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ON A RESULT OF NILPOTENT SUBGROUPS OF SOLVABLE GROUPS

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Abstract. Heineken [2] studied the order of the nilpotent subgroups of the largest order of a solvable group. We point out an error, and thus refute the proof of the main result of [2].

1. Introduction

In a paper of Heineken [2], the following bound for the nilpotent subgroups of the largest order of a solvable group is obtained. The paper has been quoted in [3] and [4]. However, we found that a key argument is invalid, and thus refute the proof of the main result of that paper. In [2], the following main result is obtained. This appears to be the best possible bound in the literature to estimate the size of the largest order of a nilpotent subgroup of a solvable group.

[2, Theorem A]. *If G is a finite solvable group with Fitting subgroup F, and if K is a nilpotent injector of G, then*

$$|K||F|^{\beta} \geq |G|,$$

where $\beta = (\log 5 + 20 \log 2)(8 \log 3)^{-1}$.

The proof of this result relies on the following.

[2, Theorem C]. *Let G be a finite solvable group with Fitting subgroup F. Denote by D and T the 2-complement and the 3-complement of F. Assume that the subgroup M of G satisfies the following conditions:*

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(I) $F \subseteq M$*,*
(II) $(C_G(T)M)/(C_G(T)F)$ *is a* 2*-Sylow subgroup of* $(C_G(T \cap D)F)/(C_G(T)F)$*,*
(III) $(C_G(D)M)/(C_G(D)F)$ *is a* 3*-Sylow subgroup of* $(C_G(T \cap D)F)/(C_G(D)F)$*.*
Then M *is nilpotent and*

$$|M||F|^k \geq |G|$$

where $k = (\log 5 + 20 \log 2) \cdot (8 \log 3)^{-1}$.

In [2, Theorem C], Heineken does not construct M, he assumes its existence. The main result of [2, Theorem A] relies on the subgroup M whose existence is assumed. As was stated in [2, p 422, lines 3-4], M is contained in a nilpotent injector of G, where G is an arbitrary solvable group, and thus the needed bound holds for all solvable groups. It appears that the argument there indicates that M exists for all the solvable groups.

However we would like to point out that the group M claimed in [2, Theorem C] need not exist, at least in some cases, thus the argument used does not work in general. The following is a counterexample.

2. A counterexample to the argument

Let $G = D \rtimes A$ where A is the dihedral group of order 8, and D is the cyclic group of order 3. Here A acts on D by inverting it, and with the kernel of this action being T, an elementary abelian group of order 4. Hence $G = AD$, the Fitting subgroup of G is $F = TD$, where D is the 2-complement of F, and T is the 3-complement of F. Now according to [2, Theorem C], if M exists, M is nilpotent and contains the Fitting subgroup. However, the Fitting subgroup has index 2, and G is not nilpotent, so that $M = F$. However, by condition (II) in [2, Theorem C], we get a contradiction because the centralizer of T in G is F, which implies that $M = G$.

3. Discussions

The author was interested in studying a problem proposed in [1, Question 17] which is related to the largest order of a nilpotent subgroup of a finite group. The results and the methods in [2] seem to be useful in studying that problem, if correct. Unfortunately, after a close look at [2], the author discovered the previous counterexample and thus the main arguments in [2] do not work, even though the statement of [2, Theorem A] might still be correct.

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