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ON CO-MAXIMAL SUBGROUP GRAPH OF $\mathbb{Z}_n$

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ABSTRACT. The co-maximal subgroup graph $\Gamma(G)$ of a group G is a graph whose vertices are non-trivial proper subgroups of G and two vertices H and K are adjacent if $HK = G$. In this paper, we study and characterize various properties like diameter, domination number, perfectness, hamiltonicity, etc. of $\Gamma(\mathbb{Z}_n)$.

1. Introduction

Over the last two decades, various graphs has been defined on groups. For a comprehensive survey on this, please refer to [2]. One of such graphs, the *co-maximal subgroup graph* of a group G was introduced by Akbari *et.al.* [1] in 2017. More results on comaximal subgroup graph can be found in [3] and [4]. In this paper, we study various properties of co-maximal subgroup graph of $\mathbb{Z}_n$. In particular, we characterize the value of n for which the graph is hamiltonian, eulerian, perfect etc.

We first recall a few definitions and results from [1] and [3].

Definition 1.1. *Let G be a group and S be the collection of all non-trivial proper subgroups of G. The co-maximal subgroup graph $\Gamma(G)$ of a group G is defined to be a graph with S as the set of vertices and two distinct vertices H and K are adjacent if and only if $HK = G$.*

Theorem 1.2. [1], [3] *Let G be a finite nilpotent group.*

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- *$\Gamma(G)$ is connected apart from possible isolated vertices, which corresponds to the non-trivial subgroups of the Frattini subgroup of G.*
- *Denote by $\Gamma^*(G)$, the graph obtained from $\Gamma(G)$ by removing the isolated vertices, if any. Then $diam(\Gamma^*(G)) \leq 3$.*
- *$\Gamma^*(G)$ is weakly perfect, i.e., the clique number and the chromatic number of $\Gamma^*(G)$ are equal.*

As cyclic groups are nilpotent, all the above results hold for $\Gamma(\mathbb{Z}_n)$. We start with some basic properties of $\Gamma(\mathbb{Z}_n)$ and $\Gamma^*(\mathbb{Z}_n)$. As for any cyclic p-group G, $\Gamma(G)$ is empty, throughout the paper, we consider $\Gamma(\mathbb{Z}_n)$ where n is not a prime power.

2. Basic Properties

In this section, we study some basic properties of $\Gamma(\mathbb{Z}_n)$ and $\Gamma^*(\mathbb{Z}_n)$ like connectedness, degree, diameter etc.

Lemma 2.1. *Let $H = \langle x \rangle$ and $K = \langle y \rangle$ be two subgroups of $\mathbb{Z}_n$ where x, y divide n. Then $H \sim K$ in $\Gamma(\mathbb{Z}_n)$ if and only if $gcd(x, y) = 1$.*

Proof. It follows from Bezout's Theorem and the observation that $HK = \{sx + ty : s, t \in \mathbb{Z}\}$. □

Theorem 2.2. *Let $n = {p_1}^{\alpha_1}{p_2}^{\alpha_2}\cdots{p_k}^{\alpha_k}$, where p_i's are distinct primes and $\alpha_i \geq 1$. Let $H = \langle {p_1}^{\beta_1}{p_2}^{\beta_2}\cdots{p_k}^{\beta_k} \rangle$ be a subgroup of $\mathbb{Z}_n$, where $\beta_i \leq \alpha_i$. Then degree of H in $\Gamma(\mathbb{Z}_n)$ is*

$$deg(H) = \begin{cases} 0, & \text{if } \beta_i \neq 0, \forall i, \\ \displaystyle\prod_{j:\beta_j=0} (\alpha_j + 1) - 1, & \text{otherwise.} \end{cases}$$

Proof. Follows from Lemma 2.1. □

Corollary 2.3. *Let $n = {p_1}^{\alpha_1}{p_2}^{\alpha_2}\cdots{p_k}^{\alpha_k}$, where p_i's are distinct primes and $\alpha_i \geq 1$. Then $\Gamma^*(\mathbb{Z}_n)$ is Eulerian if and only if n is a perfect square.*

Proof. If n is a perfect square, then each α_i is even and by Theorem 2.2, degree of every vertex of $\Gamma^*(\mathbb{Z}_n)$ is even and hence $\Gamma^*(\mathbb{Z}_n)$ is Eulerian. If n is not a perfect square, then there exists i such that α_i is odd. Let $H = \langle {p_1}^{\alpha_1}\cdots{p_{i-1}}^{\alpha_{i-1}}{p_{i+1}}^{\alpha_{i+1}}\cdots{p_k}^{\alpha_k} \rangle$. Then by Theorem 2.2, $deg(H) = \alpha_i$, which is odd. Thus $\Gamma^*(\mathbb{Z}_n)$ is not Eulerian. □

Theorem 2.4. *Let $n = {p_1}^{\alpha_1}{p_2}^{\alpha_2}\cdots{p_k}^{\alpha_k}$, where p_i's are distinct primes and $\alpha_i \geq 1$. Then $\Gamma(\mathbb{Z}_n)$ has exactly $\alpha_1\alpha_2\cdots\alpha_k - 1$ isolated vertices.*

Proof. Since G is a cyclic non p-group of order $n = {p_1}^{\alpha_1}{p_2}^{\alpha_2}\cdots{p_k}^{\alpha_k}$, $G \cong \mathbb{Z}_n$. By Lemma 2.1, $H = \langle p_1p_2\cdots p_k \rangle$ is an isolated vertex in $\Gamma(G)$. Similarly, if x is a multiple of $p_1p_2\cdots p_k$ which divides n, then $\langle x \rangle$ is an isolated vertex in $\Gamma(G)$.

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Let $A = \langle a \rangle$ with $a|n$ be a subgroup of G such that A is an isolated vertex in $\Gamma(G)$. As G has a unique subgroup of order corresponding to each factor of n and for any non-trivial proper subgroup H of G, we deduce $A \not\sim H$ in $\Gamma(G)$, we have $gcd(a, m) \neq 1$ for any factor m of $|G| = n$. Thus $p_i|a$ for all i, i.e., a is a multiple of $p_1 p_2 \cdots p_k$ which divides n.

Hence the number of isolated vertices in $\Gamma(G)$ is $\alpha_1\alpha_2\cdots\alpha_k - 1$. □

Corollary 2.5. $\Gamma(\mathbb{Z}_n)$ *is connected if and only if* n *is square-free.*

Proof. Let $n = {p_1}^{\alpha_1}{p_2}^{\alpha_2}\cdots{p_k}^{\alpha_k}$. Then the corollary follows from the fact that $\alpha_1\alpha_2\cdots\alpha_k - 1 = 0$ if and only if n is square-free. □

Theorem 2.6. *Let* G *be a cyclic non p-group of order* $n = {p_1}^{\alpha_1}{p_2}^{\alpha_2}\cdots{p_k}^{\alpha_k}$*, where* p_i*'s are distinct primes and* $\alpha_i \geq 1$*. Then* $diam(\Gamma^*(G)) = \begin{cases} 2, & \text{if } k = 2, \\ 3, & \text{if } k \geq 3. \end{cases}$

Proof. It is clear that the number of maximal subgroups of G is k. If $k = 2$, then the vertices of $\Gamma^*(G)$ are $\langle p_1\rangle, \langle {p_1}^2\rangle, \ldots, \langle {p_1}^{\alpha_1}\rangle, \langle p_2\rangle, \langle {p_2}^2\rangle, \ldots, \langle {p_2}^{\alpha_2}\rangle$ and any two non-adjacent vertices always have a common neighbour either $\langle p_1\rangle$ or $\langle p_2\rangle$. Hence its diameter is 2.

If $k \geq 3$, then $\langle p_1p_2\cdots p_{k-1}\rangle$ and $\langle p_2p_3\cdots p_k\rangle$ are non-adjacent vertices in $\Gamma^*(G)$ and they do not have any common neighbour. Thus their distance is greater than 2. Now, as $\mathbb{Z}_n$ is nilpotent, $diam(\Gamma^*(G)) = 3$. □

Theorem 2.7. *Let* G *be a cyclic non p-group of order* $n = {p_1}^{\alpha_1}{p_2}^{\alpha_2}\cdots{p_k}^{\alpha_k}$*, where* p_i*'s are distinct primes and* $\alpha_i \geq 1$*. Then* $\Gamma(G)$ *has pendant vertices if and only if* $\alpha_i = 1$ *for some* i*.*

Proof. Let G be a cyclic group of order $n = {p_1}^{\alpha_1}{p_2}^{\alpha_2}\cdots{p_k}^{\alpha_k}$, where at least one $\alpha_i = 1$, say $\alpha_1 = 1$, i.e., $n = p_1{p_2}^{\alpha_2}\cdots{p_k}^{\alpha_k}$. Then $\langle p_2p_3\cdots p_k\rangle$ is a pendant vertex in $\Gamma(G)$, which is adjacent to $\langle p_1\rangle$.

Conversely, let G be a cyclic group of order $n = {p_1}^{\alpha_1}{p_2}^{\alpha_2}\cdots{p_k}^{\alpha_k}$ such that $\Gamma(G)$ has at least one pendant vertex. If possible, let $\alpha_i \geq 2$ for all i. Let $H = \langle m\rangle$ be a pendant vertex in $\Gamma(G)$ where $m|n$. If $p_i|m$ for all i, then H is an isolated vertex, a contradiction. Thus, m misses at least one prime factor. Let $m = {p_2}^{\beta_2}\cdots{p_k}^{\beta_k}$ where $0 \leq \beta_i \leq \alpha_i$. But this implies that H is adjacent to the vertices $\langle p_1\rangle, \langle {p_1}^2\rangle, \ldots, \langle {p_1}^{\alpha_1}\rangle$. As $\alpha_1 \geq 2$, H can not be a pendant vertex. Thus, at least some α_i must be 1. □

3. Hamiltonicity, Perfectness and Dominating Sets

In this section, we characterize the values of n for which $\Gamma^*(\mathbb{Z}_n)$ is perfect and hamiltonian. We also find the domination number of $\Gamma^*(\mathbb{Z}_n)$.

Theorem 3.1. *Let* $n = {p_1}^{\alpha_1}{p_2}^{\alpha_2}\cdots{p_k}^{\alpha_k}$*, where* p_i*'s are distinct primes and* $\alpha_i \geq 1$*. Then* $\Gamma^*(\mathbb{Z}_n)$ *is Hamiltonian if and only if* $k = 2$ *and* $\alpha_1 = \alpha_2$*.*

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Proof. If $k = 2$ and $\alpha_1 = \alpha_2$, then $n = {p_1}^{\alpha_1}{p_2}^{\alpha_1}$. We now explicitly construct the hamiltonian circuit in $\Gamma^*(\mathbb{Z}_n)$:

$$\langle p_1\rangle \sim \langle p_2\rangle \sim \langle p_1^2\rangle \sim \langle p_2^2\rangle \sim \langle p_1^3\rangle \sim \langle p_2^3\rangle \sim \cdots \sim \langle p_1^{\alpha_1}\rangle \sim \langle p_2^{\alpha_1}\rangle \sim \langle p_1\rangle.$$

Conversely, let $\Gamma^*(\mathbb{Z}_n)$ be Hamiltonian. If possible, let $k \geq 3$. If $\alpha_i = 1$ for some i, then the graph has a vertex of degree 1 and hence it is not hamiltonian. Thus, we assume that $\alpha_i \geq 2$ for all i. Without loss of generality, let $\alpha_1 = min\{\alpha_1, \alpha_2, \ldots, \alpha_k\}$. Now, the vertices of the form $\langle {p_2}^{\alpha'_2}{p_3}^{\alpha'_3}\cdots{p_k}^{\alpha'_k}\rangle$ are adjacent only to the vertices of the form $\langle {p_1}^{\alpha'_1}\rangle$, where $1 \leq \alpha'_i \leq \alpha_i$, i.e., we have $\alpha_2\alpha_3\cdots\alpha_k$ distinct vertices of degree α_1. As two vertices of the form $\langle {p_2}^{\alpha'_2}{p_3}^{\alpha'_3}\cdots{p_k}^{\alpha'_k}\rangle$ are not adjacent, to complete a hamiltonian cycle, we need at least $\alpha_2\alpha_3\cdots\alpha_k$ different vertices between the vertices of the form $\langle {p_2}^{\alpha'_2}{p_3}^{\alpha'_3}\cdots{p_k}^{\alpha'_k}\rangle$. But, as $k \geq 3$, we have $\alpha_2\alpha_3\cdots\alpha_k > \alpha_1$. This leads to a contradiction. Thus $k = 2$ and $n = {p_1}^{\alpha_1}{p_2}^{\alpha_2}$.

As earlier, we can assume that $\alpha_1, \alpha_2 \geq 2$. Let, if possible, $\alpha_1 \neq \alpha_2$. Without loss of generality, let $2 \leq \alpha_1 < \alpha_2$. Now, on any hamiltonian circuit in $\Gamma^*(\mathbb{Z}_n)$, between any two vertices of the form $\langle {p_1}^i\rangle$ and $\langle {p_1}^j\rangle$ we have a vertex of the form $\langle {p_2}^t\rangle$ and between any two vertices of the form $\langle {p_2}^i\rangle$ and $\langle {p_2}^j\rangle$ we have a vertex of the form $\langle {p_1}^t\rangle$. Thus any Hamiltonian circuit should consist of an alternating run of vertices of the form $\langle {p_1}^i\rangle$ and $\langle {p_2}^j\rangle$. However, as $\alpha_1 < \alpha_2$, we have more vertices of the form $\langle {p_2}^j\rangle$ than that of the form $\langle {p_1}^i\rangle$, a contradiction. Thus $\alpha_1 = \alpha_2$. □

Theorem 3.2. *Let $n = {p_1}^{\alpha_1}{p_2}^{\alpha_2}\cdots{p_k}^{\alpha_k}$, where p_i's are distinct primes and $\alpha_i \geq 1$. Then $\Gamma^*(\mathbb{Z}_n)$ is perfect if and only if $k \leq 4$.*

Proof. We prove this by using Strong Perfect graph theorem. If $k \geq 5$, then there exists an induced 5-cycle in $\Gamma^*(\mathbb{Z}_n)$ as shown in Figure 1. Thus, in this case, $\Gamma^*(\mathbb{Z}_n)$ is not perfect. Let $k \leq 4$, i.e., n has at most 4 distinct prime factors p_1, p_2, p_3, p_4. Let, if possible, $\Gamma^*(\mathbb{Z}_n)$ admits an induced odd cycle of length $t \geq 5$, say $\langle h_1\rangle \sim \langle h_2\rangle \sim \cdots \sim \langle h_t\rangle \sim \langle h_1\rangle$. From the non-adjacency relations, we get $gcd(h_1, h_3), gcd(h_1, h_4), gcd(h_2, h_4), gcd(h_2, h_5), gcd(h_3, h_t) \neq 1$.

Let $p_1 \mid gcd(h_1, h_3)$. Then $p_1 \mid h_1$ and $p_1 \mid h_3$. Again, as $\langle h_t\rangle \sim \langle h_1\rangle$, we have $gcd(h_1, h_t) = 1$, i.e., $p_1 \nmid h_t$.

Similarly, as $\langle h_3\rangle \sim \langle h_4\rangle$, we have $p_1 \nmid h_4$, i.e., $p_1 \nmid gcd(h_1, h_4)$. Let $p_2 \mid gcd(h_1, h_4)$. Then $p_2 \mid h_1$ and $p_2 \mid h_4$. Now as $\langle h_3\rangle \sim \langle h_4\rangle$, we conclude $p_2 \nmid h_3$.

Again, as $p_1, p_2 \mid h_1$ and $\langle h_1\rangle \sim \langle h_2\rangle$, $p_1, p_2 \nmid h_2$, i.e., $p_1, p_2 \nmid gcd(h_2, h_4)$. Let $p_3 \mid gcd(h_2, h_4)$. Then $p_3 \mid h_2$ and $p_3 \mid h_4$. As $\langle h_2\rangle \sim \langle h_3\rangle$, we have $p_3 \nmid h_3$.

Thus $p_1, p_2, p_3 \nmid gcd(h_3, h_t)$. Let $p_4 \mid gcd(h_3, h_t)$. Then $p_4 \mid h_3$ and $p_4 \mid h_t$. As $\langle h_2\rangle \sim \langle h_3\rangle$, $p_4 \nmid h_2$. Again, as $\langle h_4\rangle \sim \langle h_5\rangle$, we have $p_3 \nmid h_5$.

From the above situation, we get $p_1, p_2, p_3, p_4 \nmid gcd(h_2, h_5)$. This is a contradiction, as $gcd(h_2, h_5) \neq 1$ and $k \leq 4$. Thus $\Gamma^*(\mathbb{Z}_n)$ does not admit any induced odd cycle of length $t \geq 5$.

Let, if possible, the complement of $\Gamma^*(\mathbb{Z}_n)$ admits an induced odd cycle of length $t \geq 5$, say $\langle h_1\rangle \sim \langle h_2\rangle \sim \cdots \sim \langle h_t\rangle \sim \langle h_1\rangle$. Note that in the complement graph, two vertices $\langle h_i\rangle$ and $\langle h_j\rangle$ are non-adjacent/adjacent according as $gcd(h_i, h_j)$ is equal/not equal to 1 respectively.

As $\langle h_1\rangle \sim \langle h_2\rangle$, we have $gcd(h_1, h_2) \neq 1$. Let $p_1 \mid gcd(h_1, h_2)$. Then $p_1 \mid h_1$ and $p_1 \mid h_2$. As $gcd(h_1, h_3) = 1$, $p_1 \nmid h_3$, i.e., $p_1 \nmid gcd(h_2, h_3)$. Similarly, we can conclude that p_1 does not divide any one of $gcd(h_3, h_4), gcd(h_4, h_5), gcd(h_1, h_t)$.

Let $p_2 \mid gcd(h_2, h_3)$. Then $p_2 \mid h_2$ and $p_2 \mid h_3$. As $gcd(h_2, h_4) = 1$, we have $p_2 \nmid h_4$, i.e., p_2 does not divide $gcd(h_3, h_4)$ and $gcd(h_4, h_5)$. Similarly, as $gcd(h_2, h_t) = 1$, we deduce $p_2 \nmid h_t$, i.e., $p_2 \nmid gcd(h_1, h_t)$.

As $p_1, p_2 \nmid gcd(h_3, h_4)$, let $p_3 \mid gcd(h_3, h_4)$. Then $p_3 \mid h_3$ and $p_3 \mid h_4$. As $gcd(h_1, h_3) = 1$, we have $p_3 \nmid h_1$, i.e., $p_3 \nmid gcd(h_1, h_t)$. Similarly, as $gcd(h_3, h_5) = 1$, we have $p_3 \nmid h_5$, i.e., $p_3 \nmid gcd(h_4, h_5)$.

As $p_1, p_2, p_3 \nmid gcd(h_4, h_5)$, let $p_4 \mid gcd(h_4, h_5)$. Then $p_4 \mid h_4$ and $p_4 \mid h_5$. As $gcd(h_1, h_4) = 1$, we have $p_4 \nmid h_1$, i.e., $p_4 \nmid gcd(h_1, h_t)$.

Thus $p_1, p_2, p_3, p_4 \nmid gcd(h_1, h_t)$. But this is a contradiction, as $gcd(h_1, h_t) > 1$ and n has at most four distinct prime factors. Thus $\Gamma^*(\mathbb{Z}_n)^c$ does not admit an induced odd cycle of length $t \geq 5$.

Hence, by strong perfect graph theorem, the theorem follows. □

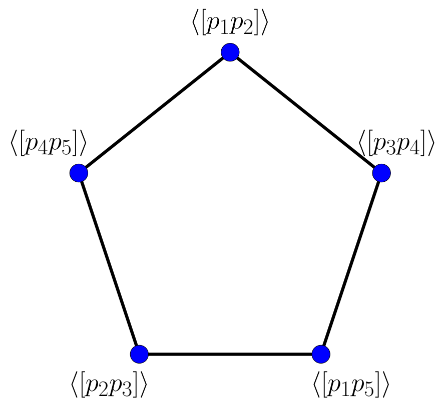

Figure 1. Induced 5-cycle in $\Gamma^*(\mathbb{Z}_n)$, for $k \geq 5$

Theorem 3.3. *Let $n = {p_1}^{\alpha_1}{p_2}^{\alpha_2}\cdots{p_k}^{\alpha_k}$, where p_i's are distinct primes, $k \geq 2$ and $\alpha_i \geq 1$. Then the domination number of $\Gamma^*(\mathbb{Z}_n)$,*

$$\gamma(\Gamma^*(\mathbb{Z}_n)) = \begin{cases} 1, & \text{if } n = {p_1}^{\alpha_1}p_2. \\ k, & \text{otherwise.} \end{cases}$$

Proof. Clearly $\{\langle p_1\rangle, \langle p_2\rangle, \ldots, \langle p_k\rangle\}$ is a dominating set for $\Gamma^*(\mathbb{Z}_n)$ of size k. Thus $\gamma(\Gamma^*(\mathbb{Z}_n)) \leq k$.

Let $S = \{\langle x_1\rangle, \langle x_2\rangle, \ldots, \langle x_{k-1}\rangle\}$ be a dominating set of $\Gamma^*(\mathbb{Z}_n)$ of size $k-1$. Let $m = p_1p_2p_3\cdots p_k$. Out of the k vertices $\langle m/p_1\rangle, \langle m/p_2\rangle, \ldots, \langle m/p_k\rangle$, at least one does not belong to S. Without loss of generality, let $\langle m/p_1\rangle \notin S$ and $\langle m/p_1\rangle \sim \langle x_1\rangle$. Thus, by Lemma 2.1, $x_1 = p_1^{\alpha'_1}$, where $1 \leq \alpha'_1 \leq \alpha_1$. Thus $\langle x_1\rangle$ is not adjacent to any of the $k-1$ vertices $\langle m/p_2\rangle, \langle m/p_3\rangle, \ldots, \langle m/p_k\rangle$. Again, by similar argument, not all of these $k-1$ vertices belong to S. Without loss of generality, let $\langle m/p_2\rangle \notin S$ and $\langle m/p_2\rangle \sim \langle x_2\rangle$. Proceeding similarly, we get $x_2 = p_2^{\alpha_2'}$, where $1 \le \alpha_2' \le \alpha_2$. Thus $\langle x_1\rangle$ and $\langle x_2\rangle$ are not adjacent to any of the $k-2$ vertices $\langle m/p_3\rangle, \dots, \langle m/p_k\rangle$. Continuing in this way, we get $x_i = p_i^{\alpha_i'}$ for $i = 1, 2, \dots, k-1$. However, in that case, $\langle m/p_k\rangle$ neither belong to S nor adjacent to any element of S, a contradiction. Hence $\gamma(\Gamma^*(\mathbb{Z}_n)) = k$.

Note that the proof does not work if $k = 2$ and exactly one of the two powers is 1. Because in that case, one of $\langle m/p_1\rangle$ and $\langle m/p_2\rangle$ is not a vertex of $\Gamma^*(\mathbb{Z}_n)$, i.e., an isolated vertex of $\Gamma(\mathbb{Z}_n)$. If $k = 2$ and $n = {p_1}^{\alpha_1} p_2$, then $\langle p_2\rangle$ dominates $\Gamma^*(\mathbb{Z}_n)$. □

4. Isomorphisms

In this section, we discuss the conditions under which co-maximal subgroup graphs defined over different cyclic groups are isomorphic. For that, we start with the following definition.

Definition 4.1. *Two positive integers n and m are said to be of same prime-factorization type if $n = {p_1}^{\alpha_1}{p_2}^{\alpha_2}\cdots{p_k}^{\alpha_k}$ and $m = {q_1}^{\beta_1}{q_2}^{\beta_2}\cdots{q_k}^{\beta_k}$ where p_i, q_i's are primes and there exists $\sigma \in S_k$ such that $\alpha_i = \beta_{\sigma(i)}$ for $i = 1, 2, \dots, k$.*

Theorem 4.2. *Let n and m be two integers. Then $\Gamma(\mathbb{Z}_n) \cong \Gamma(\mathbb{Z}_m)$ if and only if m and n are of same prime-factorization type.*

Proof. If m and n are of same prime-factorization type, then the result is obvious. Let $\Gamma(\mathbb{Z}_n) \cong \Gamma(\mathbb{Z}_m)$, then as their clique numbers are equal, both m and n have same number of distinct prime factors. Let $n = {p_1}^{\alpha_1}{p_2}^{\alpha_2}\cdots{p_k}^{\alpha_k}$ and $m = {q_1}^{\beta_1}{q_2}^{\beta_2}\cdots{q_k}^{\beta_k}$. Also as they have same number of isolated vertices, $\alpha_1 \cdot \alpha_2 \cdots \alpha_k = \beta_1 \cdot \beta_2 \cdots \beta_k$.

Without loss of generality, let $\alpha_1 = min\{\alpha_1, \alpha_2, \dots, \alpha_k, \beta_1, \beta_2, \dots, \beta_k\}$. If possible, let $\alpha_1 \notin \{\beta_1, \beta_2, \dots, \beta_k\}$. Now, note that any vertex of the form $\langle {p_2}^{\alpha_2'}{p_3}^{\alpha_3'}\cdots{p_k}^{\alpha_k'}\rangle$ $(1 \le \alpha_i' \le \alpha_i)$ in $\Gamma(\mathbb{Z}_n)$ is adjacent to only α_1 vertices, namely $\langle p_1\rangle, \langle p_1^2\rangle, \dots, \langle p_1^{\alpha_1}\rangle$. Thus $\Gamma(\mathbb{Z}_n)$ has $\alpha_2\alpha_3\cdots\alpha_k$ vertices of degree α_1. As $\alpha_1 \le min\{\beta_1, \beta_2, \dots, \beta_k\}$ and $\alpha_1 \notin \{\beta_1, \beta_2, \dots, \beta_k\}$, from Theorem 2.2, it follows that $\Gamma(\mathbb{Z}_m)$ has no vertex of degree α_1, a contradiction. Thus $\alpha_1 = \beta_i$ for some i. By suitable renaming, let $\alpha_1 = \beta_1$.

Again, without loss of generality, let $\alpha_2 = min\{\alpha_2, \dots, \alpha_k, \beta_2, \dots, \beta_k\}$. If possible, let $\alpha_2 \notin \{\beta_2, \dots, \beta_k\}$. If $\alpha_2 \ne \beta_1$, then by a similar argument, $\Gamma(\mathbb{Z}_m)$ has no vertex of degree α_2, a contradiction. Thus, we assume that $\alpha_2 = \beta_1 = \alpha_1$. Then $\Gamma(\mathbb{Z}_n)$ has $\alpha_2\alpha_3\cdots\alpha_k + \alpha_1\alpha_3\cdots\alpha_k$ of degree α_1 and $\Gamma(\mathbb{Z}_m)$ has $\beta_2\beta_3\cdots\beta_k$ of degree α_1. As $\Gamma(\mathbb{Z}_n) \cong \Gamma(\mathbb{Z}_m)$, we have

$$\alpha_2\alpha_3\cdots\alpha_k + \alpha_1\alpha_3\cdots\alpha_k = \beta_2\beta_3\cdots\beta_k,$$
$$\text{i.e., } \alpha_3\cdots\alpha_k(\alpha_1+\alpha_2) = \frac{\alpha_1\cdot\alpha_2\cdots\alpha_k}{\beta_1} \text{ (as } \alpha_1\cdot\alpha_2\cdots\alpha_k = \beta_1\cdot\beta_2\cdots\beta_k)$$
$$\text{i.e., } \beta_1(\alpha_1+\alpha_2) = \alpha_1\alpha_2, \text{i.e., } 2\alpha_1^2 = \alpha_1^2, \text{a contradiction.}$$

Thus $\alpha_2 = \beta_i$ for some $i \in \{2, 3, \dots, k\}$. By a suitable renaming, let $\alpha_2 = \beta_2$.

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Proceeding this way, suppose in the $(l-1)$-th step, we get $\alpha_i = \beta_i$ for $i = 1, 2, \ldots, l-1$. Without loss of generality, let $\alpha_l = min\{\alpha_l, \ldots, \alpha_k, \beta_l, \ldots, \beta_k\}$. If possible, let $\alpha_l \notin \{\beta_l, \ldots, \beta_k\}$. If $\alpha_l \notin \{\beta_1, \beta_2, \ldots, \beta_{l-1}\}$, then by a similar argument, $\Gamma(\mathbb{Z}_m)$ has no vertex of degree α_l, a contradiction. Thus, we assume that $\alpha_l \in \{\beta_1, \beta_2, \ldots, \beta_{l-1}\}$. Let $\alpha_l = \beta_p = \beta_{p+1} = \cdots = \beta_{l-1} = \alpha_p = \alpha_{p+1} = \cdots = \alpha_{l-1}$ for some $1 \leq p \leq l-1$.

Therefore, $\Gamma(\mathbb{Z}_n)$ has

$$(\alpha_1\alpha_2\cdots\alpha_{p-1}\alpha_{p+1}\cdots\alpha_k) + (\alpha_1\cdots\alpha_p\alpha_{p+1}\cdots\alpha_k) + \cdots + (\alpha_1\cdots\alpha_{l-1}\alpha_{l+1}\cdots\alpha_k)$$
$$= \alpha_1\cdots\alpha_k\left(\frac{1}{\alpha_p} + \frac{1}{\alpha_{p+1}} + \cdots + \frac{1}{\alpha_l}\right) \text{ vertices of degree } \alpha_l.$$

Similarly, $\Gamma(\mathbb{Z}_m)$ has

$$\beta_1\cdots\beta_k\left(\frac{1}{\beta_p} + \frac{1}{\beta_{p+1}} + \cdots + \frac{1}{\beta_{l-1}}\right) \text{ vertices of degree } \alpha_l.$$

Now, as $\Gamma(\mathbb{Z}_n) \cong \Gamma(\mathbb{Z}_m)$, we have

$$\alpha_1\cdots\alpha_k\left(\frac{1}{\alpha_p} + \frac{1}{\alpha_{p+1}} + \cdots + \frac{1}{\alpha_l}\right) = \beta_1\cdots\beta_k\left(\frac{1}{\beta_p} + \frac{1}{\beta_{p+1}} + \cdots + \frac{1}{\beta_{l-1}}\right)$$
$$\text{i.e., } \left(\frac{1}{\alpha_p} + \frac{1}{\alpha_{p+1}} + \cdots + \frac{1}{\alpha_l}\right) = \left(\frac{1}{\alpha_p} + \frac{1}{\alpha_{p+1}} + \cdots + \frac{1}{\alpha_{l-1}}\right) \Rightarrow \left(\frac{l-p+1}{\alpha_l}\right) = \left(\frac{l-p}{\alpha_l}\right),$$

a contradiction. Thus, by a suitable renaming, we get $\alpha_l = \beta_l$, and hence by induction, the theorem follows. □

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