www.ijgt.ui.ac.ir

International Journal of Group Theory
ISSN (print): 2251-7650, ISSN (on-line): 2251-7669
Vol. 11 No. 1 (2022), pp. 7-16.

www.ui.ac.ir

CHARACTERIZATION OF THE CHEVALLEY GROUP $G_2(5)$ BY THE SET OF NUMBERS OF THE SAME ORDER ELEMENTS

MARYAM JAHANDIDEH * AND MOHAMMAD REZA DARAFSHEH

ABSTRACT. Let G be a group and $\omega(G) = \{o(g)|g \in G\}$ be the set of element orders of G. Let $k \in \omega(G)$ and $s_k = |\{g \in G|o(g) = k\}|$. Let $nse(G) = \{s_k|k \in \omega(G)\}$. In this paper, we prove that if G is a group and $G_2(5)$ is the Chevalley simple group of type G_2 over $GF(5)$ such that $nse(G) = nse(G_2(5))$, then $G \cong G_2(5)$.

1. Introduction and Preliminaries

Let G be a finite group and $\omega(G)$ be the set of element orders of G. If $k \in \omega(G)$, then s_k is the number of elements of order k in G. Let $nse(G) = \{s_k|k \in \omega(G)\}$. If n is a positive integer, the set of all the prime divisors of n is denoted by $\pi(n)$. The number of the Sylow p-subgroups P_p of G is denoted by n_p or $n_p(G)$. We set $\pi(G) = \pi(|G|)$. To see notations concerning finite simple group, we refer to reader [1]. A finite group G is called a simple K_n- group, if G is a simple group and $|\pi(G)| = n$. In 1987, J. G. Thompson posed the following problem related to algebraic number fields [17].

Thompson's Problem. Let $T(G) = \{(k, s_k)|k \in \omega(G), s_k \in nse(G)\}$. Suppose that $T(G) = T(H)$ for some finite groups H, if G is a finite solvable group, is it true that H is necessarily solvable.

Communicated by Bijan Taeri.
MSC(2010): Primary 20D05; Secondary 20D06, 20D20.
Keywords: Element order, Thompson's problem, Number of the same order elements.
Received: 04 January 2020, Accepted: 20 February 2021.
Manuscript Type: Research Paper.
*Corresponding author.

http://dx.doi.org/10.22108/ijgt.2021.120906.1594

.

A finite group G is characterizable by order and nse; if H is a finite group and $|G| = |H|$ and $nse(G) = nse(H)$, then $G \cong H$. The following results have been appeared so far:

Result 1. [15, 16] Let G be a group and S be a simple K_i -group, where $i = 3, 4$. $G \cong H$ if and only if $|G| = |H|$ and $nse(G) = nse(H)$.

Result 2. [4, 5] The two groups A_{12} and A_{13} are characterizable by order and nse.

Result 3. [7] All sporadic simple groups are characterizable by nse and order.

Result 4. [14] $L_2(2^m)$ with $2^m + 1$ prime or $2^m - 1$ prime, is characterizable by nse and order.

Result 5. [16, 8] $L_2(q)$, where $q \in \{7, 8, 9, 11, 13\}$ can be characterizable by only the nse.

Result 6. [9] $L_3(4)$ is characterizable by nse.

Result 7. [10] $L_5(2)$ is characterizable by nse.

Result 8. [11] $U_3(5)$ is characterizable by nse.

Result 9. [6] $G_2(4)$ is characterizable by nse.

Result 10. [13] $L_2(81)$ is characterizable by nse.

Up to now, some groups are characterized by only the set $nse(G)$. The aim of this paper is to prove that the Chevalley group $G_2(5)$ is characterizable by nse.

Main Theorem. Let G be a group such that $nse(G) = nse(G_2(5))$, where $G_2(5)$ is the Chevalley group of type G_2 over $GF(5)$. Then $G \cong G_2(5)$.

We will give some lemmas which will be used to prove the main theorem.

Lemma 1.1. [2] *Let G be a finite group and n be a positive integer dividing $|G|$. If $L_n(G) = \{g \in G | g^n = 1\}$, then $n || L_n(G)|$.*

Lemma 1.2. [12] *Let G be a finite group and $p \in \pi(G)$ be an odd number. Suppose that P is a Sylow p-subgroup of G and $n = p^s m$ with $(p, m) = 1$. If P is not cyclic, then the number of elements of order n is always a multiple of p^s.*

Lemma 1.3. [16] *Let G be a group containing more than two elements. If the maximum number s of elements of the same order in G is finite, then G is finite and $|G| \leq s(s^2 - 1)$.*

Lemma 1.4. [3] *Let G be a finite solvable group and $|G| = mn$, where $m = {p_1}^{\alpha_1} \cdots {p_r}^{\alpha_r}$, $(m, n) = 1$. Let $\pi = \{p_1, \ldots, p_r\}$ and h_m be the number of Hall π-subgroups of G. Then $h_m = {q_1}^{\beta_1} \cdots {q_s}^{\beta_s}$ satisfies the following conditions for all $i \in \{1, 2, \ldots, s\}$:*

(1) *${q_i}^{\beta_i} \equiv 1 (mod\ p_j)$ for some p_j;*

(2) *The order of some chief factor of G is divided by $q_i^{\beta_i}$.*

http://dx.doi.org/10.22108/ijgt.2021.120906.1594

Lemma 1.5. [18] *Let G be a simple K_4-group. Then G is isomorphic to one of the following groups:*

(1) A_7, A_8, A_9 *and* A_{10};
(2) M_{11}, M_{12} *or* J_2;
(3) *one of the following:*
  (i) $L_2(r)$, *where r is a prime and $r^2-1=2^a\cdot 3^b\cdot v^c$ with $a,b,c\geq 1$, and v is a prime number greater than 3.*
  (ii) $L_2(2^m)$, *where $2^m-1=u, 2^m+1=3t^b$, with $m\geq 2, u, t$ are primes,$t>3, b\geq 1$.*
  (ii) $L_2(3^m)$, *where $3^m+1=4t, 3^m-1=2u^c$ or $3^m+1=4t^b, 3^m-1=2u, m\geq 2$, u and t are odd primes, $b\geq 1, c\geq 1$.*
  (iv) *One of the following 28 simple groups:*
  $L_2(16)$, $L_2(25)$, $L_2(49)$, $L_2(81)$, $L_3(4)$, $L_3(5)$, $L_3(7)$, $L_3(8)$, $L_3(17)$, $L_4(3)$, $S_4(4)$, $S_4(5)$, $S_4(7)$, $S_4(9)$, $S_6(2)$, ${O_8}^+(2)$, $G_2(3)$, $U_3(4)$,$U_3(5)$, $U_3(7)$, $U_3(8)$, $U_3(9)$, $U_4(3)$, $U_5(2)$, $Sz(8)$, $Sz(32)$, ${}^2D_4(2)$ *or* ${}^2F_4(2)$.

Lemma 1.6. [5] *Every simple K_5-group is isomorphic to one of the following simple groups:*

(1) $L_2(q)$ *with* $|\pi(q^2-1)|=4$;
(2) $L_3(q)$ *with* $|\pi(q^2-1)(q^3-1)|=4$;
(3) $U_3(q)$ *with q satisfies* $|\pi(q^2-1)(q^3+1)|=4$;
(4) $O_5(q)$ *with* $|\pi(q^4-1)|=4$;
(5) $Sz(2^{2m+1})$ *with* $|\pi(2^{2m+1}-1)(2^{4m+1}+1)|=4$
(6) $R(q)$, *where q is an odd power of 3,* $|\pi(q^2-1)|=3$ *and* $|\pi(q^2-q+1)|=1$;
(7) *The following 30 simple groups:*
  A_{11}, A_{12}, M_{22}, J_3, HS, He, McL, $L_4(4)$, $L_4(5)$, $L_4(7)$, $L_5(2)$, $L_5(3)$, $L_6(2)$, $O_7(3)$, $O_9(2)$, $PSP_8(2)$, $U_4(4)$, $U_4(5)$, $U_4(7)$, $U_4(9)$, $U_5(3)$, $U_6(2)$, ${O_8}^+(3)$, ${O_8}^-(2)$, ${}^3D_4(3)$, $G_2(4)$, $G_2(5)$, $G_2(7)$ *or* $G_2(9)$.

Lemma 1.7. [1] *Let G be a simple K_n-group with $n=4,5$ and $31||G|$ and $|G||2^6\cdot 3^3\cdot 5^6\cdot 7\cdot 31$. Then G is one of the following groups: $G_2(5)$, $L_2(125)$.*

2. **Main Result**

In this section, we prove that if G is a group and $nse(G)=nse(G_2(5))$, then $G\cong G_2(5)$. However, some results and lemmas have been mentioned before presenting the proof.

Lemma 2.1. *If s_n is the number of elements of order n in a group G, then $s_n=k\varphi(n)$ such that k is the number of cyclic subgroups of order n in G.*

Proof. It is straightforward. □

Lemma 2.2. *If $n > 2$, then $\varphi(n)$ is even.*

Proof. It is straightforward. □

Lemma 2.3. *If $m \in \omega(G)$, then $\varphi(m)|s_m$ and $m|\sum_{d|m} s_d$.*

Proof. It follows from Lemma 1.1. □

Lemma 2.4. *Let G be a group such that $nse(G) = nse(G_2(5))$, then G is a finite group.*

Proof. It follows from Lemma 1.3. □

Remark 2.5.

$$nse(G_2(5)) = \{1, 406875, 8153000, 24412500, 9780624, 179025000,$$

$$279000000, 488250000, 253890000, 437472000, 855900000, 558000000,$$

$$976500000, 234360000, 390600000, 945000000\}.$$

Theorem 2.6. *Let G be a group such that $nse(G) = nse(G_2(5))$, where $G_2(5)$ is the Chevalley group of type G_2 over $GF(5)$. Then $G \cong G_2(5)$.*

Proof. It follows from Lemma 2.4, G is a finite group. According to Lemma 1.1, $\pi(G) \subseteq \{2, 3, 5, 7, 11, 13, 31, 263, 313\}$. There are two claims in this argument. First claim: $\pi(G) = \{2, 3, 5, 7, 31\}$. According to Lemma 2.2, $s_2 = 406875$ and $2 \in \pi(G)$. Since s_{263}, s_{313} are not equal to none of $nse(G)$ values, $263, 313 \notin \pi(G)$. We have $\pi(G) \subseteq \{2, 3, 5, 7, 11, 13, 31\}$ and we are going to show that $11^2 \notin \omega(G)$. If $11^2 \in \omega(G)$, $s_{11} = 390600000$ or 945000000 and $s_{11^2} = 179025000$. According to Lemma 2.3, $11^2 \nmid \sum_{d|11^2} s_d$ and $11^2 \notin \omega(G)$. We have the following:

(1) If $2^a \in \omega(G)$, then $2^{a-2}|s_{2^a}$ and $1 \leq a \leq 9$;
(2) If $3^a \in \omega(G)$, then $2 \cdot 3^{a-1}|s_{3^a}$ and $1 \leq a \leq 4$;
(3) If $5^a \in \omega(G)$, then $4 \cdot 5^{a-1}|s_{5^a}$ and $1 \leq a \leq 8$;
(4) If $7^a \in \omega(G)$, then $6 \cdot 7^{a-1}|s_{7^a}$ and $1 \leq a \leq 3$;
(5) If $11^a \in \omega(G)$, then $10 \cdot 11^{a-1}|s_{11^a}$ and $a = 1$;
(6) If $13^a \in \omega(G)$, then $12 \cdot 13^{a-1}|s_{13^a}$ and $a = 1$;
(7) If $31^a \in \omega(G)$, then $30 \cdot 31^{a-1}|s_{31^a}$ and $1 \leq a \leq 2$.

We show that the prime number 13 does not belong to $\pi(G)$. There are several cases to explain.

Case(1) $\pi(G) = \{2, 3\}$.

http://dx.doi.org/10.22108/ijgt.2021.120906.1594

In this case $s_{13} = 558000000$, $|P_{13}||1 + s_{13}$ and $|P_{13}| = 13$, where P_{13} is a Sylow 13-subgroup of G.

$$n_{13} = s_{13}/\varphi(13) = 46500000 = [G : N_G(P_{13})].$$

So, $3, 5, 31 \in \pi(G)$.

Case(2) $\pi(G) = \{2, 3, 13\}$.

According to case(1), $5 \in \pi(G)$.

Case(3) $\pi(G) = \{2, 3, 5, 13\}$, $\pi(G) = \{2, 3, 5, 7, 13\}$ or $\pi(G) = \{2, 3, 5, 7, 11, 13\}$.

According to case(1), $31 \in (G)$.

Case(4) $\pi(G) = \{2, 3, 5, 7, 11, 13, 31\}$.

If $11.13 \in \omega(G)$, $11.13 \nmid 1 + s_{11} + s_{13} + s_{11\cdot 13}$. We consider Sylow 13-subgroup P_{13} of G and $\Sigma = \{g \in G | o(g) = 11\}$ such that $|\sum| = s_{11}$. If $P_{13_g} = \{a \in P_{13} | g^a = g\} \neq 1$ for some $g \in \Sigma$, then G has an element of order $11 \cdot 13$. Hence P_{13} acts fixed point freely on $\sum$ and also $|P_{13}|||\sum| = s_{11}$. This is a contradiction and $13 \notin \pi(G)$. Next, we prove that $11 \notin \pi(G)$. To prove this, we distinguish several cases.

(A) $\pi(G) = \{2, 11\}$.

According to assumption and Lemma 1.1 $s_{11} = 390600000$ or 945000000 and $|P_{11}||1+s_{11}$,$|P_{11}| = 11$. Also

$$n_{11} = s_{11}/\varphi(11) = [G : N_G(P_{11})].$$

So, $3, 5, 7 \in \pi(G)$.

(B) $\pi(G) = \{2, 3, 11\}$ or $\{2, 3, 5, 11\}$.

In this case, $7 \in \pi(G)$.

(C) $\pi(G) = \{2, 3, 5, 7, 11\}$.

If $s_{11} = 3906000$, then $31 \in \pi(G)$. If $s_{11} = 945000000$, then $|G| = 2^{n_1} \cdot 3^{n_2} \cdot 5^{n_3} \cdot 7^{n_4} \cdot 11$ and $5859000000 + 406875k_1 + 8153000k_2 + 24412500k_3 + 9780624k_4 + 179025000k_5 + 279000000k_6 + 488250000k_7 + 253890000k_8 + 437472000k_9 + 585900000k_{10} + 855900000k_{11} + 558000000k_{12} + 234360000k_{13} + 390600000k_{14} + 945000000k_{15} = |G|$,where $k_1, k_2, \ldots, k_{15}, n_1, \ldots, n_4$ are nonnegative integers and

$$0 < \sum_{i=1}^{15} k_i \leq 1784.$$

Hence $5859000000 < |G| = 1784 \cdot 976500000 + 5859000000$. With reference to prime factors of $|G|$ and $s_{11} = n_{11}.\varphi(11)$ such that $n_{11} = 11k + 1||G|/11$. So $\pi(G) \neq \{2, 3, 5, 7, 11\}$.

(D) $\pi(G) = \{2, 3, 5, 7, 11, 31\}$.

In this case, $|G| = 2^{n_1} \cdot 3^{n_2} \cdot 5^{n_3} \cdot 7^{n_4} 11 \cdot 31^{n_5}$ and $5859000000 + 406875k_1 + 8153000k_2 + 24412500k_3 + 9780624k_4 + 179025000k_5 + 279000000k_6 + 488250000k_7 + 253890000k_8 + 437472000k_9 + 585900000k_{10} + 855900000k_{11} + 558000000k_{12} + 234360000k_{13} + 390600000k_{14} + 945000000k_{15} = |G|$,where $k_1, k_2, k_{15}, n_1, \dots, n_5$ are nonnegative integers and

$$0 < \Sigma_{i=1}^{15} k_i \leq 5384.$$

Hence $5859000000 < |G| = 5384 \cdot 976500000 + 5859000000$. There are some solutions for $n_1, n_2, \dots, n_5$. Only one example is given for this case (the other cases can be ruled out as this case). For example $|G| = 2^7 \cdot 3^3 \cdot 5^7 \cdot 7^2 \cdot 11 \cdot 31$, therefore the number of Sylow 11-subgroups of G must be $11k + 1$ such that divides $|G|/11$ and $s_{11} = n_{11}.\varphi(11)$. However, all numbers of $nse(G)$ do not satisfy these conditions. So $11 \notin \pi(G)$ and $\pi(G) \subseteq \{2, 3, 5, 7, 31\}$.Now, we prove $\pi(G) = \{2, 3, 5, 7, 31\}$.To do this, we distinguish several cases.

Case(i) $\pi(G) = \{2\}$.

If $\pi(G) = \{2\}$, then $\omega(G) \subseteq \{1, 2, 2^2, \dots, 2^9\}$ and $859000000 + 406875k_1 + 8153000k_2 + 24412500k_3 + 9780624k_4 + 179025000k_5 + 279000000k_6 + 488250000k_7 + 253890000k_8 + 437472000k_9 + 585900000k_{10} + 855900000k_{11} + 558000000k_{12} + 234360000k_{13} + 390600000k_{14} + 945000000k_{15} = 2^m$, such that $k_1, k_2, \dots, k_{15}, m$ are nonnegative integers. But this equation

$$0 < \sum_{i=1}^{10} k_i \leq 6$$

has no solution in integers.

Case(ii) $\pi(G) = \{2, 3\}$.

If $2^n \cdot 3^m \in \omega(G)$, then according to Lemma 1.1, $0 \leq n \leq 9$, $0 \leq m \leq 4$. Now, $exp(P_3)$ can be $3, 9, 27$ or 81. If $exp(P_3) = 3$, then $|P_3||1 + s_3 (s_3 = 8153000)$ and $|P_3||27$. If $|P_3| = 3$, then $n_3 = s_3/\varphi(3) = 8153000/2 = 4076500$ and $[G : N_G(P_3)] = 4076500$. So, $263 \in \pi(G)$.

If $|P_3| = 9$, then $|G| = 2^m \cdot 3^2$ such that $0 \leq m \leq 9$. Also calculations show:

$859000000 + 406875k_1 + 8153000k_2 + 24412500k_3 + 9780624k_4 + 179025000k_5 + 279000000k_6 + 488250000k_7 + 253890000k_8 + 437472000k_9 + 585900000k_{10} + 855900000k_{11} + 558000000k_{12} + 234360000k_{13} + 390600000k_{14} + 945000000k_{15} = 2^m \cdot 3^2$, such that $k_1, k_2, \dots, k_{15}, m$ are nonnegative integers. But, $0 < \sum_{i=1}^{10} k_i \leq 14$.

If $|P_3| = 27$, then $|G| = 2^m \cdot 3^3$ such that $0 \leq m \leq 9$. Similar to $|P_3| = 9$, there will be contradiction. Now if $exp(P_3) = 9$, according to Lemma 1.1 $|P_3||1 + s_3 + s_9$. There are different values for s_9.

$$s_9 = \{24412500, 9780624, 279000000, 488250000, 253890000, 437472000, 585900000,$$

$$558000000, 976500000, 234360000, 390600000, 945000000\}.$$

http://dx.doi.org/10.22108/ijgt.2021.120906.1594

For all values, $n_3 = s_9/\varphi(9) = [G : N_G(P_3)]$. So, $31 \in \pi(G)$.

If $exp(P_3) = 27, 81$ similar to the previous cases, a contradiction will be obtained.

Case(iii) $\pi(G) = \{2, 5\}$.

In this case, $|G| = 2^n \cdot 5^m$ and $859000000 + 406875k_1 + 8153000k_2 + 24412500k_3 + 9780624k_4 + 179025000k_5 + 279000000k_6 + 488250000k_7 + 253890000k_8 + 437472000k_9 + 585900000k_{10} + 855900000k_{11} + 558000000k_{12} + 234360000k_{13} + 390600000k_{14} + 945000000k_{15} = 2^n \cdot 5^m$, where $k_1, k_2, \dots, k_{15}, m$ and n are nonnegative integers. According to $0 < \sum_{i=1}^{10} k_i \leq 74$ and $5859000000 < |G| \leq 74 \cdot 976500000 + 5859000000$, $n, m > 9$.

Case(iv) $\pi(G) = \{2, 7\}$.

In this case, $s_7 = 279000000$. If $7^2 \in \omega(G)$, then $s_{7^2} = 976500000$. We have:

$$|P_7||1 + s_7 + s_{49}.$$

There are two values for $|P_7|$.

If $|P_7| = 7$, then $n_7 = s_7/\varphi(7) = 46500000$ and $5 \in \pi(G)$.

If $|P_7| = 49$, then $n_7 = s_{49}/\varphi(49) = 23250000$ and $5 \in \pi(G)$.

Hence $7^2 \notin \omega(G)$ and $|P_7||1 + s_7$. Since $|P_7| = 7$, similar to mentioned solutions, we can see a contradiction.

Case(v) $\pi(G) = \{2, 31\}$.

According to assumption, $s_{31} = 945000000$ and $s_{31^2} = 179025000$. Also $|P_{31}||1 + s_{31} + s_{31^2}$ and $|P_{31}||31^2$.

If $|P_{31}| = 31$, then $n_{31} = s_{31}/\varphi(31) = 31500000$ and $5 \in \pi(G)$.

If $|P_{31}| = 31^2$, then $n_{31} = s_{31^2}/\varphi(31^2) = 192500$ and $5 \in \pi(G)$.

Hence $31^2 \notin \omega(G)$ and $|G| = 2^n \cdot 31$, $|P_{31}| = 31$. Again we see a contradiction similar to above solutions.

Case(vi) $\pi(G) = \{2, 3, 5\}$.

If $\pi(G) = \{2, 3, 5\}$, then $|G| = 2^n \cdot 3^m \cdot 5^c$ such that $0 \leq n \leq 9, 0 \leq m \leq 4, 0 \leq c \leq 8$.

$59000000 + 406875k_1 + 8153000k_2 + 24412500k_3 + 9780624k_4 + 179025000k_5 + 279000000k_6 + 488250000k_7 + 253890000k_8 + 437472000k_9 + 585900000k_{10} + 855900000k_{11} + 558000000k_{12} + 234360000k_{13} + 390600000k_{14} + 945000000k_{15} = 2^n \cdot 3^m \cdot 5^c$, where $k_1, k_2, \dots, k_{15}, m, n$ and c are nonnegative integers.

Therefore, $0 < \sum_{i=1}^{10} k_i \leq 434$ and $5859000000 < |G| \leq 434 \cdot 976500000 + 5859000000$. There is some values for $|G|$. Only one example is given for this case. Suppose that $|G| = 2^9 \cdot 3^4 \cdot 5^8$,the number of Sylow 5- subgroups of G must be $5k + 1$ such that divides $|G|/5^8$ and $s_5 = n_5 \cdot \varphi(5)$. However, all the numbers of $nse(G)$ do not satisfy these conditions.

So $\pi(G) \neq \{2, 3, 5\}$. Therefore $\pi(G) = \{2, 3, 5, 7, 31\}$.

http://dx.doi.org/10.22108/ijgt.2021.120906.1594

If $2 \cdot 7 \in \omega(G)$, let P_7 and Q_7 be Sylow 7-subgroups of G, P_7 and Q_7 are conjugate in G and $C_G(P_7)$ and $C_G(Q_7)$ are conjugate in G. So $s_{14} = \varphi(14)_7.k$, where k is the number of cyclic subgroups of order 2 in $C_G(P_7)$. Hence $n_7 = s_7/\varphi(7) = 46500000$, $s_{14} = 6 \cdot 1996800$ and $s_{14} = 27900000$.

Since $14 \nmid 1 + s_2 + s_7 + s_{14}$, is concluded that $2 \cdot 7 \notin \omega(G)$. It follows that the Sylow 2-group of G acts fixed point freely on the set of elements of order 7 so $|P_2||s_7$ and $|P_2||2^6$.

Next step $3 \cdot 31 \notin \omega(G)$.

Let P_{31} and Q_{31} be Sylow 31-subgroups of G. If $3 \cdot 31 \in \omega(G)$, then $s_{93} = \varphi(93)_{13}.k$, where k is the number of cyclic subgroups of order 3 in $C_G(P_{31})$. Therefore, $n_{31} = s_{31}/\varphi(31) = 31500000$ and $s_{93} = 60 \cdot 31500000$. $189000000|s_{93}$ are concluded in accordance with values in nse. It is impossible and it follows that the Sylow 3-group of G acts fixed point freely on the set of elements of order 31. So, $|P_3||s_{31}$ and $|P_3||3^3$.

As the same way, $|P_5||s_{31}$ and $|P_5||5^6$.

We obtain $|G| = 2^m \cdot 3^n \cdot 5^k \cdot 7 \cdot 31$ and also $\Sigma_{s_k \in nse(G)} s_k = 859000000 = 2^6 \cdot 3^3 \cdot 5^6 \cdot 7 \cdot 31 \leq |G| = 2^m \cdot 3^n \cdot 5^k \cdot 7 \cdot 31 \leq 2^6 \cdot 3^3 \cdot 5^6 \cdot 7 \cdot 31$. So, $|G| = |G_2(5)|$.

Second claim is $G \cong G_2(5)$.

Suppose that $G \ncong G_2(5)$,is shown G is not solvable. If G is supposed solvable, then by Lemma 1.4 $7 \equiv 1(mod\ 31)$, that is impossible.

Hence G has a normal series $1 \lhd K \lhd L \lhd G$ such that L/K is isomorphic to a simple k_i -group where $i = 3, 4, 5$. If L/K is isomorphic to one of the simple K_3 -group, by [18], L/K is isomorphic to $A_5, A_6, L_2(7), L_2(8), U_3(3)$ or $U_4(2)$.

If $L/K \cong A_5$, $n_2(L/K) = n_2(A_5) = 15$. Thus $n_2(G) = 15t$ such that $15 \nmid t$ and $s_2 = 15t$. Hence, $5|t$. Similarly, for the groups $A_6, L_2(7), L_2(8), U_3(3)$ and $U_4(2)$, we also can rule out these cases. If L/K is isomorphic to a simple K_n-group with $n = 4, 5$, then by Lemma 1.7, L/K is isomorphic to $G_2(5), L_2(125)$.

If $L/K \cong L_2(125)$, then $31k + 1 = a = n_{31}(L/K) = n_{31}(L_2(125))$ and $n_{31}(G) = at$ such that $31 \nmid t$ for some integer t. Hence, $s_{31} = 30.at = 30.31500000$ and $t = (2^5 \cdot 3^2 \cdot 5^6 \cdot 7)/a$. We have $t||K|$ and $|K||2^4 \cdot 3 \cdot 5^3$, then $2 \cdot 3 \cdot 5^3 \cdot 7|a|22 \cdot 32 \cdot 53 \cdot 7$. There are several choices for a as follows:

$$2 \cdot 3 \cdot 5^3 \cdot 7, 2^2 \cdot 3 \cdot 5^3 \cdot 7, 2 \cdot 3^2 \cdot 5^3 \cdot 7, 2^2 \cdot 3^2 \cdot 5^3 \cdot 7$$

All the mentioned numbers are not equal to the form $31k + 1$, therefore $L/K \ncong L_2(125)$ and $L/K \cong G_2(5)$. In this case, we put $\overline{G} = G/K$ and $\overline{L} = L/K$, then

$$\overline{L} \cong \overline{L}(C_{\overline{G}}(\overline{\overline{L}}))/(C_{\overline{G}}(\overline{\overline{L}})) \leq (\overline{G}/(C_{\overline{G}}(\overline{\overline{L}})) = (N_{\overline{G}}(\overline{\overline{L}}))/(C_{\overline{G}}(\overline{\overline{L}})) \leq Aut(\overline{L}).$$

http://dx.doi.org/10.22108/ijgt.2021.120906.1594

Set $M = \{xk | xk \in C_{\overline{G}}(\overline{\overline{L}})\}$ then $G/M \cong \overline{G}/C_{\overline{G}}(\overline{\overline{L}})$ and $G_2(5) \leq G/M \leq Aut(G_2(5)) = G_2(5)$. Therefore $G/M \cong G_2(5)$ and $|M| = 1$. Hence $K = 1$ and it is a contradiction. So $G \cong G_2(5)$. □

Acknowledgments

The first author gratefully acknowledges the financial support provided by Mahshahr Branch, Islamic Azad University, Iran to perform research project entitled: "Characterization of the Chevalley group $G_2(5)$ by the set of numbers of the same order elements".

References

1. J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker and R. A. Wilson, $\mathbb{ATLAS}$ *of finite groups. Maximal subgroups and ordinary characters for simple groups*, With computational assistance from J. G. Thackray, Oxford University Press, Eynsham, 1985.
2. G. Frobenius, *Verallgemeinerung des sylowschen satze*, Berliner Sitz, (1895) 981–993.
3. M. Jr. Hall, *The theory of groups*, The Macmillan Co., New York, N. Y., 1959.
4. M. Herzog, On finite simple groups of order divisible by three primes only, *J. Algebra*, **10** (1968) 383–388.
5. A. Jafarzadeh and A. Iranmanesh, *On simple K_n-groups for $n = 5, 6$*, In Groups St. Andrews 2005 in Oxford, **2**, Cambridge Univ. Press, Cambridge, (2007) 517–526.
6. M. Jahandideh and M. R. Darafsheh, NSE characterization of the Chevalley group $G_2(4)$, *Arab J. Math.*, **7** (2018) 21–26.
7. A. R. Khalili Asboei, S. S. S. Amiri, A. Iranmanesh and A. Tehranian, A characterization of sporadic simple groups by NSE and order, *J. Algebra Appl.*, **12** (2013) 125–158.
8. M. Khatami, B. Khosravi and Z. Akhlaghi, A new characterization for some linear groups, *Monatsh. Math.*, **163** (2011) 39–50.
9. Sh. Liu, A characterization of $L_3(4)$, *Sci. Asia*, **39** (2013) 436–439.
10. Sh. Liu, NSE characterization of projective special group $L_5(2)$, *Rend. Semin. Mat. Univ. Padova*, **132** (2014) 123–132.
11. Sh. Liu,A characterization of projective special unitary group $U_3(5)$ by nse, *Arab J. Math. Sci.*, **20** (2014) 133–140.
12. G. A. Miller, Addition to a theorem due to Frobenius, *Bull. Amer. Math. Soc.*, **11** (1904) 6–7.
13. L. Mousavi and B. Taeri, A characterization of $L_2(81)$ by *nse*, *Int. J. Group Theory.*, **1** (2016) 2935.
14. C. Shao and Q. Jiang, A new characterization of some linear groups by nse, *J. Algebra Appl.*, **13** (2014) pp. 9.
15. C. Shao, W. Shi and W. Q. Jiang, Characterization of simple K_4-groups, *Front. Math. China*, **3** (2008) 355–370.
16. R. Shen, C. Shao, Q. Jiang, W. Shi and V. Mazurov, A new characterization of A_5, *Monatsh. Math.*, **160** (2010) 337–341.
17. W. J. Shi, *A new characterization of the sporadic simple groups*, Group theory (Singapore, 1987), 531–540, de Gruyter, Berlin, 1989.

http://dx.doi.org/10.22108/ijgt.2021.120906.1594

18. W. J. Shi, On simple K_4-group, *Chin. Sci. Bul.*, **36** (1991) 1281–1283.

Maryam Jahandideh

Department of Mathematics, Mahshahr Branch, Islamic Azad University, Mahshahr, Iran

Email: maryamjahdideh2003@yahoo.com

Mohammad Reza Darafsheh

School of Mathematics, Statistics and Computer Science, College of Science, University of Tehran, Tehran, Iran

Email: mrdaraf@yahoo.com

http://dx.doi.org/10.22108/ijgt.2021.120906.1594