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VARIATIONS ON GLAUBERMAN'S ZJ THEOREM

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Abstract. We give a new proof of Glauberman's ZJ Theorem, in a form that clarifies the choices involved and offers more choices than classical treatments. In particular, we introduce two new ZJ-type subgroups of a p-group S, that contain $ZJ_r(S)$ and $ZJ_o(S)$ respectively and can be strictly larger.

Glauberman's ZJ Theorem is a basic technical tool in finite group theory [2, Theorem A]. For example, it plays a major role in the classification of simple groups having abelian or dihedral Sylow 2-subgroups. There are several versions of the theorem, depending on how one defines the Thompson subgroup. We develop the theorem in a way that clarifies the choices involved, and offers more choices than classical treatments. In this paper all groups are taken to be finite.

Writing S for a p-group, the following are new. First, Theorem 1.1 is an "axiomatic" version of the ZJ Theorem. Second, we construct ZJ-type groups $ZJ_{\text{lex}}(S)$ and $ZJ_{\text{olex}}(S)$, which contain $ZJ_r(S)$ and $ZJ_o(S)$ respectively, and can be strictly larger. Third, we establish the "normalizers grow" property of the Thompson-Glauberman replacement process, and a consequence involving the Glauberman-Solomon group $D^*(S)$; see Theorems 3.1(5) and 5.4.

1. Introduction

Suppose p is a prime, S is a p-group, $\mathfrak{Ab}(S)$ is the set of abelian subgroups of S, and $\mathcal{A} \subseteq \mathfrak{Ab}(S)$. We set

$$J_{\mathcal{A}} := \langle A : A \in \mathcal{A} \rangle \qquad I_{\mathcal{A}} := \cap_{A \in \mathcal{A}} A \qquad (I_{\mathcal{A}} = 1 \text{ if } \mathcal{A} = \emptyset).$$

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$J_{\mathcal{A}}$ is a sort of generalized Thompson subgroup, and $I_{\mathcal{A}}$ lies in its center. For $P \leq S$ we define $\mathcal{A}|_P$ as $\{A \in \mathcal{A} : A \leq P\}$ and $I_{\mathcal{A}|P}$ as $I_{\mathcal{A}|_P}$. For other notation, and the definition of a p-stable action, see Section 2.

Theorem 1.1 ("Axiomatic" ZJ Theorem)**.** *Suppose p is a prime, S is a p-group and G is a group satisfying*

(a) *S is a Sylow p-subgroup of G.*
(b) *$C_G(O_p(G)) \leq O_p(G)$.*
(c) *G acts p-stably on every normal p-subgroup of G.*

Then $I_{\mathcal{A}} \trianglelefteq G$ if $\mathcal{A} \subseteq \mathfrak{Ab}(S)$ has the following properties:

invariance (in S, for G): *$\forall P \trianglelefteq S$, $I_{\mathcal{A}|P}$ is $N_G(P)$-invariant.*

replacement (in S): *For every $B \trianglelefteq S$ with class ≤ 2, if there exist members of $\mathcal{A}$ that contain $[B, B]$ but not B, then B normalizes one of them.*

Furthermore, $I_{\mathcal{A}}$ is characteristic in G if it is characteristic in S.

Part of the point of ZJ-type theorems is to specify a subgroup of S which will be characteristic in suitable G, without referring to G. We will say that a subgroup of S has the *Glauberman property* (for S) if it is characteristic in any group G satisfying (a)–(c). Replacing $N_G(P)$ with $\mathrm{Aut}(P)$ in the definition of invariance, and quoting the theorem, lets us omit mention of G:

Corollary 1.2. *Suppose p is a prime, S is a p-group, and $\mathcal{A} \subseteq \mathfrak{Ab}(S)$ satisfies* **replacement** *(in S) and also*

full invariance (in S): *$\forall P \trianglelefteq S$, $I_{\mathcal{A}|P}$ is $Aut(P)$-invariant.*

Then $I_{\mathcal{A}}$ has the Glauberman property. □

These results allow $p = 2$, but in this case no $\mathcal{A}$ satisfying the conditions is known. Also, every $\mathcal{A}$ we consider has the following property, much stronger even than full invariance:

completeness (in S): $\mathcal{A}$ contains every subgroup of S that is isomorphic to a member of $\mathcal{A}$.

Examples 1.3. The following subsets of $\mathfrak{Ab}(S)$ are obviously complete in S. The first three are classical and the rest are new. We will abbreviate $I_{\mathcal{A}_{\dots}(S)}$ and $J_{\mathcal{A}_{\dots}(S)}$ to $I_{\dots}(S)$ and $J_{\dots}(S)$.

$$\mathcal{A}_o(S) = \{A \in \mathfrak{Ab}(S) : |A| \geq |A'| \text{ for all } A' \in \mathfrak{Ab}(S)\}$$
$$\mathcal{A}_r(S) = \{A \in \mathfrak{Ab}(S) : \mathrm{rank}(A) = \mathrm{rank}(S)\}$$
$$\mathcal{A}_e(S) = \{A \in \mathcal{A}_r(S) : A \text{ is elementary abelian}\}$$
$$\mathcal{A}_{\mathrm{lex}}(S) = \{A \in \mathfrak{Ab}(S) : A \geq_{\mathrm{lex}} A' \text{ for all } A' \in \mathfrak{Ab}(S)\}$$
$$\mathcal{A}_{\mathrm{olex}}(S) = \{A \in \mathcal{A}_o(S) : A \geq_{\mathrm{lex}} A' \text{ for all } A' \in \mathcal{A}_o(S)\}$$
$$\mathcal{A}_{O,E,\zeta}(S) = \{A \in \mathfrak{Ab}(S) : |A| = p^O, \text{ exponent}(A) \leq p^E \text{ and } A \geq_{\mathrm{lex}} \zeta\}$$

In the last case, $O, E \in \mathbb{Z}$ and $\zeta = (\zeta_1, \zeta_2, \dots)$ is a sequence of integers. For sequences $\zeta = (\zeta_1, \zeta_2, \dots)$ and $\zeta' = (\zeta'_1, \zeta'_2, \dots)$, $\zeta \geq_{\text{lex}} \zeta'$ refers to the usual lexicographic order. When an abelian p-group A appears on one side of $\geq_{\text{lex}}$, the comparison refers to the sequence

$$(\omega_1(A), \omega_2(A), \dots) := \big(|\Omega_1(A)|, |\Omega_2(A)|, \dots\big).$$

If A, A' are abelian groups of the same order, then we think of $A >_{\text{lex}} A'$ as "A is closer to being elementary abelian than A' is."

Theorem 1.4. *Suppose p is an odd prime, S is a p-group, and $D \trianglelefteq S$. Then $\mathcal{A}_o(D)$, $\mathcal{A}_r(D)$, $\mathcal{A}_e(D)$, $\mathcal{A}_{\text{lex}}(D)$, $\mathcal{A}_{\text{olex}}(D)$ and $\mathcal{A}_{O,E,\zeta}(D)$ (for any fixed O, E, ζ) have the replacement property in S.*

Corollary 1.5. *Every one of $I_o(S)$, $I_r(S)$, $I_e(S)$, $I_{\text{lex}}(S)$, $I_{\text{olex}}(S)$ and $I_{O,E,\zeta}(S)$ has the Glauberman property for S.* □

Theorem 1.4 is a wrapper around Glauberman's replacement theorem, extended to cover the last three cases (Theorem 3.1). Corollary 1.5 contains the classical forms of the ZJ Theorem. Namely: $ZJ_o(S)$ and $\Omega ZJ_e(S)$ have the Glauberman property. This follows from

$$I_o(S) = ZJ_o(S) \quad \text{and} \quad I_e(S) = \Omega ZJ_e(S).$$

The first equality uses $I_{\mathcal{A}} = Z(J_{\mathcal{A}}) = C_S(J_{\mathcal{A}})$ when every member of $\mathcal{A}$ is maximal in $\mathfrak{Ab}(S)$ under inclusion (Lemma 2.1). The second is similar. From our perspective, the Glauberman property for $ZJ_o(S)$ and $\Omega ZJ_e(S)$ derives from their coincidence with $I_o(S)$ and $I_e(S)$, and has nothing to do with $J_o(S)$ and $J_e(S)$. $\mathcal{A}_r(S)$ gives nothing new: Theorem 5.3 shows

$$I_r(S) = I_e(S) = \Omega ZJ_r(S) = \Omega ZJ_e(S).$$

We chose the new families $\mathcal{A}_{\text{lex}}(S)$ and $\mathcal{A}_{\text{olex}}(S)$ to be "small", so that $J_{\dots}(S)$ would also be "small" and $I_{\dots}(S)$ would be "large". In particular,

$$I_{\text{lex}}(S) = ZJ_{\text{lex}}(S) = C_S(J_{\text{lex}}(S)) \geq ZJ_r(S)$$
$$I_{\text{olex}}(S) = ZJ_{\text{olex}}(S) = C_S(J_{\text{olex}}(S)) \geq ZJ_o(S).$$

The equalities use Lemma 2.1. The containments follow from $\mathcal{A}_{\text{lex}}(S) \subseteq \mathcal{A}_r(S)$ and $\mathcal{A}_{\text{olex}}(S) \subseteq \mathcal{A}_o(S)$, and can easily be strict (Examples 5.1 and 5.2). The containment $ZJ_{\text{lex}}(S) \geq ZJ_r(S)$ is interesting because it is not known whether $ZJ_r(S)$ has the Glauberman property. Any two members of $\mathcal{A}_{\text{lex}}(S)$ resp. $\mathcal{A}_{\text{olex}}(S)$ are isomorphic to each other.

There is no reason to expect $\mathcal{A}_{O,E,\zeta}(S)$ to be interesting; we include it mainly to give a sense of what is possible using replacement.

Corollary 1.5 uses the $D = S$ case of Theorem 1.4. Since one can take D to be any normal subgroup there, this suggests trying to apply Theorem 1.1 to some suitable $\mathcal{A} \subseteq \mathfrak{Ab}(D)$ with $D \trianglelefteq S$. In this way we can recover some recent results of Kızmaz. Recall that $D \trianglelefteq S$ is called *strongly closed* (in S, with respect to $G \geq S$), if the only elements of S which are G-conjugate into D are the elements of D.

If this holds and $\mathcal{A} \subseteq \mathfrak{Ab}(D)$ is complete (in D), then it is not hard to see that $\mathcal{A}$ satisfies invariance (in S, for G). In fact strong closure is stronger than necessary for this argument.

Therefore Theorem 1.1 implies the following "axiomatic" version of [6, Theorem B]. Corollary 1.7 below takes $D = \Omega_i(S)$, and is our analogue of [6, Remark 1.4].

Theorem 1.6. *Suppose p is a prime, S is a p-group, G is a group satisfying (a)–(c) of Theorem 1.1, and $D \trianglelefteq S$ is strongly closed in S with respect to G. Then $I_{\mathcal{A}} \trianglelefteq G$ for any $\mathcal{A} \subseteq \mathfrak{Ab}(D)$ which is complete (in D) and satisfies replacement (in S).* □

Corollary 1.7. *Suppose p is an odd prime, S is a p-group, $i \geq 1$, and $\Omega_i(S)$ has exponent $\leq p^i$ (for example, suppose S has class $< p$). Then all of $ZJ_o\Omega_i(S)$, $\Omega ZJ_e\Omega_i(S)$, $ZJ_{\mathrm{lex}}\Omega_i(S)$, $ZJ_{\mathrm{olex}}\Omega_i(S)$ and $I_{O,E,\zeta}(\Omega_i(S))$ (for any O, E, ζ) have the Glauberman property for S.* □

2. Background and Notation

We mostly follow the conventions of [4]. Let G be a group. If $w, x \in G$, then w^x means $x^{-1}wx$ and $[w, x]$ means $w^{-1}x^{-1}wx$. Brackets nest to the left, so $[x_1, \ldots, x_n]$ means $[[x_1, \ldots, x_{n-1}], x_n]$ when $n > 2$.

Suppose p is a prime. If S is a p-group, then $\Omega_i(S)$ means the subgroup generated by all elements of order $\leq p^i$. When $i = 1$ we often write just $\Omega(S)$. The rank of an abelian group means the size of the smallest set of generators. The rank of a nonabelian group means the maximum of the ranks of its abelian subgroups. We will only use this notion for p-groups. We sometimes suppress parentheses, eg writing $\Omega ZJ_e(S)$ for $\Omega(Z(J_e(S)))$.

The largest normal p-subgroup of G is denoted $O_p(G)$. Now suppose G acts on a p-group P. We define $O_p(G \curvearrowright P) \trianglelefteq G$ as the preimage of $O_p(G/C_G(P))$ under the natural map $G \to G/C_G(P)$. This notation is nonstandard but natural; it can be pronounced "O_p of G's action on P". We say that $x \in G$ acts quadratically if $[P, x, x] = 1$. The action of G on P is called p-stable if every element of G that acts quadratically lies in $O_p(G \curvearrowright P)$. There is a simple "global" condition that guarantees this: that no subquotient of G is isomorphic to $\mathrm{SL}_2(p)$. A proof of this can be extracted from that of [2, Lemma 6.3]. One main case of interest is when p is odd and G has abelian or dihedral Sylow 2-subgroups. Having quaternionic Sylow 2-subgroups, $\mathrm{SL}_2(p)$ cannot arise as a subquotient.

We use the following elementary lemma several times.

Lemma 2.1. *Suppose S is a p-group, $\mathcal{A} \subseteq \mathfrak{Ab}(S)$, and every member of $\mathcal{A}$ is maximal in $\mathfrak{Ab}(S)$ under inclusion. Then $I_{\mathcal{A}} = Z(J_{\mathcal{A}}) = C_S(J_{\mathcal{A}})$.*

Proof. The inclusions $I_{\mathcal{A}} \leq Z(J_{\mathcal{A}}) \leq C_S(J_{\mathcal{A}})$ are obvious. Now suppose $x \in C_S(J_{\mathcal{A}})$. For any $A \in \mathcal{A}$, $\langle A, x \rangle$ is abelian, so the maximality of A forces $x \in A$. Letting A vary over $\mathcal{A}$ gives $x \in I_{\mathcal{A}}$. □

3. Replacement

Theorem 3.1 (Replacement)**.** *Suppose p is a prime, S is a p-group and $B \trianglelefteq S$. If $p = 2$ then assume B is abelian. Suppose $A \leq S$ is abelian and contains $[B, B]$.*

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Then either B normalizes A, or there exists $b \in N_B(N_S(A)) - N_B(A)$. For any such b, $A^* := (A \cap A^b)[A, b] \le AA^b$ enjoys the properties

(1) $|A^*| = |A|$.
(2) *Like A, A^* is abelian and contains $[B, B]$.*
(3) *$A^* \cap B$ strictly contains $A \cap B$ and is a proper subgroup of B.*
(4) *A^* and A normalize each other.*
(5) *$N_S(A^*)$ contains b and strictly contains $N_S(A)$.*
(6) *If $p > 2$, then $\omega_i(A^*) \ge \omega_i(A)$ for all $i \ge 1$. In particular,*

$$\text{exponent}(A^*) \le \text{exponent}(A) \qquad \text{rank}(A^*) \ge \text{rank}(A) \qquad A^* \ge_{\text{lex}} A.$$

Glauberman's replacement theorem [2, Theorem 4.1] adds the hypothesis that class$(B) \le 2$, and establishes (1)–(3). This is enough to prove that $ZJ_o(S)$ has the Glauberman property. Isaacs simplified the proof by replacing some of the counting arguments with structural ones [5]. He took B abelian, as in Thompson's replacement theorem, but with some work his arguments can be adapted. Course notes of Gagola [1] include a proof along these lines, citing long-ago unpublished work by (separately) Isaacs, Passman and Goldschmidt. This includes the exponent inequality in (6), and removed Glauberman's hypothesis on class(B). Kızmaz [6] independently adapted Isaacs' arguments from [5] and proved his own generalization of Glauberman's replacement theorem, namely [6, Theorem A]. This includes (6), although he only stated the $i = 1$ case and the rank inequality. He also clarified the overall argument by isolating the commutator calculations in [6, Lemma 2.1], from which our Lemma 3.2 grew.

To my knowledge, (5) is new. It is curious because it says that $N_S(A)$ is a measure of how well-positioned A is with respect to B, yet $N_S(A)$ is independent of B. An interesting consequence is that if $A \in \mathfrak{Ab}(G)$ has largest possible normalizer, among all abelian subgroups of S with order $|A|$, then A automatically centralizes the ZJ-type group $D^*(S)$ introduced by Glauberman and Solomon [3]. We postpone the details until Theorem 5.4, to avoid breaking the flow of ideas.

Lemma 3.2. *Suppose a group A acts on a group B and centralizes $[B, B]$. Then the commutator subgroup of $[B, A]$ is central in B.*

Furthermore, if A is abelian and $b \in B$ satisfies $[b, A, A, A] = 1$, then the commutator subgroup of $[b, A]$ is an elementary abelian 2-group.

Proof. (We do not use our blanket hypothesis that groups are finite, so A and B could be infinite.) Because A centralizes $[B, B]$, so does $[B, A]$. Two special cases of this are $[B, [B, A], [B, A]] = 1 = [[B, A], B, [B, A]]$. Now the three subgroups lemma gives $[[B, A], [B, A], B] = 1$, which is the first part of the lemma.

The commutator subgroup of $[b, A]$ is abelian because it is central. It is generated by the $[[b, x], [b, y]]$ with x, y varying over A. So it suffices to show that each has order ≤ 2. We fix x, y and abbreviate:

$$b_x = [b, x] \quad b_y = [b, y]$$
$$b_{xx} = [b, x, x] \quad b_{xy} = [b, x, y] \quad b_{yx} = [b, y, x] \quad b_{yy} = [b, y, y].$$

By hypothesis, x and y centralize the last four of these.

We will use the following identities, for any u, v, w in any group:

$$u^v = u[u, v] \qquad [u, vw] = [u, w][u, v]^w \qquad [uv, w] = [u, w]^v [v, w].$$

In particular, $b^y = bb_y$ and $b_x^y = b_x b_{xy}$. Since A centralizes $[B, B]$,

$$[b_{xy}, b] = [b_{xy}^y, b^y] = [b_{xy}, bb_y] = [b_{xy}, b_y][b_{xy}, b]^{b_y \leftarrow \text{discard}}.$$

We may discard the indicated conjugation because $[B, A]$ centralizes $[B, B]$. Canceling the $[b_{xy}, b]$ terms leaves $1 = [b_{xy}, b_y]$. Similarly,

$$\begin{aligned}[b_x, b] = [b_x^y, b^y] &= [b_x b_{xy}, bb_y] = [b_x b_{xy}, b_y][b_x b_{xy}, b]^{b_y \leftarrow \text{discard}} \\ &= [b_x, b_y]^{b_{xy} \leftarrow \text{discard}} [b_{xy}, b_y] \cdot [b_x, b]^{b_{xy} \leftarrow \text{discard}} [b_{xy}, b].\end{aligned}$$

We discard conjugations as before, and we just saw that the second commutator is trivial. The first commutator is central, so we may cancel the $[b_x, b]$ terms. This leaves ($\star$) $1 = [b_x, b_y][b_{xy}, b]$.

Next, we have $[b, xy] = [b, y][b, x]^y = b_y b_x^y = b_y b_x b_{xy}$. Exchanging x and y doesn't change the left side, so $b_y b_x b_{xy} = b_x b_y b_{yx}$. Moving two terms to the right yields $b_{xy} = [b_x, b_y] b_{yx}$. Bracketing by b, and using the centrality of $[b_x, b_y]$, gives $[b_{xy}, b] = [b_{yx}, b]$. By ($\star$) and its analogue with x and y swapped, this implies $[b_x, b_y] = [b_y, b_x]$. That is, $[b_x, b_y]^2 = 1$. □

Proof of Theorem 3.1. Suppose B does not normalize A. Since $N_B(A)$ is proper in B, it is proper in its own normalizer $N_B(N_B(A))$. Because $N_S(A)$ normalizes A and B, it also normalizes $N_B(A)$ and $N_B(N_B(A))$, hence acts on $N_B(N_B(A))/N_B(A) \neq 1$. So some $b \in N_B(N_B(A)) - N_B(A)$ is $N_S(A)$-invariant modulo $N_B(A)$, ie $[b, N_S(A)] \leq N_B(A)$. This inclusion also says that b normalizes $N_S(A)$. So $b \in N_B(N_S(A)) - N_B(A)$, as claimed.

Now set $N := N_S(A)$ and suppose $b \in N_B(N) - N_B(A)$ is arbitrary. From $[B, B] \leq A$ we have $A \cap B \trianglelefteq B$, hence $A \cap B \leq A \cap A^b$.

From $A \trianglelefteq N \trianglelefteq \langle N, b\rangle$ follows $A^b \trianglelefteq N$. So A, A^b normalize each other. Setting $H = AA^b \trianglelefteq N$, it follows that $[H, H] \leq A \cap A^b \leq Z(H)$. In particular, H has class ≤ 2. The identity $(aa'^{-1})(a')^b = a[a', b]$, for any $a, a' \in A$, shows that H is also equal to $A[A, b]$.

Using bars for images in $H/(A \cap A^b)$, obviously we have $\bar{A} \cdot \overline{[A, b]} = \bar{H}$. On the other hand, $[A, b]$ lies in $H \cap B$, and $\overline{H \cap B}$ meets $\bar{A}$ trivially. (Any element of $H \cap B$, that differs from an element of A by an element of $A \cap A^b$, lies in A, hence in $B \cap A \leq A \cap A^b$.) So $\overline{[A, b]}$ meets every coset of $\bar{A}$ in $\bar{H}$, yet lies in $\overline{H \cap B}$, which contains at most one point of each coset. Therefore $\overline{[A, b]}$ and $\overline{H \cap B}$ coincide and form a complement to $\bar{A}$ in $\bar{H}$. So

$$A^* = (A \cap A^b)[A, b] = (A \cap A^b)(H \cap B)$$

is a complement to A in H, modulo $A \cap A^b$. Since A^b is another such complement, we have $A^*/(A \cap A^b) \cong A^b/(A \cap A^b)$ and therefore $|A^*| = |A|$, proving (1).

(2) First, $[B, B] \leq A \cap B \leq A \cap A^b \leq A^*$. Now we prove A^* abelian. Because $A \cap A^b$ is central in H it is enough to prove $[A, b]$ abelian. If $p = 2$ this follows from the hypothesis that B is abelian. So take

p odd. We may apply Lemma 3.2 because $[B, B] \leq A$ and $[b, A, A, A] \leq [H, A, A] \leq [Z(H), A] = 1$. The lemma shows that the commutator subgroup of $[A, b]$ is a 2-group, hence trivial.

(3) We already saw $A \cap B \leq A \cap A^b \leq A^*$. The strict containment $A \cap B < A^* \cap B$ comes from the fact that b does not normalize A. Namely, A omits $b^{-1}ab$ for some $a \in A$, so it also omits $a^{-1}b^{-1}ab = [a, b] \in A^* \cap B$. And $A^* \cap B$ is strictly smaller than B because it lies in N and therefore omits b.

(4) Both A, A^* contain $A \cap A^b$, hence $[H, H]$, so are normal in H.

(5) N normalizes $A^* = (A \cap A^b)(H \cap B)$ because it normalizes all four terms on the right. And b normalizes $A^* = (A \cap A^b)[A, b]$ because

$$[A \cap A^b, b] \leq [A, b] \quad \text{and} \quad [[A, b], b] \leq [B, B] \leq A \cap A^b.$$

Because $b \notin N$ it follows that $N_S(A^*)$ is strictly larger than N.

(6) We fix i and write A_i for $\Omega_i(A)$. H has class ≤ 2, so the identities

$$(xy)^e = x^e y^e [x, y]^{e(e-1)/2} \quad \text{and} \quad [x, y]^e = [x, y^e]$$

hold for all $x, y \in H$. Together with the oddness of p, they show that $\Omega_i(H)$ has exponent $\leq p^i$. Therefore its subgroup

$$A_i^* := (A_i \cap A_i^b)[A_i, b] \leq A_i A_i^b \leq \Omega_i(H)$$

does too. Now we reason as follows:

$$\omega_i(A^*) \geq \omega_i(A_i^*) = |A_i^*| \geq |A_i| = \omega_i(A).$$

The first step uses the obvious containment $A^* \geq A_i^*$, and the equalities use that A_i^* and A_i have exponent $\leq p^i$. For the remaining inequality, observe that quotienting $A_i A_i^* = A_i A_i^b$ by A_i gives $A_i^*/(A_i^* \cap A_i) \cong A_i^b/(A_i^b \cap A_i)$. Because A_i^* contains $A_i^b \cap A_i$ this implies $|A_i^*| \geq |A_i|$.

The rank and lex inequalities follow. By $|A^*| = |A|$, the exponent inequality does too. □

Proof of Theorem 1.4. Write $\mathcal{A}$ for any one of $\mathcal{A}_o(D), \ldots, \mathcal{A}_{O,E,\zeta}(D)$. Supposing $B \trianglelefteq S$, and that some $U \in \mathcal{A}$ contains $[B, B]$ but not B, we will show that B normalizes some $A \in \mathcal{A}$ that also contains $[B, B]$ but not B. Among all members of $\mathcal{A}$ that contain $[B, B]$ but not B, and lie in $\langle U^S \rangle$, choose A with $|A \cap B|$ maximal. Supposing that B does not normalize A, we will derive a contradiction.

Let b and A^* be as in Theorem 3.1. In particular, A^* is abelian and lies in $AA^b \leq \langle U^S \rangle \leq D$. So $A^* \in \mathfrak{Ab}(D)$. A^* contains $[B, B]$ but not B by (2) and (3). (3) also implies $|A^* \cap B| > |A \cap B|$, so the maximality in our choice of A forces $A^* \notin \mathcal{A}$.

This is a contradiction because $A^* \in \mathcal{A}$ by other parts of Theorem 3.1. For $\mathcal{A}_o$ we use $|A^*| = |A|$. For $\mathcal{A}_r$ we use $\operatorname{rank}(A^*) \geq \operatorname{rank}(A)$. For $\mathcal{A}_e$ we use both of these properties. For $\mathcal{A}_{\text{lex}}$ we use $A^* \geq_{\text{lex}} A$. For $\mathcal{A}_{\text{olex}}$ we use this and $|A^*| = |A|$. For $\mathcal{A}_{O,E,\zeta}$ we use $|A^*| = |A|$, $\operatorname{exponent}(A^*) \leq \operatorname{exponent}(A)$ and $A^* \geq_{\text{lex}} A$. □

In fact we have proven that $\mathcal{A}$ satisfies a strengthening of the replacement axiom, got by removing "with class ≤ 2" from the statement of the axiom. We stated the axiom the way we did because only the class ≤ 2 case is needed to prove Theorem 1.1.

4. Proof of the Axiomatic ZJ Theorem

Proof of Theorem 1.1. We write I for $I_{\mathcal{A}}$. We prove $I \trianglelefteq G$ by induction starting with $1 \trianglelefteq G$, by establishing the following inductive step:

$$\text{if } \exists W \trianglelefteq G \text{ with } W < I, \text{ then } \exists B \trianglelefteq G \text{ with } W < B \leq I.$$

Fix such a W. Since S preserves I, it acts on I/W. We define $X \leq I$ as the preimage of the fixed-point subgroup. So X is normal in S, and is strictly larger than W because $I/W \neq 1$. To complete the proof we will show that $B := \langle X^G \rangle \trianglelefteq G$ lies in I. Supposing to the contrary, we will derive a contradiction.

Step 1: $B \leq O_p(G)$. Being a subgroup of I, X is abelian. Together with $X \trianglelefteq S$ this gives $[O_p(G), X, X] \leq [X, X] = 1$. Because G acts p-stably on $O_p(G)$, X lies in $O_p(G \curvearrowright O_p(G))$. This equals $O_p(G)$ because $C_G(O_p(G))$ is a p-group by hypothesis. Since X lies in $O_p(G)$, so does $B = \langle X^G \rangle$.

Step 2: $[B, B] \leq W \leq Z(B)$. By the definition of X, S acts trivially on X/W. In particular $O_p(G)$ does. Conjugation shows that $O_p(G)$ acts trivially (mod W) on every G-conjugate of X, hence trivially on B/W. That is, $[B, O_p(G)] \leq W$. By step 1 this implies $[B, B] \leq W$. And $W \leq Z(B)$ because B is generated by abelian groups that contain W.

Step 3: Set $H = O_p(G \curvearrowright B)$ and $P = H \cap S$. Then some $A \in \mathcal{A}|_P$ fails to contain B. Because we are supposing $B \not\leq I$, some $A \in \mathcal{A}$ fails to contain B. It does contain $[B, B]$, because step 2 showed $[B, B]$ lies in W, which lies in I, hence A. Step 2 also showed that $B \trianglelefteq S$ has class ≤ 2. By the replacement property, some member of $\mathcal{A}$ contains $[B, B]$ but not B, and is also normalized by B. We lose nothing by using it in place of A, because it has all the properties of A established so far. That is, we may suppose B normalizes A. By $[B, A, A] \leq [A, A] = 1$ and the p-stability of G's action on B, we have $A \leq H$. Together with $A \leq S$ this gives $A \leq P$, hence $A \in \mathcal{A}|_P$.

Step 4: $B \leq I_{\mathcal{A}|P}$. By the Frattini argument and the definition of H,

$$G = HN_G(P) = C_G(B)PN_G(P) \leq C_G(X)N_G(P).$$

So the G-conjugates of X are the same as the $N_G(P)$-conjugates. It therefore suffices to show that every $N_G(P)$-conjugate of X lies in $I_{\mathcal{A}|P}$. This follows from

$$X \leq I = \bigcap_{A' \in \mathcal{A}} A' \underset{\substack{\uparrow \\ \text{by } \mathcal{A} \supseteq \mathcal{A}|_P \neq \emptyset}}{\leq} \bigcap_{A' \in \mathcal{A}|_P} A' = I_{\mathcal{A}|P} \underset{\substack{\uparrow \\ \text{by invariance}}}{\trianglelefteq} N_G(P).$$

The contradiction. By $B \leq I_{\mathcal{A}|P}$, every member of $\mathcal{A}|_P$ contains B. But in step 3 we found one which does not.

The final claim follows from the Frattini argument: $\operatorname{Aut} G$ is generated by inner automorphisms and automorphisms that preserve S. □

Our method of "growing" the normal subgroup from W to B derives from Stellmacher's construction [7, Theorem 9.4.4][8] of a different subgroup of S that also has the Glauberman property.

5. Etc

Here we collect some results and examples we mentioned in passing. First, we claimed in the introduction that our ZJ-type groups ZJ_{lex} and ZJ_{olex} can be strictly larger than ZJ_r and ZJ_o, respectively.

Example 5.1 (ZJ_{lex} can be larger than ZJ_r). Let p be any odd prime. The group

$$S = \langle x, y, u \mid 1 = [x, y] = x^{p^2} = y^p = u^p, x^u = xy, y^u = yx^p \rangle$$

is a semidirect product $(\mathbb{Z}/p^2 \times \mathbb{Z}/p) \rtimes \mathbb{Z}/p$. (If $p = 2$ then S collapses to D_8.) For any a in $A := \langle x, y\rangle$ but outside $\langle x^p\rangle$, $C_S(a) = A$. It follows that $Z(S)$ can be no larger than $\langle x^p\rangle$. Therefore A is the unique abelian subgroup of S with order p^3, because its intersection with any other such subgroup would be central in S and have order p^2. In particular, $\operatorname{rank} S = 2$, $\mathcal{A}_{\text{lex}}(S) = \{A\}$ and $ZJ_{\text{lex}}(S) = J_{\text{lex}}(S) = A$. But $\mathcal{A}_r(S)$ also contains $\langle x^p, u\rangle \cong (\mathbb{Z}/p)^2$, so $J_r(S) = S$ and $ZJ_r(S) = \langle x^p\rangle$.

Example 5.2 (ZJ_{olex} can be larger than ZJ_o). Let p be any prime, and consider the "Heisenberg group"

$$S = \langle a, b, c \mid c = [a, b],\ 1 = [c, a] = [c, b] = a^{p^2} = b^{p^2} = c^{p^2} \rangle.$$

$\mathcal{A}_o(S)$ consists of the preimages of the p^2+p+1 order p^2 subgroups of $S/\langle c\rangle \cong (\mathbb{Z}/p^2)^2$. So $J_o(S) = S$ and $ZJ_o(S) = \langle c\rangle$. One member of $\mathcal{A}_o(S)$ is isomorphic to $(\mathbb{Z}/p)^2 \times \mathbb{Z}/p^2$, namely $A := \langle a^p, b^p, c\rangle$. The rest are isomorphic to $(\mathbb{Z}/p^2)^2$, except if $p = 2$, when there are also some isomorphic to $\mathbb{Z}/2 \times \mathbb{Z}/8$. So A is the unique lex-maximal element of $\mathcal{A}_o(S)$, and $\mathcal{A}_{\text{olex}}(S) = \{A\}$ and $J_{\text{olex}}(S) = ZJ_{\text{olex}}(S) = A$.

In the introduction we mentioned $\Omega ZJ_r = \Omega ZJ_e$. This is part of:

Theorem 5.3. *Suppose p is a prime and S is p-group. Then*

$$I_e(S) = I_r(S) = \Omega ZJ_r(S) = \Omega ZJ_e(S) = \Omega C_S(J_r(S)) = \Omega C_S(J_e(S)).$$

Proof. First, $I_r(S) \leq I_e(S)$ because $\mathcal{A}_e(S) \subseteq \mathcal{A}_r(S)$. And $I_e(S) \leq I_r(S)$ because every member of $\mathcal{A}_r(S)$ contains a member of $\mathcal{A}_e(S)$, hence $I_e(S)$. We have proven the first equality. For the others it is enough to establish the inclusions:

$$\begin{array}{ccccccc}
I_e(S) & = & I_r(S) & \hookrightarrow & \Omega ZJ_r(S) & \xhookrightarrow{\text{obvious}} & \Omega C_S(J_r(S)) \\
\downarrow{\scriptstyle\text{obvious}} & & & & & & \downarrow{\scriptstyle\text{by } J_e(S)\leq J_r(S)} \\
\Omega ZJ_e(S) & & & \xhookrightarrow{\text{obvious}} & & & \Omega C_S(J_e(S)) \hookrightarrow I_e(S).
\end{array}$$

The unlabeled inclusion in the top row is obvious, except for the fact that $I_r(S)$ has exponent $\leq p$, which holds by $I_r(S) = I_e(S)$. The unlabeled inclusion in the bottom row is standard, with proof similar to that of Lemma 2.1. □

Just before Lemma 3.2, we mentioned that "normalizers grow" during the Thompson-Glauberman replacement process, and that this forces abelian subgroups of S with "large" normalizers to centralize $D^*(S)$. Here $D^*(S)$ is the characteristic subgroup introduced by Glauberman and Solomon [3], who gave a lovely proof that it has the Glauberman property. Following Bender, $D^*(S)$ may be defined as the (unique) largest normal subgroup of S with the property that it centralizes every abelian subgroup of S that it normalizes. (It is easy to see that this exists. And considering how it acts, on abelian normal subgroups of itself, leads to a proof that $D^*(S)$ is abelian.)

Theorem 5.4. *Suppose p is a prime, S is a p-group, and $A \in \mathfrak{Ab}(S)$. Also suppose $N_S(A)$ is maximal under inclusion, among all groups $N_S(A^*)$ where $A^* \in \mathfrak{Ab}(S)$ has the same order as A. Then A centralizes $D^*(S)$.*

Proof. We apply our replacement theorem with B equal to the abelian group $D^*(S) \trianglelefteq S$. Arguing as for Theorem 1.4 shows that $D^*(S)$ normalizes A. So, by (Bender's) definition, $D^*(S)$ centralizes A.

(If $p > 2$ then this argument gives a slightly stronger result, with A^* varying over fewer elements of $\mathfrak{Ab}(S)$, namely those that satisfy $|A^*| = |A|$ and also $\omega_i(A^*) \geq \omega_i(A)$ for all i.) □

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