

ON A RESULT OF NILPOTENT SUBGROUPS OF SOLVABLE GROUPS

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ABSTRACT. Heineken [H. Heineken, Nilpotent subgroups of finite soluble groups, *Arch. Math.(Basel)*, **56** no. 5 (1991) 417–423.] studied the order of the nilpotent subgroups of the largest order of a solvable group. We point out an error, and thus refute the proof of the main result of [H. Heineken, Nilpotent subgroups of finite soluble groups, *Arch. Math.(Basel)*, **56** no. 5 (1991) 417–423].

1. Introduction

In a paper of Heineken [2], the following bound for the nilpotent subgroups of the largest order of a solvable group is obtained. The paper has been quoted in [3] and [4]. However, we found that a key argument is invalid, and thus refute the proof of the main result of that paper. In [2], the following main result is obtained. This appears to be the best possible bound in the literature to estimate the size of the largest order of a nilpotent subgroup of a solvable group.

[2, **Theorem A**]. *If G is a finite solvable group with Fitting subgroup F , and if K is a nilpotent injector of G , then*

$$|K||F|^\beta \geq |G|,$$

where $\beta = (\log 5 + 20 \log 2)(8 \log 3)^{-1}$.

The proof of this result relies on the following.

[2, **Theorem C**]. *Let G be a finite solvable group with Fitting subgroup F . Denote by D and T the 2-complement and the 3-complement of F . Assume that the subgroup M of G satisfies the following conditions:*

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- (I) $F \subseteq M$,
 (II) $(C_G(T)M)/(C_G(T)F)$ is a 2-Sylow subgroup of $(C_G(T \cap D)F)/(C_G(T)F)$,
 (III) $(C_G(D)M)/(C_G(D)F)$ is a 3-Sylow subgroup of $(C_G(T \cap D)F)/(C_G(D)F)$.
 Then M is nilpotent and

$$|M||F|^k \geq |G|$$

where $k = (\log 5 + 20 \log 2) \cdot (8 \log 3)^{-1}$.

In [2, Theorem C], Heineken does not construct M , he assumes its existence. The main result of [2, Theorem A] relies on the subgroup M whose existence is assumed. As was stated in [2, p 422, lines 3-4], M is contained in a nilpotent injector of G , where G is an arbitrary solvable group, and thus the needed bound holds for all solvable groups. It appears that the argument there indicates that M exists for all the solvable groups.

However we would like to point out that the group M claimed in [2, Theorem C] need not exist, at least in some cases, thus the argument used does not work in general. The following is a counterexample.

2. A counterexample to the argument

Let $G = D \rtimes A$ where A is the dihedral group of order 8, and D is the cyclic group of order 3. Here A acts on D by inverting it, and with the kernel of this action being T , an elementary abelian group of order 4. Hence $G = AD$, the Fitting subgroup of G is $F = TD$, where D is the 2-complement of F , and T is the 3-complement of F . Now according to [2, Theorem C], if M exists, M is nilpotent and contains the Fitting subgroup. However, the Fitting subgroup has index 2, and G is not nilpotent, so that $M = F$. However, by condition (II) in [2, Theorem C], we get a contradiction because the centralizer of T in G is F , which implies that $M = G$.

3. Discussions

The author was interested in studying a problem proposed in [1, Question 17] which is related to the largest order of a nilpotent subgroup of a finite group. The results and the methods in [2] seem to be useful in studying that problem, if correct. Unfortunately, after a close look at [2], the author discovered the previous counterexample and thus the main arguments in [2] do not work, even though the statement of [2, Theorem A] might still be correct.

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