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FINITE GROUPS WITH SOME SS-EMBEDDED SUBGROUPS

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Abstract. We call H an SS-embedded subgroup of G if there exists a normal subgroup T of G such that HT is subnormal in G and $H \cap T \leq H_{sG}$, where H_{sG} is the maximal s-permutable subgroup of G contained in H. In this paper, we investigate the influence of some SS-embedded subgroups on the structure of a finite group G. Some new results were obtained.

1. Introduction

All groups considered in this paper are finite and G denotes a group, $H \trianglelefteq \trianglelefteq G$ means that H is a subnormal subgroup of G. We use conventional notions and notation, as in Huppert [6] or Gorenstein [2]. From some subgroup's normality to investigate the structure of a finite group is a common method in the study of group theory. Recently, many new generalized normal concepts were introduced successively. Recall that a subgroup H of a group G is said to be s-permutable [7] (or s-quasinormal [1]) in G if H is permutable with every Sylow subgroup P of G. Wang in [11] introduced the concept of c-normal subgroup as follows: a subgroup H is said to be c-normal in G if G has a normal subgroup T such that $G = HT$ and $H \cap T \leq H_G$, where H_G is the normal core of H in G. Following Guo et al [5], a subgroup H of G is said to be nearly s-normal in G if there exists $N \trianglelefteq G$ such that $HN \trianglelefteq G$ and $H \cap N \leq H_{sG}$, where H_{sG} is the largest s-permutable subgroup of G contained in H. As a development, in [3] the concept of S-embedded subgroup was introduced: a subgroup H is said to be S-embedded in G if there exists a normal subgroup N such that HN is s-permutable in G and $H \cap N \leq H_{sG}$. By using the s-permutability, c-normality, nearly s-normality or S-embedded properties of some subgroups,

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many interesting results have been derived (see [12],[13],[8] etc). Basing on the above concepts, in this paper we introduce that:

Definition 1.1. *Let H be a subgroup of G. We say that H is SS-embedded in G if there exists a normal subgroup T of G such that HT is subnormal in G and $H \cap T \leq H_{sG}$, where H_{sG} is the largest s-permutable subgroup of G contained in H.*

It is easy to see that all subgroups, independently of whether they are normal, s-permutable, c-normal, nearly s-normal or S-embedded in G are SS-embedded subgroups of G. However, the converse case is not true. For example, if we let $G = [P]Q$ be a minimal non-2-nilpotent group, then it is easy to see that every maximal subgroup of P is SS-embedded in G, but some of them are not S-embedded in G (and hence they are not normal, s-permutable, c-normal or nearly s-normal in G). In this paper, we study the influence of some SS-embedded subgroups on the structure of a finite group G. Some new results are obtained.

2. Preliminaries

We list here some basic results which are useful in the sequel.

Lemma 2.1. *([7]) Suppose that H is s-permutable in G, $H \leq G$ and $N \trianglelefteq G$. Then the following holds*

(1) *If $H \leq K \leq G$, then H is s-permutable in K.*
(2) *HN and $H \cap N$ are s-permutable in G, HN/N is s-permutable in G/N.*
(3) *H is subnormal in G.*
(4) *If H is a p-group for some prime p, then $N_G(H) \geq O^p(G)$.*

Lemma 2.2. *Suppose that H is an SS-embedded subgroup of G, then*

(1) *If $H \leq K \leq G$, then H is SS-embedded in K.*
(2) *If $N \trianglelefteq G$ and $N \leq H$, then H/N is SS-embedded in G/N.*
(3) *Let H be a π-subgroup and N a normal π'-subgroup of G, then HN/N is SS-embedded in G/N.*

Proof. By the hypothesis, there exist a normal subgroup T of G and an s-permutable subgroup H_{sG} of G contained in H such that HT is subnormal in G and $H \cap T \leq H_{sG}$.

(1) It is clear that $K \cap T$ is a normal subgroup of K, $H(K \cap T) = K \cap HT$ is subnormal in K and $H \cap (K \cap T) = H \cap T \leq H_{sG}$. By Lemma 2.1(1), H_{sG} is s-permutable in K. Hence H is SS-embedded in K.

(2) Clearly, TN/N is a normal subgroup of G/N, $(H/N)(TN/N) = HT/N$ is subnormal in G/N and $(H/N) \cap (TN/N) = (H \cap TN)/N = (H \cap T)N/N \leq H_{sG}N/N$. By Lemma 2.1(2), $H_{sG}N/N$ is s-permutable in G/N. Hence H/N is SS-embedded in G/N.

(3) It is easy to see that $TN/N \trianglelefteq G/N$, $(HN/N)(TN/N) = HTN/N$ is subnormal in G/N. Since $(|H|, |N|) = 1$,

$$|H \cap TN| = \frac{|H| \cdot |TN|_\pi}{|HTN|_\pi} = \frac{|H| \cdot |T|_\pi}{|HTN|_\pi} = \frac{|H| \cdot |T|_\pi}{|HT|_\pi} = |H \cap T|.$$

This implies that $H \cap TN = H \cap T$, thus

$$(HN/N) \cap (TN/N) = (HN \cap TN)/N = (H \cap TN)N/N = (H \cap T)N/N \leq H_{sG}N/N.$$

Hence HN/N is SS-embedded in G/N. □

Lemma 2.3. ([4, Lemma 2.5]) *Let G be a group and p a prime such that $p^{n+1} \nmid |G|$ for some integer $n \geq 1$. If $(|G|, (p-1)(p^2-1)\cdots(p^n-1)) = 1$, then G is p-nilpotent.*

3. Main results

Theorem 3.1. *Let P be a Sylow p-subgroup of a group G, where p is a prime divisor of $|G|$ with $(|G|, (p-1)(p^2-1)\cdots(p^n-1)) = 1$. If $N_G(P)$ is p-nilpotent and every n-maximal subgroup of P (if exists) not having a p-nilpotent supplement in G is SS-embedded in G, then G is p-nilpotent.*

Proof. Assume that the result is false and let G be a counterexample of minimal order. Then we have:

(1) $|P| \geq p^{n+1}$ and every n-maximal subgroup of P is SS-embedded in G.

By Lemma 2.3, we may assume that $|P| \geq p^{n+1}$. If there exists an n-maximal subgroup P_1 of P which has a p-nilpotent supplement T in G, we prove that G is p-nilpotent. If not, let H be a non-p-nilpotent subgroup of G which contains P and is such that every proper subgroup of H is p-nilpotent. Then by [6, IV, Theorem 5.4], H is a minimal non-nilpotent group. Therefore, H has the following properties:

(i) $|H| = p^a q^b$, where p and q are different primes;

(ii) $H = [H_p]H_q$, where $H_p = P$ is a normal Sylow p-subgroup and H_q a non-normal cyclic Sylow q-subgroup of H;

(iii) $P/\Phi(P)$ is a chief factor of H.

Since $G = P_1T$, $H = H \cap P_1T = P_1(H \cap T)$. The fact $H \cap T \leq T$ is p-nilpotent but H is not p-nilpotent implies that $L = H \cap T$ is a proper subgroup of H, and hence L is nilpotent. Let $L = L_p \times L_q$. Obviously, L_q is also a Sylow q-subgroup of H. Since $P = P_1L_p$, L_p is not contained in $\Phi = \Phi(P)$. Now we consider the factor group H/Φ. The fact $L_q \leq N_H(L_p)$ implies that $L_q\Phi/\Phi \leq N_{H/\Phi}(L_p\Phi/\Phi)$. On the other hand, since P/Φ is an elementary abelian group, we have $L_p\Phi/\Phi \trianglelefteq P/\Phi$. Hence $L_p\Phi/\Phi \trianglelefteq H/\Phi$. Since $L_p\Phi/\Phi \neq 1$ and P/Φ is a chief factor of H, we have $L_p\Phi/\Phi = P/\Phi$. It follows that $L_p = P$. Consequently, $L = H$. This contradiction completes the proof of (1).

(2) G is soluble.

If not, then by the Feit-Thompson theorem we know $p = 2$. First, we assume that $O_2(G) \neq 1$. If $|P/O_2(G)| \leq 2^{n+1}$, then by Lemma 2.3, $G/O_2(G)$ is 2-nilpotent. Hence G is soluble, a contradiction. Thus we may suppose that $|P/O_2(G)| \geq 2^{n+1}$. Since $N_G(P/O_2(G)) = N_G(P)/O_2(G)$ is 2-nilpotent, by Lemma 2.2(2) we know $G/O_2(G)$ satisfies the hypothesis of the theorem. Hence the minimal choice of G implies that $G/O_2(G)$ is 2-nilpotent. It follows that G is soluble, a contradiction. Next, we suppose that $O_2(G) = 1$ and $|P| \geq 2^{n+1}$. Let P_1 be an n-maximal subgroup of P. By the hypothesis, P_1 is SS-embedded in G. Hence there exists $T \trianglelefteq G$ such that P_1T is subnormal in G and $P_1 \cap T \leq (P_1)_{sG}$. Since $(P_1)_{sG} \leq O_2(G) = 1$, $P_1 \cap T = 1$ and so $|T|_2 \leq 2^n$. Hence by Lemma 2.3, T is 2-nilpotent and so it is soluble. Therefore, P_1T is soluble. Since P_1T is a soluble subnormal subgroup of G, it is contained in some soluble normal subgroup M of G. Clearly, 2^{n+1} does not divide $|G/M|$. So by Lemma 2.3 and the Feit-Thompson theorem, G/M is soluble and so is G, as required.

(3) $O_{p'}(G) = 1$ and $O_p(G) \neq 1$.

If $O_{p'}(G) \neq 1$, then we know that $\overline{P} = PO_{p'}(G)/O_{p'}(G)$ is a Sylow p-subgroup of $\overline{G} = G/O_{p'}(G)$, $(|\overline{G}|, (p-1)(p^2-1)\cdots(p^n-1)) = 1$ and $N_{\overline{G}}(\overline{P}) = N_G(P)O_{p'}(G)/O_{p'}(G)$ is p-nilpotent. By (1), $|\overline{P}| \geq p^{n+1}$. Let $\overline{P_1} = P_1O_{p'}(G)/O_{p'}(G)$ be an n-maximal subgroup of $\overline{P}$. Then we may assume that P_1 is an n-maximal subgroup of P. By the hypothesis and (1), P_1 is SS-embedded in G. Then by Lemma 2.2(3), $\overline{P_1}$ is SS-embedded in $\overline{G}$. Hence $\overline{G}$ is p-nilpotent by induction. It follows that G is p-nilpotent, a contradiction. Thus we have $O_{p'}(G) = 1$. Since G is soluble, $O_p(G) \neq 1$.

(4) $O_p(G)$ is a unique minimal normal subgroup of G, $\Phi(G) = 1$ and $G/O_p(G)$ is p-nilpotent.

Let N be a minimal normal subgroup of G. By (2) and (3), N is an elementary abelian p-group and $N \leq O_p(G)$. If $|P/N| \leq p^n$, then G/N is p-nilpotent by Lemma 2.3. Now we assume that $|P/N| \geq p^{n+1}$. By Lemma 2.2(2), we know the hypothesis of the theorem holds for G/N. By the minimal choice of G, we have G/N is p-nilpotent. Since the class of all p-nilpotent groups formed a saturated formation, N is a unique minimal normal subgroup of G and $\Phi(G) = 1$. Thus there is a maximal subgroup M of G such that $G = [N]M$. Since $O_p(G) \leq F(G) \leq C_G(N)$ and $C_G(N) \cap M \trianglelefteq G$, the uniqueness of N yields that $N = O_p(G) = F(G) = C_G(N)$.

(5) $|O_p(G)| \geq p^{n+1}$.

Since $G/O_p(G)$ is p-nilpotent, let $K/O_p(G)$ be the normal p-complement of $G/O_p(G)$. If $|O_p(G)| \leq p^n$, then $|K|_p \leq p^n$ and so Lemma 2.3 implies that K is p-nilpotent. The normal p-complement of K is also a normal p-complement of G. This contradiction shows that $|O_p(G)| \geq p^{n+1}$.

(6) The final contradiction.

Since $\Phi(G) = 1$, there exists a maximal subgroup M of G such that $G = [O_p(G)]M$. Let $P = O_p(G)M_p$ be a Sylow p-subgroup of G, where M_p is a Sylow p-subgroup of M. Since $|O_p(G)| \geq p^{n+1}$, we can pick an n-maximal subgroup P_1 of P containing M_p such that $P_1 \cap O_p(G) \neq 1$. Clearly, $O_p(G) \nleq P_1$. By the hypothesis, there exists a normal subgroup T of G such that $P_1T \trianglelefteq\trianglelefteq G$ and $P_1 \cap T \leq (P_1)_{sG}$. Since $O_p(G)$ is an elementary abelian group and $M_p \leq P_1$, we get that $P = O_p(G)P_1 \leq N_G(P_1 \cap O_p(G))$. If $T = 1$, then $P_1 = P_1T \trianglelefteq\trianglelefteq G$, so $P_1 \leq O_p(G)$ which implies that $P = O_p(G)P_1 = O_p(G)$. In this case, $G = N_G(P)$ is p-nilpotent, a contradiction. Next, we assume that $T \neq 1$. By Lemma 2.1(3), $(P_1)_{sG}$ is subnormal in G and so $(P_1)_{sG} \leq O_p(G)$. Therefore, $P_1 \cap T \leq (P_1)_{sG} \leq O_p(G)$. Assume that $P_1 \cap T \neq 1$, then $(P_1)_{sG} \neq 1$ and Lemma 2.1(4) shows that $N_G((P_1)_{sG}) \geq O^p(G)$. Thus $1 < ((P_1)_{sG})^G = ((P_1)_{sG})^{PO^p(G)} = ((P_1)_{sG})^P \leq (P_1 \cap O_p(G))^P = P_1 \cap O_p(G) \leq O_p(G)$. Hence by (4) we have $((P_1)_{sG})^G = O_p(G) = P_1 \cap O_p(G)$. It follows that $O_p(G) \leq P_1$ and then $P = O_p(G)P_1 = P_1$, a contradiction. Therefore, $P_1 \cap T = 1$ and so p^{n+1} does not divide $|T|$. Since $T \neq 1$ and $O_p(G)$ is the unique minimal normal subgroup of G, $O_p(G) \leq T$. Hence p^{n+1} does not divide $|O_p(G)|$, which contradicts (5). This final contradiction completes the proof of the theorem. □

Remark: Theorem 3.1 does not hold if we delete the condition that "$N_G(P)$ is p-nilpotent". To see this, let $G = [P]Q$ be a minimal non-2-nilpotent group, then it is easy to see that every maximal subgroup of P is SS-embedded in G but G is not 2-nilpotent.

From Theorem 3.1, we can easily deduce that:

Corollary 3.2. *Let P be a Sylow p-subgroup of G, where $p = \min\pi(G)$. If $N_G(P)$ is p-nilpotent and every maximal subgroup of P not having a p-nilpotent supplement in G is SS-embedded in G, then G is p-nilpotent.*

Next, we prove that:

Theorem 3.3. *Let p be a prime divisor of $|G|$ and P a Sylow p-subgroup of G. If $N_G(P)$ is p-nilpotent and every maximal subgroup of P not having a p-nilpotent supplement in G is SS-embedded in G, then G is p-nilpotent.*

Proof. If $p = \min\pi(G)$, then by Corollary 3.2 we know that G is p-nilpotent. Hence we only need to consider the case when p is not the minimal prime divisor of $|G|$ (so it is an odd prime). Assume that the result is false and let G be a counterexample of minimal order. Then we have:

(1) Every maximal subgroup of P is SS-embedded in G.

See the proof of step (1) in Theorem 3.1.

(2) $O_{p'}(G) = 1$.

Suppose that $O_{p'}(G) \neq 1$. Clearly, $\overline{P} = PO_{p'}(G)/O_{p'}(G)$ is a Sylow p-subgroup of $\overline{G} = G/O_{p'}(G)$ and $N_{\overline{G}}(\overline{P}) = N_G(P)O_{p'}(G)/O_{p'}(G)$ is p-nilpotent. Let $T/O_{p'}(G)$ be a maximal subgroup of $PO_{p'}(G)/O_{p'}(G)$. Then $T = P_1O_{p'}(G)$ for some maximal subgroup P_1 of P. By (1) and Lemma 2.2(3), we know $P_1O_{p'}(G)/O_{p'}(G)$ is SS-embedded in $\overline{G}$. This shows that $\overline{G}$ satisfies the hypothesis of the theorem. Thus $G/O_{p'}(G)$ is p-nilpotent by induction. It follows that G is p-nilpotent, a contradiction.

(3) If M is a proper subgroup of G containing P, then M is p-nilpotent.

Since $N_M(P) \leq N_G(P)$ is p-nilpotent. Now by (1) and Lemma 2.2(1), we see that M satisfies the hypothesis. The minimal choice of G implies that M is p-nilpotent.

(4) $G = PQ$ is soluble and $1 < O_p(G) < P$, where Q is a Sylow q-subgroup of G with $q \neq p$.

Since G is not p-nilpotent, by Thompson's theorem [9, Theorem 10.4.1], there is a nonidentity characteristic subgroup H of P such that $N_G(H)$ is not p-nilpotent. Since $N_G(P)$ is p-nilpotent, we may choose a characteristic subgroup H of G such that $N_G(H)$ is not p-nilpotent, but $N_G(K)$ is p-nilpotent for every characteristic subgroup K of P containing H. Obviously, $N_G(P) \leq N_G(H)$. Then by (3), $N_G(H) = G$. Therefore, $H \leq O_p(G) \neq 1$ and $O_p(G) < K$. Now by the Thompson theorem again, we see that $G/O_p(G)$ is p-nilpotent, and so G is p-soluble. By [2, VI, Theorem 3.5], there exists a Sylow q-subgroup Q of G such that PQ is a subgroup of G, where q is a prime divisor of G and $q \neq p$. If $PQ < G$, then PQ is p-nilpotent by (3). This implies that $Q \leq C_G(O_p(G)) \leq O_p(G)$ by [2, VI, Theorem 3.2], a contradiction. Thus $G = PQ$ and (4) holds.

(5) G has a unique minimal normal subgroup N such that $G = [N]M$, where M is a maximal subgroup of G and $N = O_p(G) = F(G)$.

Let N be a minimal normal subgroup of G. Then by (2) and (4), N is an elementary abelian p-group and $N \leq O_p(G)$. It is easy to see that G/N satisfies the hypothesis. The minimal choice of G implies that G/N is p-nilpotent. Since the class of all p-nilpotent groups is a saturated formation, N is a unique minimal normal subgroup of G and $N \nleq \Phi(G)$. Thus, we can see that (5) holds.

(6) $|N| = p$.

Obviously, $P = NM_p$, where M_p is a Sylow p-subgroup of M. Let P_1 be a maximal subgroup of P containing M_p. If $P_1 = 1$, then $|N| = |P| = p$. Now suppose that $P_1 \neq 1$. By (1), there exists $K \trianglelefteq G$ such that $P_1K \trianglelefteq\trianglelefteq G$ and $P_1 \cap K \leq (P_1)_{sG}$. If $K = 1$, then $P_1 = P_1K \trianglelefteq\trianglelefteq G$ which implies that $P_1 \leq O_p(G) = N$, and so $P = NP_1 = N$, a contradiction. Thus we have $K \neq 1$ and then $N \leq K$. Therefore, $P_1 \cap N \leq P_1 \cap K \leq (P_1)_{sG}$. Since $(P_1)_{sG}$ is subnormal in G by Lemma 2.1(3), $(P_1)_{sG} \leq O_p(G) = N$ and so $P_1 \cap N = (P_1)_{sG}$. If $P_1 \cap N = 1$, then $|N| = p$. If $P_1 \cap N \neq 1$, then $1 \neq (P_1 \cap N)^G = ((P_1)_{sG})^G = ((P_1)_{sG})^{O^p(G)P} = ((P_1)_{sG})^P = (P_1 \cap N)^P = P_1 \cap N \leq N$. It follows that $(P_1 \cap N)^G = P_1 \cap N = N$, i.e., $N \leq P_1$, a contradiction. Hence (6) holds.

(7) The final contradiction.

By (5) and (6), we know $M \cong G/N = N_G(N)/C_G(N)$ is isomorphic to a subgroup of $Aut(P)$, which is a cyclic group of order $p-1$. Hence M and in particularly Q, is a cyclic group. It follows form [6, IV, Theorem 2.8] that G is q-nilpotent. Thus, $P \trianglelefteq G$. Then by the hypothesis, $N_G(P) = G$ is p-nilpotent. This contradiction completes the proof of the theorem. □

Let G be a group and $|G| = p_1^{r_1}p_2^{r_2}\cdots p_s^{r_s}$, where $p_1, p_2, \cdots, p_r$ are different primes. Recall that G is said to be a Sylow tower group, if there exists a normal series $1 = G_0 \leq G_1 \leq G_2 \leq \cdots \leq G_s = G$ of G such that $|G_i : G_{i-1}| = p_i^{r_i}$ for $1 \leq i \leq s$. If, in addition that, $p_1 > p_2 > \cdots > p_s$, then G is called a Sylow tower group of supersoluble type.

Theorem 3.4. *Let G be a finite group. If for any non-cyclic Sylow p-subgroup P of G, $N_G(P)$ is p-nilpotent and every maximal subgroup of P is SS-embedded in G, then G is a Sylow tower group of supersoluble type.*

Proof. Let p_1 be the minimal prime divisor of $|G|$ and $P_1 \in Syl_{p_1}(G)$, we first prove that G is p_1-nilpotent. If P_1 is cyclic, then by [9, Theorem 10.1.9] we know that G is p_1-nilpotent. If P_1 is not cyclic, then by hypothesis $N_G(P_1)$ is p_1-nilpotent and every maximal subgroup of P_1 is SS-embedded in G. By Corollary 3.2, we can also conclude that G is p_1-nilpotent. Now we let K be the normal p_1-complement of G. By the hypothesis and Lemma 2.2, we know that for any non-cyclic Sylow q-subgroup Q of K, $N_K(Q) \leq N_G(Q)$ is q-nilpotent and every maximal subgroup of Q is SS-embedded in K. Thus, by induction we can deduce that K is a Sylow tower group of supersoluble type. It follows that G is a Sylow tower group of supersoluble type, as required. □

Next, by using the SS-embedded properties of some subgroups, we give some criteria for the solubility of a group G.

Theorem 3.5. *Let P be a Sylow 2-subgroup of G. If P is SS-embedded in G or every maximal subgroup of P is SS-embedded in G, then G is soluble.*

Proof. Assume that the result is false and let G be a counterexample of minimal order. If $O_2(G) = P$, then $G/O_2(G)$ is a group of odd order, which is soluble by the Feit-Thompson theorem, thus G is also soluble. If $1 < O_2(G) < P$, then $P/O_2(G)$ is a Sylow 2-subgroup of $G/O_2(G)$. By the hypothesis and Lemma 2.2, $P/O_2(G)$ is SS-embedded in $G/O_2(G)$ or every maximal subgroup of $P/O_2(G)$ is SS-embedded in $G/O_2(G)$. So $G/O_2(G)$ is soluble by induction, and hence G is soluble. Next, we may assume that $O_2(G) = 1$. Let H denote P or a maximal subgroup of P, respectively. By the hypothesis, there exists a normal subgroup K of G such that $HK \trianglelefteq\trianglelefteq G$ and $H \cap K \leq H_{sG} \leq O_2(G) = 1$. Thus $|K|_2 \leq 2$, which is 2-nilpotent and so soluble. Since HK/K is a 2-group, HK is soluble. Then HK is contained in some soluble normal subgroup M of G. Since $|G/M|_2 \leq 2$, it is soluble. Therefore G is soluble, as required. □

Theorem 3.6. *Let G be a group. Then G is soluble if and only if every maximal subgroup of G is SS-embedded in G.*

Proof. Suppose every maximal subgroup of G is SS-embedded in G, we prove that G is soluble. For any maximal subgroup M of G, there exists a normal subgroup K of G such that $MK \trianglelefteq\trianglelefteq G$ and $M \cap K \leq M_{sG}$. If G is a simple group, then $K = 1$ or $K = G$. In both case, we can conclude that $M = 1$ and G is a cyclic group of prime order. Thus it is soluble, as required. Next, we let N be a minimal normal subgroup of G. Clearly the hypothesis holds for G/N by Lemma 2.2, so by induction we have G/N is soluble. Hence we may assume that N is non-abelian and it is the only minimal normal subgroup of G. By the Frattini argument [2, I, Theorem 3.7], for any prime q dividing $|N|$ and for any Sylow q-subgroup Q of N, there is a maximal subgroup M of G such that $NM = G$ and $N_G(Q) \leq M$. It is clear that $M_G = 1$ and q does not divide $|G : M|$. Then by [10, Lemma 2.8], $M_{sG} = 1$. Since M is SS-embedded in G, G has a normal subgroup T and an subnormal subgroup K of G such that $K = MT$ and $M \cap T = 1$. Since a subnormal maximal subgroup is normal in G, it follows that $K = G$ and $|G : M| = |T|$. Therefore, q does not divide $|T|$. But $N \leq T$ as N is the only minimal normal subgroup of G. Thus, q divides $|T|$. This contradiction shows that G is soluble.

Conversely, suppose that G is soluble and let M be a maximal subgroup of G. If M is normal in G, then obviously M is SS-embedded in G. Assume that M is not normal in G and let T/K be a chief factor of G such that $K \leq M$ and $TM = G$. Then clearly $M \cap T = K \leq M_{sG}$, so M is SS-embedded in G, as required. □

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