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GELFAND PAIRS ASSOCIATED WITH THE ACTION OF GRAPH AUTOMATON GROUPS

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Abstract. Graph automaton groups constitute a special class of automaton groups constructed from a graph. In this paper, we show that the action of any graph automaton group on each level of the rooted regular tree gives rise to a Gelfand pair. In particular, we determine the irreducible submodules of the action of such a group on the space of functions defined on each level of the tree, and we exhibit the corresponding spherical functions.

1. Introduction

Automaton groups can be regarded as automorphism groups of rooted regular trees and they are important examples of groups that have peculiar and exotic properties. Strongly developed after the introduction of the Grigorchuk group [13], which was the first example of a group of intermediate growth (i.e., faster than polynomial and slower than exponential), the theory of automaton groups (and more generally the theory of self-similar groups) has led to the discovery of new examples of amenable groups, of groups with intermediate growth, and of iterated monodromy groups of complex maps (we refer the interested reader to [2, 14, 15] and references therein for more details on this theory). In the last decades, automaton groups have been studied from different points of view: algebraic, geometric and combinatorial (e.g., in relation to the structure of the associated Schreier graphs), dynamical (looking at their action on the boundary of the tree), and algorithmic (in relation to decision problems). Gelfand pairs in the context of self-similar groups were studied in [1, 3] and

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further investigated in [9, 10, 11]. The theory of Gelfand pairs is strictly related to harmonic analysis and representation theory and has very important applications also in probability and statistics (see, for instance, [12]). Let G be a group and let K be a subgroup of G. Then (G, K) is said to be a Gelfand pair if the algebra of bi-K-invariant functions on G is commutative under convolution, or, equivalently, when the associated permutation representation, that is, the representation of G on the space $L(X)$ of functions defined on the homogeneous space $X = G/K$, is multiplicity-free (for more details about the theory of Gelfand pairs of finite groups we refer the reader to [7, 8] and to [6] for the case of the action of group compositions on trees and related substructures).

In this paper, we consider a class of automaton groups, called graph automaton groups, introduced in [4]. Such groups are built starting from a finite graph and have a number of interesting properties: e.g., they are amenable with exponential growth, they are fractal and regular weakly branch over their commutator subgroup, and they are strictly related to the theory of right-angled Artin groups. In [5], we studied Schreier graphs of the action of graph automaton groups obtained by restricting the general construction introduced in [4] to the special case where the finite graph is a star, giving an explicit description of their spectrum and a classification up to isomorphism.

By taking the quotient modulo the stabilizer of a given level of the tree, one can focus the attention on the action of a graph automaton group on the n-th level L_n of the tree. In this way, one obtains a finite group G_n. We show that (G_n, K_n) is a Gelfand pair, where K_n is the stabilizer of a fixed vertex of L_n. This follows from the fact that the action of each graph automaton group on the tree levels is 2-points homogeneous and this allows to establish that (G_n, K_n) is a symmetric Gelfand pair and to explicitly determine the associated spherical functions.

2. Preliminaries

2.1. The rooted q-ary tree T_q. For each integer $q \geq 2$, let T_q denote the rooted q-ary tree, i.e., the rooted tree in which each vertex has q children. If $X = \{0, 1, \ldots, q-1\}$ is an alphabet of q letters, we denote by $X^n = \{x_1 \cdots x_n : x_i \in X\}$ the set of words of length n over X, and put $X^* = \bigcup_{n=0}^{\infty} X^n$, where $X^0 = \{\emptyset\}$ and $\emptyset$ denotes the empty word. In this way, each vertex in the n-th level L_n of T_q can be naturally identified with a word of X^n.

Notice that the level L_n of T_q or, equivalently, the set X^n, can be endowed with an ultrametric distance d, defined in the following way: if $v = x_1 \cdots x_n$ and $w = y_1 \cdots y_n$, then

$$d(v, w) = n - |\{i : x_k = y_k, \forall k \leq i\}|.$$

In this way, the space (L_n, d) becomes an ultrametric space, and in particular a metric space, on which the automorphism group $Aut(T_q)$ acts by isometries. Note that the diameter of (L_n, d) is exactly n. Moreover, one can observe that $d = d'/2$, where d' denotes the usual geodesic distance on T_q. In order to indicate the action of an automorphism $g \in Aut(T_q)$ on a vertex v of T_q, we will use the notation $g(v)$.

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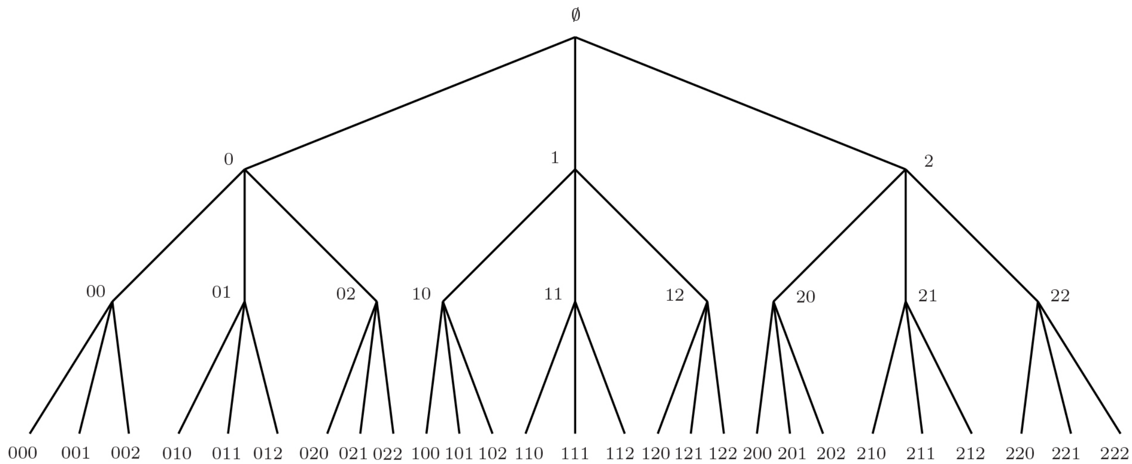

FIGURE 1. The first three levels of the tree T_3.

Example 2.1. In Fig 1 the first three levels of the tree T_3 are depicted. If we focus, for instance, on the third level L_3, then its vertices are identified with the word set $\{0,1,2\}^3$. One has, for instance, $d(000,002) = 1; d(000,021) = 2; d(000,211) = 3$. In particular, all vertices associated with words starting with 1 or 2 have ultrametric distance 3 from the vertex 000; the diameter of (L_3, d) is 3.

2.2. **Self-similar and fractal groups.** The following definitions about self-similar groups, fractal groups, automaton groups can be found, for instance, in [2, 15].

An automorphism group G of T_q is *self-similar* if, for every $g \in G$, $x \in X$, there exist $g_x \in G$, $x' \in X$ such that

$$g(xw) = x' g_x(w), \tag{2.1}$$

for all $w \in X^*$. This allows to embed a self-similar group G into the wreath product $G \wr X = (G^q) \rtimes Sym(q)$, where $Sym(q)$ is the symmetric group on q elements. In words, the automorphism g maps the subtree of T_q rooted at the vertex x of the first level of T_q to the (isomorphic) subtree rooted at the vertex x', and the automorphism $g_x \in G$ represents the restriction of the action of G on such a subtree. Therefore, it is possible to represent any automorphism $g \in G$ as

$$g = (g_0, \dots, g_{q-1})\sigma$$

where $\sigma \in Sym(q)$ is the permutation induced by the action of g on the vertices of the level L_1 of T_q and $g_i \in G$ is the restriction of the action of g on the subtree rooted at the vertex i of L_1, for each $i = 0, \dots, q-1$.

For a self-similar group $G \leq Aut(T_q)$, the *stabilizer* of the vertex $v \in T_q$ is the subgroup of G defined as $Stab_G(v) = \{g \in G : g(v) = v\}$ and the stabilizer of the n-th level is given by $Stab_G(L_n) = \bigcap_{v \in L_n} Stab_G(v)$. Observe that $Stab_G(L_n)$ is a normal subgroup of G of finite index, for each $n \geq 1$. In particular, an automorphism $g \in Stab_G(L_1)$ is completely determined by the q-tuple $(g_0, \dots, g_{q-1})$,

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where g_i describes the action of g on the subtree rooted at the vertex i of L_1, which is clearly isomorphic to the entire tree T_q. Therefore, we get the following embedding

$$\varphi : Stab_G(L_1) \longrightarrow \underbrace{G \times G \times \cdots \times G}_{q \text{ times}} \tag{2.2}$$

that associates with g the q-tuple $(g_0, g_1, \ldots, g_{q-1})$. Moreover, the group G is said to be *fractal* if the map φ in Eq. (2.2) is a subdirect embedding, that is, it is surjective on each factor. Finally, we say that the action of $G \leq Aut(T_q)$ is *spherically transitive* if it is transitive on each level of the tree. It is easy to check that, if the action of G on T_q is spherically transitive, then the subgroups $Stab_G(v)$ are all conjugate, for each $v \in L_n$. Let G be a spherically transitive group and suppose that there exists a nontrivial normal subgroup K in G such that $\varphi(K \cap Stab_G(L_1)) \geq K \times K \times \cdots \times K$. Then G is said to be *regular weakly branch over* K. It is worth mentioning that this property for the subgroup K is stronger than fractalness, since the map φ is surjective on the whole product $K \times K \times \cdots \times K$.

2.3. **Automaton groups.** Self-similar groups can be alternatively represented as automaton groups. A finite *automaton* is a quadruple $\mathcal{A} = (S, X, \lambda, \mu)$, where:

(1) S is a finite set, called the set of *states*;
(2) X is a finite set, called the *alphabet*;
(3) $\lambda : S \times X \to S$ is the *restriction* map;
(4) $\mu : S \times X \to X$ is the *output* map.

The automaton $\mathcal{A}$ is *invertible* if, for all $s \in S$, the transformation $\mu(s, \cdot) : X \to X$ is a permutation of X. An automaton $\mathcal{A}$ can be visually represented by its *Moore diagram*: this is a directed labeled graph whose vertices are identified with the states of $\mathcal{A}$. For every state $s \in S$ and every letter $x \in X$, the diagram has an arrow from s to $\lambda(s, x)$ labeled by $x|\mu(s, x)$. A *sink* in $\mathcal{A}$ is a state $id \in S$ with the property that $\lambda(id, x) = id$ and $\mu(id, x) = x$, for any $x \in X$.

The action of $\mathcal{A}$ can be recursively extended to the infinite set X^* as follows:

$$\lambda(s, xw) = \lambda(\lambda(s, x), w) \qquad \mu(s, xw) = \mu(s, x)\mu(\lambda(s, x), w),$$

for each $s \in S$, $x \in X$ and $w \in X^*$. According to the self-similar notation of Eq. (2.1), if $g(xw) = x'g_x(w)$, with $x, x' \in X$ and $g, g_x \in G$, one has:

$$\lambda(s, x) = s_x \qquad \mu(s, x) = s(x) = x'.$$

Notice that, in a similar way, one can compose the action of the states in S extending the maps λ and μ to the set S^*. Analogously, since each s represents a bijection of X^* one can describe the inverse action s^{-1} and then consider the *automaton group* generated by the set S.

2.4. **Gelfand pairs and their spherical functions.** We conclude this preliminary section by recalling some basic elements of the theory of finite Gelfand pairs (see [7], or the monograph [8] and references therein).

Let G be a finite group and let $K \leq G$. Let us denote by $X = G/K = \{gK : g \in G\}$ the associated homogeneous space, obtained by considering the left multiplication action of G on the set of left cosets. In this way, G acts transitively on X and K is the stabilizer of the coset $K \in X$. The space $L(G) = \{f : G \longrightarrow \mathbb{C}\}$ becomes an algebra when endowed with the standard convolution product defined as

$$(f_1 * f_2)(g) = \sum_{h \in G} f_1(gh) f_2(h^{-1}),$$

for each $f_1, f_2 \in L(G)$ and $g \in G$. Notice that $L(X)$ is a subalgebra of $L(G)$, consisting of all functions which are K-invariant to the right, on which the group G acts as $f^g(x) = f(g^{-1}x)$, for each $f \in L(X)$, $g \in G$ and $x \in X$. Similarly, the subspace of $L(X)$ consisting of all K-invariant functions, denoted by $L(K\backslash G/K)$, is a subalgebra of $L(X)$. In formulae, we have

$$L(K\backslash G/K) = \{f : G \longrightarrow \mathbb{C} : f(kgk') = f(g), \ \forall g \in G, k, k' \in K\}.$$

Then (G, K) is said to be a *Gelfand pair* if the algebra $L(K\backslash G/K)$ of bi-K-invariant functions on G is commutative with respect to the convolution product inherited from $L(G)$. Also, notice that $L(G)$ is a commutative algebra if and only if G is an Abelian group. In particular, it follows that if G is Abelian, then (G, K) is a Gelfand pair for any subgroup $K \leq G$. It can be shown that the following properties are equivalent:

(1) (G, K) is a Gelfand pair;
(2) the decomposition of the space $L(X)$ into irreducible submodules under the action of G is multiplicity-free, i.e., each irreducible submodule occurs with multiplicity 1;
(3) given an irreducible representation V of G, the dimension of the subspace of K-invariant vectors $V^K = \{v \in V : k(v) = v \ \forall k \in K\}$ is less than or equal to 1, and it is 1 if and only if V occurs in the decomposition of $L(X)$ into irreducible submodules.

A particular example of a Gelfand pair (G, K) is provided by the so called *symmetric Gelfand pairs*: this is the case when, for every $g \in G$, one has $g^{-1} \in KgK$. It can be shown that this condition is equivalent to require that, for all $x, y \in X$, the pairs (x, y) and (y, x) belong to the same orbit under the diagonal action of G on the set $X \times X$.

Example 2.2. A symmetric Gelfand pair is obtained when the group G acts isometrically on a metric space (X, d) and its action is 2-*points homogeneous*: this means that, for all $x, x', y, y' \in X$ satisfying the condition $d(x, y) = d(x', y')$, there exists $g \in G$ such that $g(x) = x'$ e $g(y) = y'$. Under this condition, if K is the stabilizer of an element $x_0 \in X$, then the pairs (x, y) and (y, x) always belong to a same orbit, since $d(x, y) = d(y, x)$ for any choice of x and y.
We can observe that in this case the K-orbits in X (which can be identified with the double cosets of K in G) are the spheres centered at x_0

$$S_j = \{x \in X : d(x_0, x) = j\}.$$

Hence, a function $f \in L(X)$ is K-invariant if and only if it is constant on the spheres S_j.

If (G,K) is a Gelfand pair and $L(X)=\bigoplus_{i=0}^{n} V_i$ is the decomposition of $L(X)$ into irreducible submodules, then for each i there exists a unique (up to normalization) bi-K-invariant function ϕ_i whose G-translates generate V_i. In particular, one requires that these functions take value exactly 1 on the element $x_0 \in X$ stabilized by K. The functions ϕ_i, $i=0,1,\ldots,n$ are called *spherical functions* and they form a basis for the algebra $L(K\backslash G/K)$. In particular, the number of K-orbits under the action of G on X equals the number of spherical functions. As an example, the function $\phi_0 \equiv 1$ is a spherical function: this corresponds to the fact that the trivial representation always occurs in the decomposition of the space $L(X)$ into irreducible submodules.

3. Gelfand pairs associated with graph automaton groups

In [4] we introduced a new construction associating an invertible automaton with a given finite graph. The corresponding self-similar group is called graph automaton group. The aim of this section is to prove that the action of a graph automaton group on the rooted tree gives rise to symmetric Gelfand pairs. Let us start by recalling the definition of graph automaton group. Observe that the definition is given here for simple graphs, since the presence of loops and multiedges does not affect the structure of the group (see [4, Remark 3.2]).

Definition 3.1. *Let $\Gamma=(V,E)$ be a finite simple graph, where $V=\{x_1,\ldots,x_k\}$ denotes the vertex set and E is the edge set, endowed with an orientation. Let us denote by $e=(x_i,x_j)$ an edge connecting the vertices x_i and x_j of Γ, and oriented from x_i to x_j. We define an automaton $\mathcal{A}_\Gamma=(E\cup\{id\},V,\lambda,\mu)$ such that:*

- *$E\cup\{id\}$ is the set of states, where id is a sink;*
- *V is the alphabet;*
- *$\lambda: E\times V\to E\cup\{id\}$ is the restriction map such that, for each $e=(x_i,x_j)\in E$, one has*

$$\lambda(e,x_k)=\begin{cases} e & \text{if } k=i\\ id & \text{if } k\neq i;\end{cases}$$

- *$\mu: E\times V\to V$ is the output map such that, for each $e=(x_i,x_j)\in E$, one has*

$$\mu(e,x_k)=\begin{cases} x_j & \text{if } k=i\\ x_i & \text{if } k=j\\ x_k & \text{if } k\neq i,j.\end{cases}$$

In words, any oriented edge $e=(x_i,x_j)$ is a state of the automaton $\mathcal{A}_\Gamma$ and it has just one nontrivial restriction to itself, and all other restrictions to the sink id. Its action is nontrivial only on the letters x_i and x_j, which are switched since $e(x_i)=x_j$ and $e(x_j)=x_i$. It is easy to check that $\mathcal{A}_\Gamma$ is invertible for any given graph $\Gamma=(V,E)$ and any orientation of E. The group generated by the transformations of V^* associated with the states of $\mathcal{A}_\Gamma$ is called *graph automaton group* associated with the graph Γ, and it is denoted by $\mathcal{G}_\Gamma$. It follows from the definition that such a group is an automorphism group of the rooted regular tree $T_{|V|}$ of degree $|V|$. Moreover, one can check that the action of $\mathcal{G}_\Gamma$ on the tree is spherically transitive if and only if Γ is connected.

Note that the graph automaton group associated with $\Gamma = (V, E)$ is well defined since it is independent of the orientation of the edges. In fact, changing the orientation of an edge corresponds to consider the inverse of the associated generator in the group. In [4, Proposition 3.4, Theorem 3.7] the following properties of graph automaton groups have been proved:

- the graph automaton group $\mathcal{G}_{\Gamma'}$ constructed starting from a subgraph $\Gamma' = (V', E')$ of the graph $\Gamma = (V, E)$ is a subgroup of the graph automaton group $\mathcal{G}_\Gamma$;
- if the graph $\Gamma = (V, E)$ contains at least 2 edges, then the associated graph automaton group is nonabelian, amenable, fractal, and it is regular weakly branch over its commutator subgroup.

Notice that the case where Γ is the complete graph K_2 on two vertices (so that it has only one edge) gives rise to a graph automaton group, often called the Adding Machine in the context of self-similar groups, which is isomorphic to the infinite cyclic group (see [4, Example 3.6, Part 1]), which is Abelian, so that it trivially provides Gelfand pairs. For this reason, we will focus our attention on graphs containing at least two edges.

We need the following lemma about the action of a spherically transitive group G on the n-th level L_n of the tree T_q.

Lemma 3.2. *Let G act spherically transitively on the rooted regular tree T_q of degree q. Denote by G_n the quotient group $G/Stab_G(L_n)$ and by K_n the stabilizer in G_n of a fixed vertex $x_0 \in L_n$. Then the action of G_n on L_n is 2-points homogeneous if and only if K_n acts transitively on each sphere of L_n centered at x_0.*

Proof. Suppose that K_n acts transitively on each sphere of L_n centered at x_0 and consider four elements $x, y, x', y' \in L_n$ such that $d(x, y) = d(x', y')$. Since the action of G_n is transitive on L_n, there exists an automorphism $g \in G_n$ such that $g(x) = x'$. Now

$$d(x', g(y)) = d(g^{-1}(x'), y) = d(x, y) = d(x', y')$$

and so $g(y)$ and y' are in the same sphere of center x' and radius $d(x', y')$. By transitivity, the stabilizer K_n is conjugate with $Stab_{G_n}(x')$ and so there exists an automorphism $g' \in Stab_{G_n}(x')$ carrying $g(y)$ to y'. The composition of g and g' is the required automorphism.

Suppose now that the action of G_n on L_n is 2-points homogeneous and consider two elements x and y in the sphere of center x_0 and radius r. Then $d(x_0, x) = d(x_0, y) = r$. Therefore there exists an automorphism $g \in K_n = Stab_{G_n}(x_0)$ such that $g(x) = y$. This completes the proof. □

Proposition 3.3. *Let $\Gamma = (V, E)$ be a finite, connected, simple graph with at least 2 edges and let $\mathcal{G}_\Gamma$ be the associated graph automaton group. Then the commutator subgroup $\mathcal{G}'_\Gamma$ is spherically transitive on $T_{|V|}$.*

Proof. As we have already mentioned, if $\Gamma = (V, E)$ is a graph that contains at least two edges, then $\mathcal{G}_\Gamma$ is regular weakly branch over its commutator subgroup $\mathcal{G}'_\Gamma$. This means that

$$\varphi(\mathcal{G}'_\Gamma \cap Stab_{\mathcal{G}_\Gamma}(L_1)) \geq \underbrace{\mathcal{G}'_\Gamma \times \cdots \times \mathcal{G}'_\Gamma}_{|V| \text{ times}},$$

where φ is as in Eq. (2.2). Let us prove our statement by induction on the depth n of the levels of the rooted regular tree of degree $|V|$.

Let x, y be elements in L_1. Recall that the vertices of L_1 are identified with the vertices of Γ, by definition of $\mathcal{G}_\Gamma$. Since Γ is supposed to be connected, there exists a (not necessarily directed) path from x to y in Γ. Let $e_1, e_2, \ldots, e_k$ be such a path. Let us suppose that e_i joins the vertices x_i and x_{i+1}, so that $x_1 = x$ and $x_{k+1} = y$. Then a direct computation shows that, for each $i = 1, \ldots, k-1$, the commutator $[e_i, e_{i+1}]$ acts on L_1 as the permutation $(i, i+1, i+2) \in Sym(|V|)$. This implies that

$$g := \prod_{j=1}^{k-1} [e_j, e_{j+1}]$$

is an element of $\mathcal{G}'_\Gamma$ such that $g(x) = y$. The assertion is verified since x and y are arbitrary.

For $n > 1$, let $v = x_1 x_2 \cdots x_n$ and $w = y_1 y_2 \cdots y_n$ be vertices of L_n. We have two cases: either $x_1 = y_1 = x$ or $x_1 \neq y_1$. First, suppose $x = x_1 = y_1$. Then there exists, by the inductive hypothesis, an element $h \in \mathcal{G}'_\Gamma$ such that $h(x_2 \cdots x_n) = y_2 \cdots y_n$ (regarded as elements in L_{n-1}). By weakly branchness there exists also $g \in \mathcal{G}'_\Gamma$ such that $g_x = h$ and $g(x) = x$. This ensures $g(v) = w$. Now suppose $x_1 \neq y_1$. We have already proven that there exists $g \in \mathcal{G}'_\Gamma$ such that $g(x_1) = y_1$. Hence $g(v)$ and w are words of length n starting with the same letter. The previous argument implies that there exists $h \in \mathcal{G}'_\Gamma$ such that $h(g(v)) = hg(v) = w$ and this completes the proof since $hg \in \mathcal{G}'_\Gamma$. □

Remark 3.4. It is well known that in any connected graph Γ there exists a vertex x such that $\Gamma \setminus \{x\}$ is still connected. Without loss of generality, we can always assume that such vertex corresponds to the letter 0 of the alphabet V (the vertex set of Γ). In fact, the name of the vertices can be changed by taking conjugation with an element of the symmetric group, and this does not change the structure of the associated graph automaton group.

Proposition 3.5. *Let $\mathcal{G}_\Gamma$ be the graph automaton group associated with a connected simple graph $\Gamma = (V, E)$ with at least 2 edges. Let 0 be a vertex such that $\Gamma \setminus \{0\}$ is connected, and let n be a positive integer. Then, for every $k \in \{0, \ldots, n-1\}$, for all $x, y \in V$ such that $x, y \neq 0$, there exists $g_{k,x,y} \in \mathcal{G}_\Gamma$ such that*

- $g_{k,x,y}(0^n) = 0^n$,
- $g_{k,x,y}(0^k x) = 0^k y$.

Proof. Observe that it is possible to construct an element h of $\mathcal{G}_\Gamma$ such that $h(0) = 0$ and $h(x) = y$. In fact, we can remove from Γ the vertex 0 and consider the remaining connected subgraph Γ'. In Γ', we can connect x and y through a directed path $e_1 \cdots e_k$, where each of the e_i's belongs to Γ' and

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it is not incident to the vertex 0. In particular, regarded as a group element, each of the e_i's acts trivially on the word 0^n, for any $n \geq 1$. Moreover, by Definition 3.1, if we put $h := e_1 \cdots e_k$, we have $h(x) = y$. Since $\mathcal{G}_\Gamma$ is fractal, there exists $g \in Stab_{\mathcal{G}_\Gamma}(0^k)$ such that $g_{0^k} = h$, for $k \in \{0, 1, \ldots, n-1\}$. In particular $g(0^n) = 0^n$ and $g(0^k x) = 0^k y$. Thus $g = g_{k,x,y}$ is the desired element. □

Now we have all ingredients to state and prove our main result.

Theorem 3.6. *Let $\mathcal{G}_\Gamma$ be the graph automaton group associated with a connected simple graph $\Gamma = (V, E)$ with at least 2 edges. Let $G_n = \mathcal{G}_\Gamma / Stab_{\mathcal{G}_\Gamma}(L_n)$ and $K_n = Stab_{G_n}(0^n)$. Then (G_n, K_n) is a symmetric Gelfand pair, for all $n \geq 1$.*

Proof. By Example 2.2 and Lemma 3.2, it is enough to prove that $K_n = Stab_{G_n}(0^n)$ acts transitively on each sphere of L_n with center 0^n. Notice that two vertices $v, w \in L_n$ belong to the same sphere if and only if $v = 0^k x v'$ and $w = 0^k y w'$ for some $k \in \{0, \ldots, n-1\}$, $x, y \in V \setminus \{0\}$, and $v', w' \in V^{n-k-1}$. Let g be the image in G_n of the element $g_{k,x,y} \in \mathcal{G}_\Gamma$ from Proposition 3.5. In particular, $g \in K_n$ and it satisfies $g(0^k x) = 0^k y$. Therefore the vertices $g(v) = 0^k y v''$ and w both belong to the subtree rooted at the vertex $0^k y$ of L_{k+1}. Now Proposition 3.3 guarantees that there exists $h \in \mathcal{G}'_\Gamma$ such that $h(v'') = w'$. By weak branchness, there exists an element $l \in \mathcal{G}'_\Gamma$ such that l acts nontrivially only on the subtree rooted at $0^k y$ and $l_{0^k y} = h$. Let ℓ be its image in G_n. In particular, $\ell \in K_n$ and therefore the element $\ell g \in K_n$ is such that

$$\ell g(v) = \ell g(0^k x v') = \ell(0^k y v'') = 0^k y h(v'') = 0^k y w' = w.$$

This concludes the proof. □

Let $\Gamma = (V, E)$ be a graph with $|V| = q$ and let $\mathcal{G}_\Gamma$ be the associated graph automaton group. Let us consider the action of $\mathcal{G}_\Gamma$ (or, equivalently, of the group $G_n = \mathcal{G}_\Gamma / Stab_{\mathcal{G}_\Gamma}(L_n)$) on $L(L_n)$. Recall that each vertex of L_n can be identified with a word $x_1 x_2 \cdots x_n$, where $x_i \in \{0, 1, \ldots, q-1\}$. Denote by $V_0 \cong \mathbb{C}$ the trivial representation of $\mathcal{G}_\Gamma$ and for every $j = 1, \ldots, n$, define the following subspace

$$V_j = \{f \in L(L_n) : f = f(x_1 \cdots x_j), \ \sum_{x=0}^{q-1} f(x_1 x_2 \cdots x_{j-1} x) = 0\}$$

of dimension $q^{j-1}(q-1)$. In words, each function of V_j takes a constant value on the leaves of each subtree rooted at a vertex of L_j, and the sum of these values is 0. One can verify that the V_j's are $\mathcal{G}_\Gamma$-invariant pairwise orthogonal submodules (in fact, they are $Aut(T_q)$-invariant); now, since the number of the V_j's equals the number of $K_n = G_n / Stab_{G_n}(0^n)$-orbits in L_n (that is, the number of spheres of L_n centered at 0^n), the Wielandt Lemma (see [7, Lemma 2.21]) ensures that they are also irreducible submodules, and that the following decomposition holds:

$$L(L_n) = \bigoplus_{j=0}^{n} V_j.$$

We get the following description of the spherical functions.

Corollary 3.7. *Let $\Gamma = (V, E)$ be a graph with $|V| = q$ and let $\mathcal{G}_\Gamma$ be the corresponding graph automaton group. Then, for each $n \geq 1$, the spherical functions associated with the Gelfand pair (G_n, K_n) are*

$$\phi_j(x) = \begin{cases} 1 & \text{if } d(x, 0^n) < n - j + 1 \\ \frac{1}{1-q} & \text{if } d(x, 0^n) = n - j + 1 \\ 0 & \text{if } d(x, 0^n) > n - j + 1 \end{cases}$$

with $j = 0, \ldots, n$.

Proof. It is enough to observe that, for any $j = 0, \ldots, n$, the function ϕ_j belongs to the space V_j, it is K_n-invariant and it satisfies the property $\phi_j(0^n) = 1$. □

Example 3.8. If Γ is the cyclic graph C_3 on 3 vertices, the associated automaton $\mathcal{A}_{C_3}$ is represented in Fig 2 (see [4, Example 3.6, Part 3]).

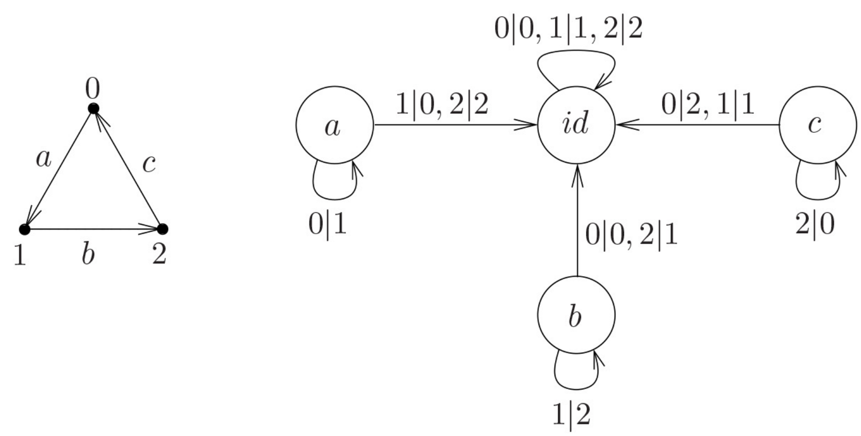

FIGURE 2. The cycle C_3 and the associated automaton $\mathcal{A}_{C_3}$.

More precisely, the automaton $\mathcal{A}_{C_3}$ generates the group $\mathcal{G}_{C_3}$ whose generators have the self-similar representation:

$$a = (a, id, id)(0,1) \qquad b = (id, b, id)(1,2) \qquad c = (id, id, c)(2,0),$$

where $(0,1), (1,2), (2,0)$ are transpositions in $Sym(3)$.

For each $n \geq 1$, the action of $\mathcal{G}_{C_3}$ on each level L_n of the rooted ternary tree gives rise to a Gelfand pair (G_n, K_n) such that

$$L(L_n) = \bigoplus_{j=0}^{n} V_j$$

where

$$V_j = \{f \in L(L_n) : f = f(x_1 \cdots x_j),\ \ \sum_{x=0}^{2} f(x_1 x_2 \cdots x_{j-1} x) = 0\}.$$

The spherical functions in this case are given by

$$\phi_j(x) = \begin{cases} 1 & \text{if } d(x, 0^n) < n - j + 1 \\ -\frac{1}{2} & \text{if } d(x, 0^n) = n - j + 1 \\ 0 & \text{if } d(x, 0^n) > n - j + 1 \end{cases}$$

with $j = 0, \dots, n$.

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