)} \\ &= \prod_{t=1}^{n} \prod_{\omega\in\hat{\Delta}} \lambda_t^{m_t(\omega)} \\ &= \prod_{t=1}^{n} \lambda_t^{q_t}, \end{aligned}$$

where $q_t := \sum_{\omega\in\hat{\Delta}} m_t(\omega)$ and $m_t(\omega)$ is the multiplicity of t in ω. We first prove the amount q_t is independent of t. Since $s_{\alpha\sigma} = s_\alpha$, $m_t(\alpha\sigma) = m_t(\alpha)$ for all $\alpha \in \Gamma_{m,n}$ and $\sigma \in G$, also for every $\tau \in S_n$,

$m_t(\tau\alpha) = m_{\tau^{-1}(t)}(\alpha),\ G_{\tau\alpha} = G_\alpha$ and $s_{\tau\alpha} = s_\alpha$, so we have

$$\begin{aligned} q_t &= \sum_{\omega\in\hat{\Delta}} m_t(\omega) \\ &= \sum_{\alpha\in\bar{\Delta}} s_\alpha m_t(\alpha) \\ &= \sum_{\alpha\in\Delta} s_\alpha m_t(\alpha) \\ &= \frac{1}{|G|}\sum_{\alpha\in\Delta}\frac{1}{|G_\alpha|}\sum_{\sigma\in G}|G_\alpha|s_\alpha m_t(\alpha) \\ &= \frac{1}{|G|}\sum_{\alpha\in\Delta}\frac{1}{|G_\alpha|}\sum_{\sigma\in G}|G_{\alpha\sigma}|s_{\alpha\sigma} m_t(\alpha\sigma) \\ &= \frac{1}{|G|}\sum_{\gamma\in\Gamma_{m,n}}|G_\gamma|s_\gamma m_t(\gamma). \end{aligned}$$

Then for any $\tau \in S_n$,

$$q_t = \frac{1}{|G|}\sum_{\alpha\in\Gamma_{m,n}}|G_{\tau\alpha}|s_{\tau\alpha}m_t(\tau\alpha) = \frac{1}{|G|}\sum_{\alpha\in\Gamma_{m,n}}|G_\alpha|s_\alpha m_{\tau^{-1}(t)}(\alpha) = q_{\tau^{-1}(t)}.$$

We set $q_t = q,\ \ t = 1,\dots,n$. Hence

$$nq = \sum_{t=1}^n q_t = \sum_{t=1}^n\sum_{\omega\in\hat{\Delta}} m_t(\omega) = \sum_{\omega\in\hat{\Delta}}\sum_{t=1}^n m_t(\omega) = m|\hat{\Delta}|.$$

Therefore

$$\det\,(K_\Lambda(T)) = \prod_{t=1}^n \lambda_t^{\frac{m}{n}|\hat{\Delta}|} = \left(\prod_{t=1}^n \lambda_t\right)^{\frac{m}{n}|\hat{\Delta}|} = (det\ (T))^{\frac{m}{n}|\hat{\Delta}|}.$$

□

4. Some results on the generalized Schur functions

In this section we deduce some results on the generalized Schur functions (see [2]).

Theorem 4.1. *Suppose λ is the corresponding character of an irreducible unitary representation Λ of G and $\lambda(1) = r$. Then*

(i) $D_\Lambda(I_m) = I_r$.

(ii) *$Tr\ D_\Lambda(A) = d_G^\lambda(A)$, where d_G^λ is the generalized matrix function.*

(iii) $D_\Lambda(A^*) = D_\Lambda(A)^*$.

Proof. (i) $D_\Lambda(I_m) = \sum_{\sigma\in G}\Lambda(\sigma)\prod_{i=1}^m \delta_{i\sigma(i)} = \Lambda(1) = I_r$.

(ii) According to the definition of the generalized Schur function we have

$$\text{Tr}\ D_\Lambda(A) = \text{Tr}\left(\sum_{\sigma\in G}\Lambda(\sigma)\prod_{i=1}^m\ a_{i\sigma(i)}\right) = \sum_{\sigma\in G}\lambda(\sigma)\prod_{i=1}^m\ a_{i\sigma(i)} = d_G^\lambda(A).$$

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(iii)

$$\begin{aligned} D_\Lambda(A^*) &= \sum_{\sigma\in G}\Lambda(\sigma)\prod_{i=1}^{m} a^*_{i\sigma(i)} \\ &= \sum_{\sigma\in G}\Lambda(\sigma)\prod_{i=1}^{m} \bar{a}_{\sigma(i)i} \\ &= \sum_{\sigma\in G}\Lambda(\sigma^{-1})^*\prod_{i=1}^{m} \bar{a}_{i\sigma^{-1}(i)} \\ &= D_\Lambda(A)^*. \end{aligned}$$

□

Theorem 4.2. *If A is an upper triangular matrix, then*

$$D_\Lambda(A) = h(A)I_r = per(A)I_r = det(A)I_r,$$

where $h(A)=\prod_{i=1}^{m} a_{ii}$ is Hadamard function.

Proof. Since A is an upper triangular matrix, so

$$det(A) = h(A) = per(A) = \prod_{i=1}^{m} a_{ii}.$$

Also

$$D_\Lambda(A) = \sum_{\sigma\in G}\Lambda(\sigma)\prod_{i=1}^{m} a_{i\sigma(i)} = (\prod_{i=1}^{m} a_{ii})I_r.$$

□

Theorem 4.3. [6] *Suppose $u, v \in U$ and $x_1, \dots, x_m, y_1, \dots, y_m \in V$ are arbitrary vectors and $A = (a_{ij}) \in \mathbb{C}_{m\times m}$ such that $a_{ij} = (x_i, y_j)$. Then*

$$(D_\Lambda(A)u, v) = |G|(u \circledast x^{\circledast},\ v \circledast y^{\circledast}).$$

Theorem 4.4. *Let $\mathbb{E} = \{e_1, \dots, e_n\}$ be an orthonormal basis of V and $T \in End\ (V)$. If $[T]_{\mathbb{E}} = A^t$ then for all $\alpha, \beta \in \Gamma_{m,n}$ and $u, v \in U$,*

$$(D_\Lambda(A[\alpha|\beta])u, v) = |G|\left(K_\Lambda(T)u \circledast e^{\circledast}_\alpha, v \circledast e^{\circledast}_\beta\right).$$

Proof. By Theorem (4.3), we have

$$\begin{aligned} \left(K_\Lambda(T)u \circledast e^{\circledast}_\alpha, v \circledast e^{\circledast}_\beta\right) &= (u \circledast Te_{\alpha(1)} \circledast \cdots \circledast Te_{\alpha(m)}, v \circledast e_{\beta(1)} \circledast \cdots \circledast e_{\beta(m)}) \\ &= \frac{1}{|G|}(D_\Lambda(B)u, v), \end{aligned}$$

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where

$$\begin{aligned} b_{ij} &= (Te_{\alpha(i)}, e_{\beta(j)}) \\ &= (\sum_{k=1}^{n} a_{\alpha(i)k} e_k, e_{\beta(j)}) \\ &= \sum_{k=1}^{n} a_{\alpha(i)k} \delta_{k\beta(j)} \\ &= a_{\alpha(i)\beta(j)} = (A[\alpha|\beta])_{ij}, \end{aligned}$$

so the result holds. □

Corollary 4.5. *Suppose $u, v \in U$. Suppose $\mathbb{E} = \{e_1, \ldots, e_m\}$ is an orthonormal basis of a unitary space of V and $T \in End(V)$. If $[T]_{\mathbb{E}} = A^t$ then*

$$(D_\Lambda(A)u, v) = |G| \left(K_\Lambda(T)u \circledast e^\circledast, v \circledast e^\circledast \right).$$

Corollary 4.6. *Suppose $G \leq S_m$ and Λ is an irreducible unitary representation of G over a unitary space U. If $A, B \in \mathbb{C}_{m\times m}$ are positive semi-definite matrices, then*

(i) $D_\Lambda(A + B) \geqslant D_\Lambda(A) + D_\Lambda(B)$,
(ii) *If $A \geqslant B$ then $D_\Lambda(A) \geqslant D_\Lambda(B)$.*

Proof. (i) Let $\mathbb{E} = \{e_1, \ldots, e_m\}$ be an orthonormal basis of a unitary space V. Let S and T be linear operators on V whose $A^t = [T]_E$ and $B^t = [S]_E$. Then by Theorem (3.4) and Corollary (4.5), for any $u \in U$, we have

$$\begin{aligned} (D_\Lambda(A + B)u, u) =& |G| \left(K_\Lambda(T + S)u \circledast e^\circledast, u \circledast e^\circledast \right) \\ \geq& |G| \left(K_\Lambda(T)u \circledast e^\circledast, u \circledast e^\circledast \right) + \left(K_\Lambda(S)u \circledast e^\circledast, u \circledast e^\circledast \right) \\ =& (D_\Lambda(A)u, u) + (D_\Lambda(B)u, u), \end{aligned}$$

so the assertion holds.

(ii) The result follows from corollaries (3.5) and (4.5).

□

Theorem 4.7. *Let $A, B \in \mathbb{C}_{m\times m}$. Then for every $u, v \in U$, we have the following inequality*

$$|(D_\Lambda(AB^*)u, v)|^2 \leq (D_\Lambda(AA^*)u, u)(D_\Lambda(BB^*)v, v).$$

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Proof. Let $\mathbb{E} = \{e_1, \ldots, e_m\}$ be an orthonormal basis of V. Let S and T be linear operators on V whose $A^t = [T]_{\mathbb{E}}$ and $B^t = [S]_{\mathbb{E}}$. Then

$$\begin{aligned}\frac{1}{|G|}(D_\Lambda(AB)u, v) =&(K_\Lambda(ST)u \circledast e^{\circledast}, v \circledast e^{\circledast})\\ =&(K_\Lambda(S)K_\Lambda(T)u \circledast e^{\circledast}, v \circledast e^{\circledast})\\ =&(K_\Lambda(T)u \circledast e^{\circledast}, K_\Lambda(S^*)v \circledast e^{\circledast})\end{aligned}$$

Using the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}(\frac{1}{|G|}|(D_\Lambda(AB)u, v)|)^2 &\leq ||K_\Lambda(T)u \circledast e^{\circledast}||^2||K_\Lambda(S^*)v \circledast e^{\circledast}||^2\\ &= (K_\Lambda(T^*T)u \circledast e^{\circledast}, u \circledast e^{\circledast})(K_\Lambda(SS^*)v \circledast e^{\circledast}, v \circledast e^{\circledast})\\ &= \frac{1}{|G|}(D_\Lambda(AA^*)u, u)\frac{1}{|G|}(D_\Lambda(B^*B)v, v).\end{aligned}$$

By switching B to B^*, the result holds. □

Corollary 4.8. *Suppose Λ is an irreducible unitary representation of G over unitary space of U. If $A \in \mathbb{C}_{m\times m}$ and $u, v \in U$ then*

$$|(D_\Lambda(A)u, v)|^2 \leq (D_\Lambda(AA^*)u, u)(v, v)$$

Proof. We only need to put $B = I_m$ in Theorem (4.7). □

The following theorem extends the Schur inequality to the generalized Schur functions.

Theorem 4.9. *(The generalized Schur inequality)*
If $A \in \mathbb{C}_{m\times m}$ is positive semi-definite, then $D_\Lambda(A) \geq (det A)I_r$.

Proof. Since A is positive semi-definite, so there is an upper triangular matrix L such that $A = LL^*$. Applying Corollary (4.8), we obtain

$$|(D_\Lambda(L)u, u)|^2 \leq |(D_\Lambda(LL^*)u, u)|(u, u) = |(D_\Lambda(A)u, u)|(u, u),$$

for every $u \in U$. Since L is upper triangular, so by Theorem (4.2), $D_\Lambda(L) = det\ (L)I_r$. Hence

$$|det\ (L)|^2(I_r u, u) \leq |(D_\Lambda(A)u, u)|.$$

According to Corollary (4.5), $D_\Lambda(A)$ is positive semi-definite, so the result holds. □

The following theorem generalizes [5, Corollary 7.27].

Theorem 4.10. *Let V be a unitary space of dimension m. If $A \in \mathbb{C}_{m\times m}$ is positive semi-definite, then there exist $v_1, \ldots, v_m$ in V such that $a_{ij} = (v_i, v_j)$ and for each $u \in U$ we have:*

$$||u \circledast v^{\circledast}||^2 = \frac{1}{|G|}(D_\Lambda(A)u, u).$$

Proof. Suppose $\mathbb{E} = \{e_1, \ldots, e_m\}$ is an orthonormal basis for V. Since A is positive semi-definite, so there exists a matrix B such that $A = BB^*$. Define

$$v_i = \sum_{j=1}^{m} b_{ij} e_j,\ 1 \le i \le m.$$

Then

$$\begin{aligned}(v_i, v_j) =& (\sum_{k=1}^{m} b_{ik} e_k, \sum_{s=1}^{m} b_{js} e_s) \\ =& \sum_{k,s=1}^{m} b_{ik} \bar{b}_{js} (e_k, e_s) = \sum_{k,s=1}^{m} \delta_{k,s} b_{ik} \bar{b}_{js} \\ =& \sum_{k=1}^{m} b_{ik} b^*_{kj} = a_{ij}.\end{aligned}$$

Now, the assersion follows from Theorem (4.3). □

Theorem 4.11. *(The generalized Cauchy-Binet formula)*
Let Λ be an irreducible unitary representation of the subgroup G of S_m over a unitary space U. If $A, B \in \mathbb{C}_{n\times n}$ and $m \le n$ then

$$D_\Lambda((AB)[\alpha|\beta]) = \frac{1}{|G|} \sum_{\gamma \in \Gamma_{m,n}} D_\Lambda(B[\gamma|\beta]) D_\Lambda(A[\alpha|\gamma])$$

for all $\alpha, \beta \in \Omega$.

Proof. Let V be a unitary space of dimension n. Suppose $\mathbb{E} = \{e_1, \ldots, e_n\}$ and $\mathbb{F} = \{u_1, \ldots, u_r\}$ are orthonormal bases for V and U, respectively. Then

$$\mathbb{B} = \{u_i \otimes e^{\otimes}_\alpha \mid 1 \le i \le r,\ \alpha \in \Gamma_{m,n}\}$$

is an orthonormal basis for $U \otimes V^{\otimes m}$. Suppose $A^t = [T]_{\mathbb{E}}$ and $B^t = [S]_{\mathbb{E}}$. Let $u, v \in U$. Applying Theorem (4.4), [5, Parseval's Identity 2.14] and $S_\Lambda^2 = S_\Lambda = S_\Lambda^*$, we have

$$\begin{aligned}
&\frac{1}{|G|}(D_\Lambda(AB)[\alpha|\beta]u, v)\\
&= (K_\Lambda(ST)u \circledast e_\alpha^\circledast, v \circledast e_\beta^\circledast)\\
&= (K_\Lambda(S)K_\Lambda(T)u \circledast e_\alpha^\circledast, v \circledast e_\beta^\circledast)\\
&= (K_\Lambda(T)u \circledast e_\alpha^\circledast, K_\Lambda(S)^* v \circledast e_\beta^\circledast)\\
&= \sum_{j=1}^{r} \sum_{\gamma\in\Gamma_{m,n}} (K_\Lambda(T)u \circledast e_\alpha^\circledast, u_j \otimes e_\gamma^\otimes)(u_j \otimes e_\gamma^\otimes, K_\Lambda(S)^* v \circledast e_\beta^\circledast)\\
&= \sum_{j=1}^{r} \sum_{\gamma\in\Gamma_{m,n}} (K_\Lambda(T)u \circledast e_\alpha^\circledast, u_j \circledast e_\gamma^\circledast)(K_\Lambda(S)u_j \circledast e_\gamma^\circledast, v \circledast e_\beta^\circledast)\\
&= \sum_{j=1}^{r} \sum_{\gamma\in\Gamma_{m,n}} (K_\Lambda(T)u \circledast e_\alpha^\circledast, u_j \circledast e_\gamma^\circledast)(K_\Lambda(S)u_j \circledast e_\gamma^\circledast, v \circledast e_\beta^\circledast)\\
&= \frac{1}{|G|^2} \sum_{\gamma\in\Gamma_{m,n}} \sum_{j=1}^{r} (D_\Lambda(A[\alpha|\gamma])u, u_j)(D_\Lambda(B[\gamma|\beta])u_j, v)\\
&= \frac{1}{|G|^2} \sum_{\gamma\in\Gamma_{m,n}} \sum_{j=1}^{r} (D_\Lambda(A[\alpha|\gamma])u, u_j)(u_j, D_\Lambda(B[\gamma|\beta])^* v) \ (Theorem\ (4.1))\\
&= \frac{1}{|G|^2} \sum_{\gamma\in\Gamma_{m,n}} \sum_{j=1}^{r} (D_\Lambda(A[\alpha|\gamma])u, u_j)(u_j, D_\Lambda(B^*[\beta|\gamma])v)\\
&= \frac{1}{|G|^2} \sum_{\gamma\in\Gamma_{m,n}} (D_\Lambda(A[\alpha|\gamma])u, D_\Lambda(B^*[\beta|\gamma])v)\\
&= \frac{1}{|G|^2} \sum_{\gamma\in\Gamma_{m,n}} (D_\Lambda(B^*[\beta|\gamma])^* D_\Lambda(A[\alpha|\gamma])u, v) \ (Theorem\ (4.1))\\
&= \frac{1}{|G|^2} \sum_{\gamma\in\Gamma_{m,n}} (D_\Lambda(B[\gamma|\beta]) D_\Lambda(A[\alpha|\gamma])u, v).
\end{aligned}$$

Therefore

$$D_\Lambda((AB)[\alpha|\beta]) = \frac{1}{|G|} \sum_{\gamma\in\Gamma_{m,n}} D_\Lambda(B[\gamma|\beta]) D_\Lambda(A[\alpha|\gamma]).$$

□

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References

[1] R. Bhatia and J. A. Dias da Silva, Variation of induced linear operators, *Linear Algebra Appl.*, **341** (2002) 391–402.

[2] T. G. Lei, Generalized Schur functions and generalized decompossble symmetric tensors, *Linear Algebra Appl.*, **263** (1997) 311–332.

[3] C. K. Li and T. Y. Tam, Operator properties of T and $K(T)$, *Linear Algebra Appl.*, **401** (2005) 173–191.

[4] C. K. Li and A. Zaharia, Induced operators on symmetry classes of tensors, *Trans. Amer. Math. Soc.*, **342** (2001) 807–836.

[5] R. Merris, *Multilinear Algebra*, Gordon and Breach Science Publisher, Amsterdam, 1997.

[6] G. Rafatneshan and Y. Zamani, Generalized symmetry classes of tensors, *Czechoslovak Math. J.*, **Published online July 8** (2020) 1–13.

[7] G. Rafatneshan and Y. Zamani, *On the orthogonal basis of the generalized symmetry classes of tensors*, to submitted.

[8] M. Ranjbari and Y. Zamani, Induced operators on symmetry classes of polynomials, *Int. J. Group Theory*, **6** no. 2 (2017) 21–35.

[9] Y. Zamani and S. Ahsani, On the decomposable numerical range of operators, *Bull. Iranian. Math. Soc.*, **40** no. 2 (2014) 387–396.

[10] Y. Zamani and M. Ranjbari, Representations of the general linear group over symmetry classes of polynomials, *Czechoslovak Math. J.*, **68** no. 143 (2018) 267–276.

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