International Journal of Group Theory
ISSN (print): 2251-7650, ISSN (on-line): 2251-7669
Vol. 11 No. 2 (2022), pp. 109-119.

http://ijgt.ui.ac.ir
www.ui.ac.ir

NEW LOWER BOUNDS FOR THE NUMBER OF CONJUGACY CLASSES IN FINITE NILPOTENT GROUPS

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Abstract. P. Hall's classical equality for the number of conjugacy classes in p-groups yields $k(G) \geq (3/2)\log_2|G|$ when G is nilpotent. Using only Hall's theorem, this is the best one can do when $|G| = 2^n$. Using a result of G.J. Sherman, we improve the constant $3/2$ to $5/3$, which is best possible across all nilpotent groups and to $15/8$ when G is nilpotent and $|G| \neq 8, 16$. These results are then used to prove that $k(G) > \log_3(|G|)$ when G/N is nilpotent, under natural conditions on $N \trianglelefteq G$. Also, when G' is nilpotent of class c, we prove that $k(G) \geq (\log|G|)^t$ when $|G|$ is large enough, depending only on (c, t).

1. Introduction

Let $k(G)$ denote the number of conjugacy classes of the finite group G. One of the fundamental problems in group theory is whether there exists a positive constant c such that $k(G) \geq c\ \log_2|G|$ for all finite groups G. The classification of finite groups according to their number k of conjugacy classes (studied since around 1910) is now complete for $k \leq 14$ and these all satisfy $k(G) > \log_3|G|$ (see [15, 16, 17] and the Table on page 10). 25 of the more than 350 (nonisomorphic) groups G with $k(G) \leq 14$ satisfy $k(G) < \log_2|G|$. And there are five solvable groups G with $k(G) \in \{6, 8, 9\}$ all of which satisfy $k(G) < \log_2|G|$. When G is abelian-by-nilpotent, $k(G) \geq \frac{3}{4}\log_2|G| > \log_3|G|$ (see [8]). If K is normal in G, G/K is nilpotent and N is *minimal* normal in G with $K' \leq N \leq K \trianglelefteq G$ (so $K' = 1$ or N) then $k(G) \geq \log_3|G|$ for $|G|$ large enough [4, Example 3.14(a)]. Is $k(G) \geq \log_3|G|$ whenever G contains a normal subgroup N with G/N nilpotent? One result here is that if $N \trianglelefteq G$, G/N is nilpotent *and* $k(G) \geq \log_2|N|$ then $k(G) \geq \log_3|G|$. When G has trivial solvable radical (e.g.

Communicated by Attila Maróti.

MSC(2010): 20D15; 20E45; 20F14.
Keywords: Nilpotent, conjugacy, derived series.
Received: 29 April 2021, Accepted: 10 July 2021.
Article Type: Research Paper.
http://dx.doi.org/10.22108/IJGT.2021.128396.1687

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when G is simple), Baumeister, Maróti and Tong-Viet [1] proved that $k(G) > \log_3|G|$. Note that $k(G) = \lceil \log_3|G| \rceil$ when $G = \mathrm{PSL}(3,4)$ or M_{22} (both simple groups).

Question: Is $k(G) \geq \log_3|G|$ for every finite group G?

2. G Nilpotent

Every (finite) nilpotent group G is the direct product of p-groups, so $|G| = \prod_{i=1}^{n} p_i^{\alpha_i}$ where each p_i is a prime. Thus the number of conjugacy classes $k(G)$ equals the product of the number of classes of each Sylow p-subgroup P of G.

Lemma 2.1. *P. Hall, see [9, V,15.2]: If $|P| = p^{2n+e}$, p is a prime and $n \geq 0$, with $e \in \{0,1\}$, then there exists an integer $r \geq 0$ such that $k(P) = (n + r(p-1))(p^2-1) + p^e$.*

Hall originally proved this using representation theory and it was later proved by A. Mann [12] using direct methods. In particular $k(P) \geq n(p^2-1) + p^e$. When $|P| = p$ or p^2, then P is abelian, so $k(P) = |P|$. When $|P| \geq p^3$ it follows from Hall's theorem [3, Theorem 4.2] that $k(P) > \left(\frac{3}{4}\right) p \log_2|P|$. Thus the best we can do when $|G| = 2^n$ using Hall's theorem is $k(G) > \left(\frac{3}{2}\right)\log_2|G|$.

Lemma 2.2. [12, G. J. Sherman]*: If G is a nilpotent group of nilpotence class c, then ($c < \log_2|G|$ and) $k(G) \geq c|G|^{1/c} - c + 1$.*

Theorem 2.3. *Every (finite) nilpotent group G satisfies $k(G) \geq \frac{5}{3}\log_2|G|$, and $\frac{5}{3}$ is best possible.*

Proof. First, check that when $|G| \leq 27$, then *except* for $|G| = 8$, 16, 24, or 27, if G is nilpotent then G is abelian. Next, check that when $|G| = 8$, $k(G) \geq 5 = \frac{5}{3}\log_2|G|$; when $|G| = 16$, $k(G) \geq 7 > \frac{5}{3}\log_2|G|$; when $|G| = 24$, $k(G) \geq 15 > \frac{5}{3}\log_2|G|$, and when $|G| = 27$, $k(G) \geq 11 > \frac{5}{3}\log_2|G|$. So we assume that $|G| \geq 28$ and G is not abelian. The proof concludes using Sherman's Lemma. First we need the following:

Lemma 2.4. *If $\alpha \geq (|G|e)^{-1/2}$, then $c|G|^{1/c} - c + 1 > (1-\alpha)c|G|^{1/c}$*

□

Proof. The inequality to be proved is equivalent to $\alpha c|G|^{1/c} > c - 1$, that is $\alpha^c|G| > (1-1/c)^c$. Noting that $(1-1/c)^c \uparrow 1/e$ when $c \to \infty$, the inequality follows from $\alpha^c|G| \geq 1/e$, that is $\alpha \geq (1/e|G|)^{1/c}$, and our assumption that $c \geq 2$. □

Proof. Using Sherman's theorem and Lemma 2.4, $k(G) > (1-\alpha)\cdot c|G|^{1/c}$, as long as $\alpha \geq 1/\sqrt{28e} = 0.1146\cdots$. When g is constant and $2 \leq x \leq \log_2 g$, the function $xg^{\frac{1}{x}}$ attains its minimum value $e\ln g$ at $x = \ln g$. With $\alpha = 0.11543$ we conclude that

$$k(G) > 0.88457 \cdot e\ln|G| = (0.88457 \cdot e \cdot \ln 2)\log_2|G| > \left(\frac{5}{3}\right)\log_2|G|$$

□

Corollary 2.5. *If G is a p-group P, then $k(P) \geq (3/2)\log_2|P| + 1/2$, with equality only when ($p = 2$ and) $|P| = 2$ or 8. Otherwise $k(P) \geq (3/2)\log_2|P| + 1$.*

In general, if G is nilpotent and $|G| \neq 8, 16$ the best lower bound of this type so far is:

Theorem 2.6. *If G is nilpotent and $|G| \neq 8, 16$ then $k(G) \geq (15/8)\log_2|G|$.*

We use Lemma 2.2: If G is nilpotent of nilpotence class c, then ($c < \log_2|G|$ and) $k(G) \geq c|G|^{1/c} - c + 1$. Our proof depends on showing that $c|G|^{1/c} - c + 1 \geq (1-\alpha)c|G|^{1/c} \geq (15/8)\log_2|G|$, when $c \geq 2$ and $\alpha \geq (1/e|G|)^{1/c}$, and using Lemma 2.4. The proof of the Theorem concludes by showing that $(1-\alpha)\cdot c|G|^{1/c} \geq \{(1-\alpha)e\ln 2\}\log_2|G|$ and $(1-\alpha)e\ln 2 \geq 15/8$. As mentioned earlier, $c|G|^{1/c}$ attains its minimum value $e\ln|G|$ when $c = \ln|G|$, so $(1-\alpha)c|G|^{1/c} \geq \{(1-\alpha)e\ln 2\}\log_2|G|$. The latter is $\geq (15/8)\log_2|G|$ when $\alpha \leq 1 - 15/(8e\ln 2)$. So we select $\alpha = 4.8665398 \times 10^{-3} (< 1 - 15/(8e\ln 2))$. According to Lemma 2.4, we also need $(|G|e)^{1/2} \geq 1/\alpha$, that is $|G| \geq 1/e\alpha^2 > 15{,}533$. Thus as long as $|G| \geq 15{,}534$, we have $k(G) \geq (15/8)\log_2|G|$.

Finally, we assume that $|G| \leq 15,533, |G| \neq 8, 16$ and G is nilpotent. Since $|G| \geq 2$ implies that $|G| > (15/8)\log_2|G|$, we may assume that G is non-abelian. If $|G| = p$ or p^2 (for p a prime), then G is abelian. If $|G| = \Pi p_i^{\alpha_i} (p_i \neq p_{i+1}, p_i$ primes), then G is the direct product of its normal Sylow p_i-subgroups $P_i, |P_i| = p_i^{\alpha_i}$, and thus $k(G) = \Pi k(P_i)$.

Lemma 2.7. *Suppose $B, C \trianglelefteq G$, $G = B \times C$ (for $B \neq C$) and $k(B) \geq a\log_2|B|, k(C) \geq a\log_2|C|$. Then*

$$k(G) \geq d\log_2|G|$$

as long as $d \leq (\frac{bc}{b+c})a^2$, where $b := \log_2|B|$ and $c := \log_2|C|$.

Proof. We have $k(G) = k(B)\cdot k(C) \geq a^2\log_2|B|\cdot\log_2|C|$.
Thus $k(G) \geq d\log_2|G| = d\log_2(|B|\cdot|C|) = d(\log_2|B| + \log_2|C|) = d(b+c)$ as long as $a^2bc \geq d(b+c)$. □

In particular, suppose G is nilpotent and $G = B \times C (B \neq C)$, with exactly one of B, C abelian. Then $k(G) \geq (15/8)\log_2|G|$ since $15/8 > 1 + 1/3$ and we are assuming $|B| \geq 2, |C| \geq 8$. Now we assume that neither B nor C is abelian. Factoring each $|G| \leq 15,533$ into the product of prime powers: $2^\alpha, 3^\beta, \cdots (\alpha, \beta \cdots \geq 3)$, only the product of at most two prime powers need be checked, and the list of 58 such products $\leq 15,533$ is listed in the tables below:

$8(2^3)$	$16(2^4)$	$27(3^3)$	$32(2^5)$	$64(2^6)$	$81(3^4)$	$125(5^3)$
$128(2^7)$	$243(3^5)$	$256(2^8)$	$343(7^3)$	$512(2^9)$	$625(5^4)$	$729(3^6)$
$1024(2^{10})$	$1331(11^3)$	$2048(2^{11})$	$2187(3^7)$	$2197(13^3)$	$2401(7^4)$	$3125(5^5)$
$4096(2^{12})$	$4913(17^3)$	$6561(3^8)$	$6859(19^3)$	$8192(2^{13})$	$12167(23^3)$	$14641(11^4)$

$216(2^3 \cdot 3^3)$	$432(2^4 \cdot 3^3)$	$648(2^3 \cdot 3^4)$	$864(2^5 \cdot 3^3)$	$1000(2^3 \cdot 5^3)$
$1296(2^4 \cdot 3^4)$	$1728(2^6 \cdot 3^3)$	$1944(2^3 \cdot 3^5)$	$2000(2^4 \cdot 5^3)$	$2592(2^5 \cdot 3^4)$
$2744(2^3 \cdot 7^3)$	$3375(3^3 \cdot 5^3)$	$3456(2^7 \cdot 3^3)$	$3888(2^4 \cdot 3^5)$	$4000(2^5 \cdot 5^3)$
$5000(2^3 \cdot 5^4)$	$5184(2^6 \cdot 3^4)$	$5488(2^4 \cdot 7^3)$	$5832(2^3 \cdot 3^6)$	$6912(2^8 \cdot 3^3)$
$7776(2^5 \cdot 3^5)$	$8000(2^6 \cdot 5^3)$	$9261(3^3 \cdot 7^3)$	$10000(2^4 \cdot 5^4)$	$10125(3^4 \cdot 5^3)$
$10368(2^7 \cdot 3^4)$	$10648(2^3 \cdot 11^3)$	$10976(2^5 \cdot 7^3)$	$11664(2^4 \cdot 3^6)$	$13824(2^9 \cdot 3^3)$

Using P. Hall's theorem (see Lemma 2.1) for the number of conjugacy classes in p-groups, we show in [3, Theorem 4.2] that if P is a p-group of order p^n, $n \geq 3$ then $k(P) > (\frac{3}{4}p) \cdot \log_2|P|$. We assume that P and Q are of prime power order, and $|Q| > |P|$ in Lemma 2.8.

Lemma 2.8. *Suppose $G = P \times Q$ is nilpotent, $|G|$ is odd, and neither P nor Q is abelian. Then*

$$k(G) > 20 \log_2|G|$$

Proof. Without loss of generality, we may assume that $|P| \geq 3^3$ and $|Q| \geq 5^3$ so $k(P) > (9/4)\log_2|P|$ and $k(Q) > (15/4)\log_2|Q|$. Then $k(G) = k(P) \cdot k(Q) > (135/16)\log_2|P| \cdot \log_2|Q|$.

Thus, $k(G) > d \cdot \log_2|G| = d\log_2(|P| \cdot |Q|)$ when $(135/16)\log_2|P| \cdot \log_2|Q| \geq d(\log_2|P| + \log_2|Q|)$, i.e. when $\{(135/16) \cdot \log_2|P| - d\}\log_2|Q| \geq d\log_2|P|$. Since $|Q| > |P|$, the last inequality follows from $\{(135/16) \cdot \log_2|P| - d\} \geq d$, that is $d \leq (135/32)\log_2|P|$. But $\log_2|P| \geq \log_2 27$, so $d \leq 20$ is sufficient. □

Lemma 2.9. *Suppose $|G|$ is even and $G = P \times Q$ is nilpotent with $|P| = 2^n \geq 2^3$ and $|Q|$ (is odd and) $\geq 3^3$. Again neither P nor Q is abelian. Then $k(G) \geq 5\log_2|G|$.*

Proof. P. Hall's theorem $k(P) \geq (3/2)\log_2|P|$ and our earlier comments yield $k(Q) > (9/4)\log_2|Q|$. Thus $k(G) = k(P) \cdot k(Q) > (27/8)\log_2|P| \cdot \log_2|Q|$, and

$$k(G) \geq d\log_2|G| = d\log_2(|P| \cdot |Q|) = d(\log_2|P| + \log_2|Q|)$$

if $(27/8) \cdot \log_2|P| \cdot \log_2|Q| \geq d(\log_2|P| + \log_2|Q|)$.

If $|Q| > |P|$, then $(27/8) \cdot \log_2|P| \cdot \log_2|Q| \geq d(\log_2|P| + \log_2|Q|)$ follows from

$$\{(27/8)\log_2|P| - d\}\log_2|Q| > \{(27/8)\log_2|P| - d\}\log_2|P| \geq d\log_2|P|$$

that is, $d \leq (27/16)\log_2|P|$. Since $|P| \geq 2^3$, $d \leq 5$ suffices.

On the other hand, if $|P| > |Q|$ then $(27/8) \cdot \log_2|P| \cdot \log_2|Q| \geq d(\log_2|P| + \log_2|Q|)$ follows from $\{(27/8)\log_2|Q| - d\}\log_2|P| \geq d\log_2|Q|$ which is satisfied when $((27/8)\log_2|Q| - d) \cdot \log_2|Q| \geq d \cdot \log_2|Q|$, that is $d \leq (27/16)\log_2|Q|$. Since $|Q| \geq 3^3, d \leq 8$ $(< (27/16)\log_2 27) \leq (27/16)\log_2|Q|)$ suffices. Comparing the two cases, we see that $d \leq 5$ is sufficient, whether $|Q| > |P|$ or $|P| > |Q|$. □

3. Groups of order 2^n

In 2000, N. Boston and J. L. Walker [8] investigated the least number c_n of conjugacy classes among all groups of order 2^n. For each $n \leq 14$ they computed c_n using the software package Magma [7], and found a lower bound for c_{15}. The database for Magma is the SmallGroups library (see [6]), which contains descriptions of *all* of the (nearly 424 million) groups of order at most 2000, excluding the (nearly 50 billion) groups of order 2^{10}. It also contains descriptions of many groups of orders between 2001 and $20,000$.

If G_n contains the *smallest number* (c_n) of conjugacy classes among all groups of order 2^n, and $ncl(n)$ is (defined as) the *smallest* nilpotency class among all such G_n, Boston and Walker proved that $ncl(n+1) \leq ncl(n)+1$ for $n = 1, 2, \ldots$ [8, Theorem 2.2].

The next two tables are based on their results. The first table contains $c_n, ncl(n)$ and $\#(n)$ for $3 \leq n \leq 9$, where $\#n$ stands for the SmallGroups library* number of a group of order 2^n with nilpotency class $ncl(n)$ and c_n conjugacy classes (thanks to Eamonn O'Brien, Auckland, N. Z.). The second table contains $c(n)$ and $ncl(n)$ for $10 \leq n \leq 14$.

$\|G\|$	=	8	16	32	64	128	256	512
n	=	3	4	5	6	7	8	9
c_n	=	5	7	11	13	14	19	26
$ncl(n)$	=	2	3	3	4	5	6	5
$\#(n)$	=	3	7	6	32	138	511	58394

$\|G\|$	=	1,024	2,048	4,096	8,192	16,384
n	=	10	11	12	13	14
c_n	=	28	29	34	35	37
$ncl(n)$	=	6	7	8	9	10

* The SmallGroups library did not exist in 1998, when [7] was submitted. [7] had examples of such groups G_n having the nilpotency classes shown, which were available on request. Moreover for $9 \leq n \leq 12$ they had 2-generated as well as 3-generated examples of such G_n. But for $n = 13$ and $n = 14$ they did not report G_n examples with 3 generators. Mike Newman (ANU) found 2-generated as well as 3-generated examples for each $n : 10 \leq n \leq 14$ [private communication] using the p-group generation algorithm [13] in Magma. He found:

24 groups of order 2^{10} having 28 classes; 12 are 3-generator groups with $ncl = 6$.
12 groups of order 2^{11} having 29 classes; 6 are 3-generator groups with $ncl = 7$.
46 groups of order 2^{12} having 34 classes; 38 are 3-generator groups with $ncl = 8$.
12 groups of order 2^{13} having 35 classes; 6 are 3-generator groups with $ncl = 9$, and
12 groups of order 2^{14} having 37 classes; 8 are 3-generator groups with $ncl = 10$.

The other examples are 2-generator groups, with $ncl = 7, 8, 9, 10$ and 11 respectively. This is as far as he took a full search; the last search took about 40 minutes.

Conjecture. When $|G| = 2^n$, $n \geq 5$ then $k(G) \geq 2\log_2|G|$.

Remark. Note that there exists a group G of order 2^7 with $k(G) = 14 = 2\log_2|G|$.

Comment. A. J.-Zapirain [10] proved that there exists an (explicitly computable) constant c $(= 10^{-4})$ such that every finite nilpotent group G of order $n \geq 8$ satisfies:

$$k(G) > c\left(\frac{\log_2 \log_2 n}{\log_2 \log_2 \log_2 n}\right) \log_2 n.$$

4. G solvable and G/N Nilpotent

When N is a normal subgroup of the finite group G, we need some preliminary results, involving inequalities of the form:

$$k(G/N) \geq [G:N]^{\alpha} \text{ (for } 0 < \alpha \leq 1) \text{ and } k(G)/(\log|N \cap G'|)^t \geq 1+\epsilon \text{ (for } \epsilon, t > 0).$$

These will lead to theorems concluding that $k(G) \geq \log_3 |G|$, assuming that $k(G) \geq \log_2 |N|$ under natural assumptions on $N \trianglelefteq G$ (e.g. G/N is nilpotent).

Lemma 4.1. *(a) If $N \trianglelefteq G, \alpha > 0$ and $k(G/N) \geq [G:N]^{\alpha}$, then $k(G/N \cap G') \geq [G:N \cap G']^{\alpha}$.*

(b) If $k(G/N) \geq \log_2[G:N]$, then $k(G/N \cap G') \geq \log_2[G:N \cap G']$.

Proof. (a) We always have $k(G/N \cap G') = [N:N \cap G']k(G/N)$ [3, Proposition 3.7(b)] which is

$$\geq [N:N\cap G'][G:N]^{\alpha} = |N|^{1-\alpha}|G|^{\alpha}/|N \cap G'| \geq |N\cap G'|^{1-\alpha}|G|^{\alpha}/|N\cap G'| = [G:N\cap G']^{\alpha}$$

using our assumption on $k(G/N)$.

(b) We may assume that $N \not\leq G' \not\leq N$, so in particular $[G:N] \geq 6$ and

$$[G:N \cap G'] = [G:N][N:N\cap G'] \geq 2[G:N].$$

Since $n^{\log_2 \log_2 n/\log_2 n} = \log_2 n$, our assumption here is the same as $k(G/N) \geq [G:N]^{\alpha}$, where

$$\alpha = \log_2 \log_2 [G:N]/\log_2[G:N]$$

and from (a) we may conclude that $k(G/N \cap G') \geq [G:N\cap G']^{\alpha}$. It is straightforward to prove that when $\log(\cdot) = \log_2(\cdot)$ and $n \geq 2^e$ (and hence for $n \geq 6$) that $\log_2 \log_2 n/\log_2 n \geq \log_2 \log_2 (mn)/\log_2(mn)$ when $m \geq 2$. Thus

$$k(G/N\cap G') \geq [G:N\cap G']^{\log_2 \log_2 [G:N]/\log_2[G:N]} \geq [G:N\cap G']^{\log_2\log_2[G:N\cap G']/\log_2[G:N\cap G']} = \log_2[G:N\cap G']$$

□

Although Theorem 4.2, Corollaries 4.3, 4.4, 4.7, 4.8, 4.9 and Lemma 4.10 are true in any base ≥ 2, our main interest is in base 3.

Theorem 4.2. *Suppose $N \trianglelefteq G$ and $k(G/N) \geq [G:N]^{\alpha}$ $(0 < \alpha \leq 1)$.*

(a) If $k(G) \geq (1+\epsilon)(\log_3|N \cap G'|)^t$ $(\epsilon, t > 0)$ then for all $|G|$ large enough, depending only on (α, ϵ, t), $k(G) > (\log_3|G|)^t$.

b) If $N \cap G'$ is abelian, then $k(G) > (\log_3|G|)^t$ for all $|G|$ large enough, depending only on (α, t).

http://dx.doi.org/10.22108/IJGT.2021.128396.1687

Proof. Since G/N is solvable, $[G:N] \geq 2|(G/N)'| = 2[G':N\cap G']$ so

$$k(G) > k(G/N) \geq [G:N]^\alpha \geq 2^\alpha [G':N\cap G']^\alpha.$$

We always have $k(G) > [G:G']$, so we may assume that $|G'| > |G|/(\log_3|G|)^t$. Since $2^\alpha > 1$, we have $k(G) > (|G|/(\log_3|G|)^t)^\alpha/|N\cap G'|^\alpha$. Thus $k(G) > (\log_3|G|)^t$ as long as $|G|^\alpha/(\log_3|G|)^{t(1+\alpha)} \geq |N\cap G'|^\alpha$, that is $|N\cap G'| \leq |G|/(\log_3|G|)^{t(1+1/\alpha)}$. Otherwise, $|N\cap G'| \geq |G|/(\log_3|G|)^{t(1+1/\alpha)}$, that is $\log_3|N\cap G'| \geq \log_3|G| - t(1+1/\alpha)\log_3\log_3|G|$. From (a) we obtain $k(G) \geq (1+\epsilon)\{\log_3|G| - t(1+1/\alpha)\log_3\log_3|G|\}^t \geq (\log_3|G|)^t$ as long as $(1+\epsilon)^{1/t}\{\log_3|G| - t(1+1/\alpha)\log_3\log_3|G|\} \geq \log_3|G|$, that is

$$\left((1+\epsilon)^{1/t} - 1\right)\log_3|G| \geq (1+\epsilon)^{1/t}\cdot t(1+1/\alpha)\log_3\log_3|G|$$

The latter is true for all $|G|$ large enough, depending only on (α,ϵ,t).

Assume (b). Since $k(G/N) \geq [G:N]^\alpha$, it follows from Lemma 4.1(a) that $k(G/N\cap G') \geq [G:N\cap G']^\alpha$. Since $N\cap G'$ is abelian, $k(G) \geq |G|^{\alpha/(2+\alpha)}$ follows from [2, Lemma 3(i)]. Thus, $k(G) \geq (\log_3|G|)^t$ as long as $|G|^{\alpha/(2+\alpha)} \geq (\log_3|G|)^t$, that is $\log_3|G|/\log_3\log_3|G| \geq t(1+2/\alpha)$, which is true for all $|G|$ large enough depending only on (α,t). □

Setting $t=1$, we have the following Corollaries:

Corollary 4.3. *Suppose* $N \trianglelefteq G$ *and* $k(G/N) \geq [G:N]^\alpha$ $(0<\alpha\leq 1)$ *with* $k(G) \geq (1+\epsilon)\log_3|N\cap G'|$ $(\epsilon>0)$. *Then* $k(G) > \log_3|G|$ *as long as*

either (i) $\log_3|G|/\log_3\log_3|G| \geq (1+1/\alpha)(1+1/\epsilon)$,

or (ii) $k(G) \geq (1+1/\alpha)(1+1/\epsilon)\log_3\log_3|G|$.

Proof. Setting $t=1$ in the statement and proof of (a) in the Theorem, we see that (i) yields $k(G) > \log_3|G|$. The conclusion also follows when (i) is false and (ii) is true. □

Corollary 4.4. *(a) If* $k(G) \geq (1+\epsilon)\log_3|G'|$ $(\epsilon>0)$, *and*

either (i) $\log_3|G|/\log_3\log_3|G| \geq 1+2/\epsilon$

or (ii) $k(G) \geq (1+2/\epsilon)\log_3\log_3|G|$,

then $k(G) > \log_3|G|$.

(b) If $k(G) \geq 3\log_3|G''|$, *then* $k(G) > \log_3|G|$.

Proof. (a) Set $N = G'$ and $\alpha = 1$ in Corollary 4.3. If $|G'| \geq (\log_3|G|)^{2/\epsilon}$ then $(1+\epsilon)\log_3|G'| \geq 2(1+1/\epsilon)\cdot\log_3\log_3|G|$, so $k(G) \geq (1+1/\alpha)(1+1/\epsilon)\cdot\log_3\log_3|G|$ and (ii) of Corollary 4.3 is satisfied. Otherwise $|G'| < (\log_3|G|)^{2/\epsilon}$, so $k(G) > [G:G'] > |G|/(\log_3|G|)^{2/\epsilon}$. If assumption (i) is true then $\log_3|G| \geq (1+2/\epsilon)(\log_3\log_3|G|)$, that is $|G| > (\log_3|G|)^{1+2/\epsilon}$ so $k(G) > |G|/(\log_3|G|)^{2/\epsilon} > \log_3|G|$. If assumption (i) is false then assumption (ii) yields $k(G) > \log_3|G|$.

(b) In Corollary 4.3, let $N = G'', \alpha = 1/3$ and $\epsilon = 2$. Since $(G/N)' = G'/G''$ is abelian, we have $k(G/N) > [G:N]^{1/3}$ [2, Theorem 3]. We are assuming $|G| > 3^{15}$, so $\log_3|G|/\log_3\log_3|G| > 6 = (1+1/\alpha)(1+1/\epsilon)$ and (i) of Corollary 4.3 is satisfied. □

Corollary 4.5. *If* $k(G) \geq \log_2 |G'|$, *then* $k(G) > \log_3 |G|$.

Proof. In Corollary 4.4, set $1 + \epsilon = \log_2 3$. Since $|G| \geq 3^9$, $\log_3 |G| / \log_3 \log_3 |G| \geq 4.5 > 1 + 2/\epsilon = 1 + 2/\log_2(3/2)$ and (i) of Corollary 4.4 is satisfied. Thus $k(G) > \log_3 |G|$. □

Corollary 4.6. *If* $k(G) \geq 2^{s-1} \log_2 |G^{(s)}|$ $(s \geq 2)$, *then* $k(G) > \log_3 |G|$ *when*

$$\log_3 |G| / \log_3 \log_3 |G| \geq 3 \cdot 2^{s-1}$$

Proof. In Corollary 4.3, let $N = G^{(s)}$. Then G/N has derived length s, so $k(G/N) \geq [G : N]^{1/(2^s-1)}$ [2, Theorem 1]. By assumption, $k(G) \geq 2^{s-1} \log_2 |G^{(s)}| = (2^{s-1} \log_2 3) \log_3 |G^{(s)}|$. Setting $\alpha = (2^s - 1)^{-1}$ yields $1 + 1/\alpha = 2^s$. So in Corollary 4.3, the assumption on $k(G)$ is met when $1 + \epsilon = 2^{s-1} \log_2 3$, which yields $(1 + 1/\alpha)(1 + 1/\epsilon) = 2^s / \left(1 - \frac{1}{(\log_2 3) 2^{s-1}}\right) < 3 \cdot 2^{s-1}$ since $s \geq 2$. □

We generalize Theorem 4.2(b) with:

Corollary 4.7. *Suppose* $N \trianglelefteq G$, $k(G/N) \geq [G : N]^\alpha$ $(0 < \alpha \leq 1)$ *and* $N \cap G'$ *is nilpotent of class* $c \geq 1$. *Then* $k(G) \geq (\log_3 |G|)^t$ $(t > 0)$ *if*

either (*i*) $\log_3 |G| / \log_3 \log_3 |G| \geq t\left(\frac{c+1}{\alpha} + c\right)$
or (*ii*) $k(G) \geq \left\{t\left(\frac{c+1}{\alpha} + c\right) \log_3 \log_3 |G|\right\}^t$.

Proof. As in the proof of Theorem 4.2, we know that $k(G) \geq (\log_3 |G|)^t$ when $|N \cap G'| \leq |G| / (\log_3 |G|)^{t\left(1+\frac{1}{\alpha}\right)}$. So suppose that $|N \cap G'| > |G| / (\log_3 |G|)^{t\left(\frac{1}{\alpha}+1\right)}$. Since $N \cap G'$ is nilpotent of class $c \geq 1$, $k(N \cap G') \geq |N \cap G'|^{1/c}$ [14]. Lemma 4.1(a) and our assumption on $k(G/N)$ imply that $k(G/N \cap G') \geq [G : N \cap G']^\alpha$. Using of [2, Lemma 3(i)] we conclude that $k(G) \geq |G|^{\alpha/c(1+\alpha)+1}$. Thus $k(G) \geq (\log_3 |G|)^t$ when $|G|^{\alpha/c(1+\alpha)+1} \geq (\log_3 |G|)^t$, which is equivalent to (i). Note that if (i) is false and (ii) is true, then again $k(G) \geq (\log_3 |G|)^t$. □

In particular, setting $N = G'$ we have:

Corollary 4.8. *If* G' *is nilpotent of class* $c \geq 1$ *and*
either (*i*) $\log_3 |G| / \log_3 \log_3 |G| \geq t(2c + 1)$ $(t > 0)$
or (*ii*) $k(G) \geq \{t(2c + 1) \log_3 \log_3 |G|\}^t$
then $k(G) \geq (\log_3 |G|)^t$.

Thus, since we assume $|G| > 3^{15}$, we conclude that if G' is nilpotent of class ≤ 2 then $k(G) \geq \log_3 |G|$. In [4, Theorem 3.5(a)] we proved that (assuming $|G| > 3^{15}$) if $C_G(G') \not\leq G'$ and $k(G/C_G(G')) \geq \log_3[G : C_G(G')]$ then $k(G) \geq \log_3 |G|$. Now assume that $C_G(G') \leq G'$, that is $C_G(G') = Z(G')$ is abelian. Setting $N = C_G(G')$ and $c = t = 1$ in Corollary 4.7 we have:

Corollary 4.9. *If* $k(G/C_G(G')) \geq [G : C_G(G')]^\alpha$ $(0 < \alpha < 1)$ *and*

either (*i*) $\log_3 |G| / \log_3 \log_3 |G| \geq 1 + 2/\alpha$
or (*ii*) $k(G) \geq (1 + 2/\alpha) \log_3 \log_3 |G|$,

then $k(G) \geq \log_3 |G|$.

Thus when $C_G(G') \le G'$ and $k(G/C_G(G')) \ge [G : C_G(G')]^{2/5}, k(G) \ge \log_3 |G|$ (assuming $|G| > 3^{15}$).

Lemma 4.10. *Suppose $N \trianglelefteq G$ and $a, b > 0$ with*

(*i*) $k(G/N) \ge a \log_3[G : N]$ *and*

(*ii*) $k(G) \ge b \log_3|N|$.

Then $k(G) \ge (ab/(a + b)) \log_3|G|$

Proof. If $|N| \ge |G|^{a/a+b}$, then by (ii) $k(G) \ge (ab/(a+b)) \log_3|G|$. On the other hand if $|N| < |G|^{a/a+b}$, then $[G : N] > |G|^{b/a+b}$ and by (i) $k(G) > k(G/N) \ge (ab/(a + b)) \log_3|G|$. □

The next Theorem shows why improving the constant 3/2 in P. Hall's theorem to 15/8 in Theorem 2.6 is very useful. When assuming that G/N is nilpotent and trying to prove that $k(G) \ge \log_3|G|$ when $|G| > 3^{15}$, we only need to assume that $k(G) \ge \log_2|N|$. Theorem 4.11 generalizes Corollary 4.5.

Theorem 4.11. *If G/N is nilpotent and $k(G) \ge \log_2|N|$, then $k(G) > \log_3|G|$.*

Proof. If $[G : N] \le 16$ then $|N| \ge |G|/16$ and $k(G) \ge \log_2|N| \ge \log_2|G| - 4 > \log_3|G|$ since $\log_2|G| - 4 = (\log_2 3) \log_3|G| - 4 > \log_3|G|$ when $|G| > 3^{15}$. Since G/N is nilpotent and we may assume that $[G : N] > 16$, Theorem 2.6 concludes that $k(G/N) \ge (15/8) \log_2[G : N]$. Since we also assumed that $k(G) \ge \log_2|N|$, by Lemma 4.10 we conclude that

$$k(G) \ge \{(15/8) \cdot 1/(\frac{15}{8} + 1)\} \log_2|G| = (15/23) \log_2|G| > \log_3|G|.$$

□

The *nilpotent residual* N of G is defined as the smallest normal subgroup N of G such that G/N is nilpotent. The nilpotent residual is thus the intersection of all normal subgroups L of G such that G/L is nilpotent, so the nilpotent residual is contained in G' and is the last term of the lower central series.

The next Table shows that when G is solvable and $k(G) \le 14, k(G) > \log_2 |G|$ except when $k(G) = 6, 8$ or 9. When $k(G) = 6$ or 8, $k(G) > \log_2 |G'|$. But the largest solvable group G with 9 classes satisfies $k(G) < \log_2 1176 = \log_2 |G'|$, and here it turns out that G' is the nilpotent residual of G.

Corollary 4.12. *If the nilpotent residual N of G satisfies $|N| \le |G|^{5/8}$, then $k(G) > \log_3|G|$.*

Proof. G/N is nilpotent, so by Theorem 2.3, $k(G/N) \ge \frac{5}{3} \log_2[G : N] \ge \frac{5}{3} \log_2(|G|^{3/8}) = \frac{5}{8} \log_2|G| \ge \log_2|N|$ by assumption. By the Theorem we conclude that $k(G) > \log_3|G|$. □

The Largest Solvable Groups G with $k(G) = k$

k	Largest solvable G with $k(G) = k$	Small Gp. $(\lvert G\rvert, \#)$	$k(G'); G'$	Small Gp. $(\lvert G'\rvert, \#)$
3	Sym(3)= $C_3 \times_f C_2$	(6,1)	3; *Alt(3)	(3,1)
4	Alt(4) = $V_4 \times_f C_3$	(12, 3)	4; *$C_2 \times C_2$	(4,2)
5	*Sym(4)	(24,12)	4; Alt(4) = $V_4 \times_f C_3$	(12,3)
6	$(C_3 \times C_3) \times_f Q8$	(72,41)	6; $(C_3 \times C_3) \times_f C_2$	(18,4)
7	$C_{11} \times_f C_5$	(55,1)	11; *C_{11}	(11,1)
8	$(C_5 \times C_5) \times_f$SL(2,3)	(600,150)	8;$(C_5 \times C_5) \times_f Q8$	(200,44)
9	$(C_7 \times C_7) \times_f$ SL(2,3)·C_4	2352	9; $(C_7 \times C_7) \times_f$SL(2,3)	(1176,215)
10	$(C_7 \times C_7) \times_f Q16$	(784,162)	16;$(C_7 \times C_7) \times_f C_4$	(196,8)
11	*,[1]Hol($C_3 \times C_3$)	(432, 734)	10*,[2]Hol(C_3^2,SL(2, 3))	(216,153)
12	$C_3^4 \times_f (C_5 \times_\lambda C_8)$	3240	21; $C_3^4 \times_f C_5$	(405,15)
13	*$C_9^2 \rtimes$ SL(2,3)	(1944,2289) (1944,2290)	15;$C_9^2 \times_f Q8$	(648,253)
14	$(C_{17} \times C_{17}) \times_f$ SL(2,3)·C_4	13,872	19; $C_{17}^2 \times_f$SL(2,3)	6936

* These are **not** Frobenius groups

[1] Hol(G) = $G \rtimes$Aut(G)

[2] Hol(G, K), where K is a subgroup of Aut(G), is the "Relative Holomorph" of G and K, and is a subgroup of Sym(G). The notation is also $G \rtimes K$. In fact, Hol($(C_3)^2$,SL(2,3)) = ASL($2, \mathbb{F}_3$), the *Affine Special Linear Group* over $\mathbb{F}_3$.

Many thanks to Eamonn O'Brien (U. Auckland) for his help with this Table, using the SmallGroups library. The SmallGroups library contains all G with $|G| \leq 2000$, and many others. The Table is based on the classification of all groups G with $k(G) \leq 14$ by A. Vera-Lopez, et al. [15, 16, 17].

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