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RESTRICTIONS ON SETS OF CONJUGACY CLASS SIZES IN ARITHMETIC PROGRESSIONS

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Abstract. We continue the investigation, that began in [M. Bianchi, A. Gillio and P. P. Pálfy, A note on finite groups in which the conjugacy class sizes form an arithmetic progression, *Ischia group theory 2010, World Sci. Publ.*, Hackensack, NJ (2012) 20–25.] and [M. Bianchi, S. P. Glasby and Cheryl E. Praeger, Conjugacy class sizes in arithmetic progression, *J. Group Theory*, **23** no. 6 (2020) 1039–1056.], into finite groups whose set of nontrivial conjugacy class sizes form an arithmetic progression. Let G be a finite group and denote the set of conjugacy class sizes of G by $\mathrm{cs}(G)$. Finite groups satisfying $\mathrm{cs}(G) = \{1, 2, 4, 6\}$ and $\{1, 2, 4, 6, 8\}$ are classified in [M. Bianchi, S. P. Glasby and Cheryl E. Praeger, Conjugacy class sizes in arithmetic progression, *J. Group Theory*, **23** no. 6 (2020) 1039–1056.] and [M. Bianchi, A. Gillio and P. P. Pálfy, A note on finite groups in which the conjugacy class sizes form an arithmetic progression, *Ischia group theory 2010, World Sci. Publ.*, Hackensack, NJ (2012) 20–25.], respectively, we demonstrate these examples are rather special by proving the following. There exists a finite group G such that $\mathrm{cs}(G) = \{1, 2^\alpha, 2^{\alpha+1}, 2^\alpha 3\}$ if and only if $\alpha = 1$. Furthermore, there exists a finite group G such that $\mathrm{cs}(G) = \{1, 2^\alpha, 2^{\alpha+1}, 2^\alpha 3, 2^{\alpha+2}\}$ and α is odd if and only if $\alpha = 1$.

1. Introduction

Let G be a finite group and $\mathrm{cs}(G)$ the set of conjugacy class sizes of elements of G. That is, $\mathrm{cs}(G) = \{|x^G| : x \in G\}$ where x^G denotes the conjugacy class of x in G. The question as to which sets of natural numbers can occur as $\mathrm{cs}(G)$ for some finite group G, and also the relation between $\mathrm{cs}(G)$ and the algebraic structure of G, are questions that have long interested mathematicians, see [9] for an overview of results. For example in 1953 Itô [15] proved that if $\mathrm{cs}(G) = \{1, n\}$ then $n = p^a$ for some prime p and $G \cong P \times A$

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where P is a p-group and A is an abelian p'-group, more recently Ishikawa proved that such a group has nilpotency class at most 3 [14]. It is worth noting that if A is an abelian group then $\mathrm{cs}(G \times A) = \mathrm{cs}(G)$ and thus, when working with $\mathrm{cs}(G)$, we are always working modulo abelian direct factors. In [10] the authors prove that for a given prime p if S is any set of p-powers containing 1 then $S = \mathrm{cs}(P)$ for some finite p-group P of nilpotency class 2. However, in general, there are many restrictions on the sets that can occur as sets of conjugacy class sizes. In particular, in [2] the authors prove that if the largest two conjugacy class sizes m and n are coprime, then $\mathrm{cs}(G) = \{1, m, n\}$ and G is quasi-Frobenius (that is $G/Z(G)$ is a Frobenius group).

Recently there has been interest in when $\mathrm{cs}(G)$ or $\mathrm{cs}^*(G) = \mathrm{cs}(G) \setminus \{1\}$ can be an arithmetic progression and, when this is so, the algebraic implications on the group have been investigated. That is, we ask when can $\mathrm{cs}(G)$ or $\mathrm{cs}^*(G)$ be of the form $\{a, a+d, a+2d, \ldots, a+kd\}$ for natural numbers a, d and k? A simple argument shows that if $\mathrm{cs}(G)$ is an arithmetic progression with at least 3 terms then we are in the situation described above, that is the largest two conjugacy class sizes are coprime and thus $\mathrm{cs}(G) = \{1, m, n\}$. This situation has been analysed further in [4] where the authors prove that if $\mathrm{cs}(G)$ is an arithmetic progression (with at least 3 terms) then necessarily $\mathrm{cs}(G) = \{1, 2, 3\}$. The authors classify all such G.

A less restrictive condition is to insist $\mathrm{cs}^*(G)$ is an arithmetic progression. We call $|\mathrm{cs}^*(G)|$ the conjugate rank of G and note that groups with conjugate rank 2 have been classified [12]. We suspect it is quite rare for $\mathrm{cs}^*(G)$ to be an arithmetic progression of size at least 3, simple arguments (see the next section) show that such a group must have a centre but cannot be nilpotent. In [4] the authors classify all groups which satisfy $\mathrm{cs}^*(G) = \{2, 4, 6\}$ and groups which satisfy $\mathrm{cs}^*(G) = \{2, 4, 6, 8\}$ are classified in [3]. It is interesting to note that if G is a group with order divisible by only two primes, conjugate rank at least 4 and $\mathrm{cs}^*(G)$ an arithmetic progression then $\mathrm{cs}^*(G) = \{m, 2m, 3m, 4m\}$ for some $m = 2^\alpha 3^\beta$ and natural numbers α and β (this follows from the analysis of primitive arithmetic progressions in [4, Lemma 5]). Thus we are led to considering this scenario, we restrict to the case when $\beta = 0$ and also consider the conjugate rank 3 case. We prove the following theorem, indicating that the examples above are rather special. Our proofs are quite different to those used in [4] and [3].

Theorem 1.1. *There exists a finite group G with*

(i) $\mathrm{cs}(G) = \{1, 2^\alpha, 2^{\alpha+1}, 2^\alpha 3\}$ *if and only if* $\alpha = 1$.

(ii) $\mathrm{cs}(G) = \{1, 2^\alpha, 2^{\alpha+1}, 2^\alpha 3, 2^{\alpha+2}\}$ *and α odd if and only if* $\alpha = 1$.

We note that, for $\alpha = 1$, in (i) the Sylow 3-subgroup of G is normal, whereas in (ii) the Sylow 2-subgroup of G is normal. The α even, conjugate rank 4 case remains open. If such a G were to exist it would follow, from the analysis in this paper, that neither the Sylow 2 or 3-subgroup would be normal and moreover $G/F(G)$ would be isomorphic to S_3, where $F(G)$ denotes the Fitting subgroup of G. Thus we are led to the following (increasingly general) questions.

Question 1. *Suppose α is even. Does there exist a finite group G satisfying* $\mathrm{cs}^*(G) = \{2^\alpha, 2^{\alpha+1}, 2^\alpha 3, 2^{\alpha+2}\}$?

Question 2. *For which $m = 2^\alpha 3^\beta$ does there exist a finite group G satisfying* $\mathrm{cs}^*(G) = \{m, 2m, 3m, 4m\}$*?*

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Question 3. *For which natural numbers a, k and d does there exist a finite group G satisfying $\mathrm{cs}^*(G) = \{a, a+d, a+2d, \ldots, a+kd\}$?*

Notation and terminology used throughout the paper are mostly standard. For example for G a group and $x \in G$ we denote the centraliser of x in G by $C_G(x)$, $Z(G)$ denotes the centre of G, $\Phi(G)$ is the Frattini subgroup of G and $O_2(G)$ denotes the 2-core of G, that is, the largest normal 2-subgroup in G. For p a prime a p-element is a non-trivial element of order p-power and a mixed element is an element whose order is divisible by at least two distinct primes. However we also use the terminology of Itô and call $|x^G|$ the *index of x in G* or simply the *index of x* when G is clear. Similarly $\mathrm{cs}(G)$ is called the *set of indices of G*. All groups considered are finite.

2. Preliminaries & Discussion

Thoughout this paper we will assume well-known properties of indices. Namely that if x and y are commuting elements of coprime order then $C_G(xy) = C_G(x) \cap C_G(y)$, that if N is a normal subgroup of G then both $|x^N|$ and $|(xN)^{G/N}|$ divide $|x^G|$ and finally if all indices of a group G are coprime to a prime p then the Sylow p-subgroup of G is an abelian direct factor [6]. We start with a definition.

Definition 2.1. *We say a group G has property $\mathcal{AP}$, or is an $\mathcal{AP}$ group, if G has conjugate rank at least 3 and $\mathrm{cs}^*(G) = \{a, a+d, a+2d, \ldots, a+kd\}$ for some natural numbers a, d and k.*

The following lemma shows that if G is an $\mathcal{AP}$ group then there exists a prime p which divides all nontrivial indices of elements of G.

Lemma 2.2. *Suppose $\mathrm{cs}^*(G) = \{a, a+d, a+2d, \ldots, a+kd\}$ with a and d coprime, then $|\mathrm{cs}^*(G)| \leq 2$.*

Proof. Note $\gcd(a+(k-1)d, a+kd) = \gcd(a, d) = 1$. The result follows from [2]. □

Lemma 2.3. *Suppose p is a prime and p^α divides the index of all noncentral elements of a finite group G. Let P be a Sylow p-subgroup of G then $Z(P) = P \cap Z(G)$ and p^α divides $|Z(P)|$.*

Proof. By the class equation p^α divides $|Z(G)|$. Also, any element in the centre of a Sylow p-subgroup must be central and thus $Z(P) = P \cap Z(G)$. The result follows. □

The previous two lemmas show that if G is an $\mathcal{AP}$ group then G has a nontrivial centre. The next lemma shows that nilpotent groups do not have property $\mathcal{AP}$.

Lemma 2.4. *Suppose $\mathrm{cs}^*(G) = \{a, a+d, a+2d, \ldots, a+kd\}$ and $|\mathrm{cs}^*(G)| \geq 3$. Then G is not nilpotent.*

Proof. Suppose G is nilpotent. As $|\mathrm{cs}^*(G)| \geq 3$ we know by Lemma 2.2 that there exists a prime p such that p divides all nontrivial indices of G, this forces G to be a p-group. Thus $\mathrm{cs}^*(G) = \{p^{a_1}, p^{a_2}, \ldots, p^{a_k}\}$ for some positive integers $a_1 < a_2 < \cdots < a_k$. But then $p^{a_3} = p^{a_2} + d$ and so p^{a_2} divides d. However also $p^{a_2} = p^{a_1} + d$ which leads to p^{a_2} dividing p^{a_1}, a contradiction. □

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A group is called an F-group if given any noncentral elements x and y then $C_G(x) \not< C_G(y)$. It follows that a sufficient condition to be an F-group is to require that given any noncentral elements x and y then $|y^G|$ does not divide $|x^G|$. The F-groups have been classified [16], a check of the list shows that no F-group has property $\mathcal{A}P$. Thus any example of an $\mathcal{A}P$ group will have to include some divisibility of indices. However even with divisibility we run into obstacles. Applying the main result in [11] shows that many arithmetic progressions cannot occur as sets of indices. For example it follows that there does not exist a group G with $\mathrm{cs}^*(G) = \{2, 6, 10, 14\}$ or more generally with $\mathrm{cs}^*(G) = \{a, 3a, 5a, 7a\}$ and a coprime to 105.

We now focus on proving our main result. The following lemmas will be useful.

Lemma 2.5. *If p is the highest power of p which divides any index of G then $\Phi(P)$ is central in G for any Sylow p-subgroup P of G.*

Proof. Let $x \in G$ then there exists a Sylow p-subgroup $\hat{P}$ and a subgroup P_0 of index p in $\hat{P}$ such that $C_G(x) \geq P_0$. There exists $g \in G$ such that $\hat{P}^g = P$ and thus $C_G(x^g) \geq P_0^g \geq \Phi(P)$. So $x^g \in C_G(\Phi(P))$. So, we have shown that any element of G is conjugate to an element in $C_G(\Phi(P))$ and thus $C_G(\Phi(P)) = G$ by a result of Burnside [5, §26]. □

The following is adapted from [8, Proposition 1].

Lemma 2.6. *Suppose P is an abelian Sylow p-subgroup and Z is a central p-subgroup of a group G. Then $\mathrm{cs}(G) = \mathrm{cs}(G/Z)$.*

Proof. We proceed by induction on the order of Z. First suppose Z has order p and denote G/Z by $\bar{G}$. Suppose there exists an element $x \in G$ such that $|x^G| \neq |\bar{x}^{\bar{G}}|$. Then $x^g = xz$ for some $g \in G$ where $Z = \langle z \rangle$. As the order of x equals the order of $x^g = xz$ it follows that p, the order of z, divides the order of x. Write $x = x_p x_{p'}$ as a product of its p- and p'-parts. By considering x^m, where m is the order of $x_{p'}$, we can assume x is a p-element and thus has p'-index. Now $x^{g^p} = xz^p = x$ and so $g^p \in C_G(x)$. Writing $g = g_p g_{p'}$ as a product of its p- and p'- parts gives that $g_{p'} \in C_G(x)$ and so $x^{g_p} = x^g = xz$. Thus $\langle x, z, g_p \rangle$ is a p-group and so abelian, a contradiction. Thus $\mathrm{cs}(G) = \mathrm{cs}(G/Z)$ in this case.

Now suppose $|Z| > p$. Let Z_0 be a subgroup of index p in Z then, by induction, $\mathrm{cs}(G) = \mathrm{cs}(G/Z_0)$. As $(G/Z_0)/(Z/Z_0) \cong G/Z$, it follows, again by induction, that $\mathrm{cs}(G/Z) = \mathrm{cs}(G/Z_0) = \mathrm{cs}(G)$, as required. □

3. Proof of Main Result

Suppose G is a finite group with either $\mathrm{cs}^*(G) = \{2^\alpha, 2^{\alpha+1}, 2^\alpha 3\}$ or $\mathrm{cs}^*(G) = \{2^\alpha, 2^{\alpha+1}, 2^\alpha 3, 2^{\alpha+2}\}$ and $\alpha \geq 1$. As mentioned in the previous section we can assume G is a $\{2,3\}$-group. As $2^\alpha 3 \in \mathrm{cs}^*(G)$ and $3 \notin \mathrm{cs}^*(\mathrm{G})$, it follows that G is not nilpotent. We let P denote a Sylow 3-subgroup of G. As in [7], we call a group G a q-Baer group if q is a prime dividing the order of G and all q-elements have prime power index.

Step 1 *Let $\bar{G} = G/O_2(G)Z(G)$. The Sylow 3-subgroup $\bar{P}$ of $\bar{G}$ is normal, elementary abelian and $|\bar{G} : \bar{P}| \leq 2$.*

Note $PZ(G)/Z(G)$ is elementary abelian by Lemma 2.5. Since $P \not\leq Z(G)$, the group $\bar{G}$ has an elementary abelian Sylow 3-subgroup where every 3-element has index a power of 2. Thus $\bar{G}$ is a 3-Baer group, and

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as $O_2(\bar{G}) = 1$ so the Sylow 3-subgroup of $\bar{G}$ is normal by [7, Theorem A(a)]. Note that any 2-element of 2-power index in G lies in $O_2(G)$, by Wielandt [1, Lemma 6]. Thus $\bar{G}/\bar{P}$ is a 2-group acting faithfully on an elementary abelian 3-group so that every 2-element has index 3 in its action on $\bar{P}$. If t is any 2-element acting non-trivially on $\bar{P}$ we get $\bar{P} = [\bar{P}, t] \oplus C_{\bar{P}}(t)$ where $|[\bar{P}, t]| = 3$. So t inverts $[\bar{P}, t]$ and has order 2. Hence the determinant of the linear transformation induced by t is -1. This holds for every element in $\bar{G}/\bar{P}$ and thus $|\bar{G}/\bar{P}| \leq 2$.

Step 2 *Suppose x is a 3-element and t is a 2-element with $[x, t] = 1$ and $C_{O_2(G)}(x) \geq C_{O_2(G)}(t)$, then x centralises $O_2(G)$.*

This is a direct consequence of Thompson's $P \times Q$ lemma [13, Theorem 4.31].

Step 3 *We can assume $|P| = 3$.*

If all 2-elements have 2-power index then the Sylow 2-subgroup of G is a direct factor [7, Lemma 3], which is clearly false. Choose t a 2-element of index $2^\alpha 3$. If $O_2(G)$ is not a Sylow 2-subgroup of G choose $t \notin O_2(G)$ (this is possible by [1, Lemma 6]) otherwise choose $t \in O_2(G)$. Suppose x is a non-central 3-element in $C_G(t)$. Then, as $C_G(xt) = C_G(x) \cap C_G(t)$, it follows that xt has index $2^\alpha 3$ and $C_G(x) \geq C_G(t)$. Applying Step 2 gives that x centralises $O_2(G)$. Also x centralises t and so, by Step 1, centralises a Sylow 2-subgroup and thus has index prime to 2, which is false. So $C_G(t)$ contains no non-central 3-elements. If P_0 is the Sylow 3-subgroup of $C_G(t)$ it is central in G. Let P be a Sylow 3-subgroup of G containing P_0 then $|P : P_0| = 3$. Thus P is abelian. Then $\mathrm{cs}(G/P_0) = \mathrm{cs}(G)$ by Lemma 2.6 and it is enough to consider G/P_0, equivalently that $|P| = 3$.

Step 4 *If P is normal then $\mathrm{cs}^*(G) = \{2, 4, 6\}$.*

We show that P normal implies $\alpha = 1$. The result then follows by the analysis in [3] and [4]. Since in [3] it is shown that a group with $\mathrm{cs}^*(G) = \{2, 4, 6, 8\}$ does not have a normal Sylow 3-subgroup. However a group with $\mathrm{cs}^*(G) = \{2, 4, 6\}$ exists and does indeed have a normal Sylow 3-subgroup, see [4].

As P is normal then $O_2(G)$ cannot be the whole Sylow 2-subgroup of G as this would imply G is nilpotent, which is false. Thus, $O_2(G)$ has index 2 in the Sylow 2-subgroup of G by Step 1. As P is abelian it follows that any 3-element of G has index 2, i.e. $\alpha = 1$ as claimed.

Step 5 *Suppose P is not normal. Then the 2-elements are either central or have index 2^α or $2^\alpha 3$, and these all occur. Furthermore the 3-elements have index $2^{\alpha+1}$.*

Clearly any element central in a Sylow 2-subgroup has 3-power index and so is central in G as $\mathrm{cs}^*(G)$ contains no 3-powers. So we have central 2-elements. If all 2-elements have 2-power index then the Sylow 2-subgroup is a direct factor [7, Lemma 3], which is clearly false, so there exist 2-elements of index $2^\alpha 3$. Also as any element of index 2^α lies in the Fitting subgroup of G [12, Proposition 3.1], we have 2-elements of index 2^α. Note 3-elements do not lie in the Fitting subgroup so do not have index 2^α. Furthermore, as P is abelian 3-elements do not have index $2^\alpha 3$.

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In the rank 4 case suppose there exists a 2-element of index $2^{\alpha+2}$. However then G has a normal 2-complement [6, Theorem 1], contradicting our hypothesis.

We need to consider the case that t is a 2-element of index $2^{\alpha+1}$. Since 3 divides $|G|$ and not $|t^G|$, it divides $|C_G(t)|$. Let x be a 3-element in $C_G(t)$. By the first paragraph the index of x is either $2^{\alpha+1}$ or $2^{\alpha+2}$. Now $C_G(xt) = C_G(x) \cap C_G(t)$ so xt has index $2^{\alpha+1}$ or $2^{\alpha+2}$. If xt has index $2^{\alpha+1}$ then $C_G(xt) = C_G(x) = C_G(t)$ and x centralises $O_2(G)$ by Step 2, giving x has index 2, a contradiction. Suppose xt has index $2^{\alpha+2}$. Then $C_G(xt)$ is normal in $C_G(t)$. Choose $g \in C_G(t) \setminus C_G(xt)$ and let $P = \langle x \rangle \leq C_G(xt)$. Then g acts on P and so inverts x. If $g \in O_2(G)$ then $[g, x] \in O_2(G)$ forces $x \in O_2(G)$, a contradiction. Thus all elements in $C_G(t) \cap O_2(G)$ centralise x. Applying Step 2 gives that x centralises $O_2(G)$, again a contradiction.

As a Sylow 3-subgroup P is cyclic of order 3 it follows that all 3-elements have the same index. As mentioned previously 3-elements do not have index 2^α or $2^\alpha 3$. Thus, in the conjugate rank 3 case, the result follows. For the conjugate rank 4 case suppose the 3-elements have index $2^{\alpha+2}$. But then any element of mixed order also has this index and there are no elements of index $2^{\alpha+1}$.

Step 6 *Suppose* $\mathrm{cs}^*(G) = \{2^\alpha, 2^{\alpha+1}, 2^\alpha 3\}$ *then* $\alpha = 1$.

Let $P = \langle x \rangle$, where x is an element of order 3 by Step 3. If P is normal the result holds by Step 4, so assume P not normal. Thus x has index $2^{\alpha+1}$ by Step 5. Let t be a noncentral 2-element in $C_G(x)$, then $|t^G| = 2^\alpha$ by Step 5. Now $C_G(xt) = C_G(x) \cap C_G(t) \geq P$ and thus $|(xt)^G| = 2^{\alpha+1}$ and, in particular, $C_G(x) = C_G(xt)$ has index 2, and so is normal, in $C_G(t)$. Now we proceed as in Step 5 and show that all elements in $C_G(t) \cap O_2(G)$ centralise x, then applying Step 2 gives that x centralises $O_2(G)$, this contradicts $|x^G| = 2^{\alpha+1}$.

Thus the first part of our Theorem is proved. From now on we suppose G is a finite group with $\mathrm{cs}^*(G) = \{2^\alpha, 2^{\alpha+1}, 2^\alpha 3, 2^{\alpha+2}\}$. By Step 4 we know P is not normal in G.

Step 7 *The number of Sylow 3-subgroups of* G *is* $2^{\alpha+1}$ *if* α *is odd and* 2^α *if* α *is even. Furthermore,* α *is odd if and only if the Sylow 2-subgroup of* G *is normal.*

Suppose $P = \langle x \rangle$, where x is an element of order 3 by Step 3, then $C_G(P) = C_G(x)$. Further, $|N_G(P) : C_G(P)| = 1$ or 2 since $|P| = 3$. The number of Sylow 3-subgroups, denoted n_3, satisfies $n_3 \equiv 1 \bmod 3$ and $|G : C_G(x)| = 2^{\alpha+1}$ by Step 5, hence $n_3 = 2^{\alpha+1}$ if α is odd and $n_3 = 2^\alpha$ if α is even. Furthermore, if $|N_G(P) : C_G(P)| = 1$ then G has a normal 3-complement by Burnside's p-complement theorem [13, Theorem 5.13], so a normal Sylow 2-subgroup. Finally note that if $t \in N_G(P) \setminus C_G(P)$ then $t \notin O_2(G)$, so if G has a normal Sylow 2-subgroup then α is odd.

Step 8 *Suppose* α *is odd, then* $\alpha = 1$.

Since α is odd the Sylow 2-subgroup of G is normal by Step 7. As G is not nilpotent, by Step 5 it follows that the 2-elements are central or have index 2^α or $2^\alpha 3$. Let T denote the unique Sylow 2-subgroup of G and suppose t is a non-central 2-element then $|C_G(t) : C_T(t)| = |TC_G(t) : T|$ which is equal to 1 or 3. Further,

$$|G : C_G(t)||C_G(t) : C_T(t)| = |G : T||T : C_T(t)| = 3|T : C_T(t)|,$$

and so $|T : C_T(t)| = 2^\alpha$. Thus $T/Z(T)$ has exponent 2, by [14, Corollary 2.2], and is therefore abelian. Also $Z(T) = Z(G)$. Let $P = \langle x \rangle$ be a Sylow 3-subgroup of G. Then $C_T(P)$ contains $Z(T)$ so is normalised by T and P so $C_T(P)$ is normal. But then $C_T(P)$ centralises all Sylow 3-subgroups. By Step 5 an element of index $2^{\alpha+2}$ is a mixed element. Thus we can choose $y \in C_T(P)$ such that xy has index $2^{\alpha+2}$. So, $|C_G(y) : C_G(x) \cap C_G(y)| = 4$ and thus y commutes with exactly 4 Sylow 3-subgroups. However $y \in C_T(P)$ so y centralises all Sylow 3-subgroups. By the previous step, G has $2^{\alpha+1}$ Sylow 3-subgroups. So $2^{\alpha+1} = 4$ and $\alpha = 1$.

This completes the proof of our Theorem.

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