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ISOMETRY GROUPS OF SIX-DIMENSIONAL FILIFORM NILMANIFOLDS

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Abstract. In this paper, we classify up to isometry the connected and simply connected Riemannian nilmanifolds on six-dimensional filiform Lie groups and we compute the corresponding isometry groups.

1. Introduction

A connected Riemannian manifold M which admits a transitive nilpotent Lie group of isometries is called *a Riemannian nilmanifold*. For every Riemannian nilmanifold M there exists a unique nilpotent Lie subgroup $\aleph$ of the group $\mathcal{I}(M)$ of isometries of M which acts simply transitively on M (see [9]). Therefore the Riemannian nilmanifold M can be treated as the nilpotent Lie group $\aleph$ equipped with a left invariant metric $\langle .,. \rangle_{\aleph}$. Furthermore, the full group $\mathcal{I}(\aleph)$ of isometries of $(\aleph, \langle .,. \rangle_{\aleph})$ is the semi-direct product $\aleph \rtimes \mathcal{OA}(\mathfrak{n})$ of the group $\mathcal{OA}(\mathfrak{n})$ of automorphisms of the Lie algebra $\mathfrak{n}$ of $\aleph$ which preserves the inner product on $\mathfrak{n}$ determined by the left-invariant metric $\langle .,. \rangle_{\aleph}$ and the group $\aleph$. Hence the classification of the connected and simply connected Riemannian nilmanifolds up to isometry is equivalent to the determination of the classes of isometrically isomorphic metric nilpotent Lie algebras, that is nilpotent Lie algebras endowed with an inner product. This means the determination of the isometry equivalence classes of nilmanifolds can be successfully carried out by working in their metric Lie algebras. Using this procedure the connected and simply connected two-step Riemannian nilmanifolds of dimension at most 6 were intensively studied. Their isometry equivalence classes and isometry groups are determined in [3], [6], [8]. It turns out, that this method is

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also fruitful to classify the at most 5-dimensional Riemannian nilmanifolds of higher nilpotency class and to determine their isometry groups ([4], [8]). Among nilpotent Lie groups with higher nilpotency class the filiform Lie groups play an essential role. An n-dimensional filiform Lie algebra has the maximal possible nilpotency class $n-1$. The geometry of filiform metric Lie algebras are described in [1], [2], [7]. In [4] the isometry equivalence classes and the isometry groups of Riemannian nilmanifolds corresponding to filiform metric Lie algebras of arbitrary dimension were thoroughly investigated. It is proved there that every filiform metric Lie algebra $(\mathfrak{n}, \langle.,.\rangle)$ has a decomposition into orthogonal direct sum of 1-dimensional subspaces such that any orthogonal automorphism of $(\mathfrak{n}, \langle.,.\rangle)$ preserves this decomposition. Hence the isometry groups of the filiform nilmanifolds have the same dimension as the nilmanifolds. In particular the classification of standard filiform metric Lie algebras and that of the isometry groups of the corresponding nilmanifolds are given in [4]. In this paper we follow the approach of [4] and deal with 6-dimensional filiform metric Lie algebras. Our aim is to classify the isometry equivalence classes of nilmanifolds on 6-dimensional filiform Lie groups and to determine their full isometry groups.

2. Preliminaries

The *lower central series* of a Lie algebra ℓ is $\mathcal{C}^0\ell \supset \mathcal{C}^1\ell \supset \cdots \supset \mathcal{C}^j\ell \supset \mathcal{C}^{j+1}\ell \supset \cdots$ such that $\mathcal{C}^0\ell = \ell$ and $\mathcal{C}^{j+1}\ell = [\ell, \mathcal{C}^j\ell], j \in \mathbb{N}$. A Lie algebra ℓ is called *k-step nilpotent* if $\mathcal{C}^k\ell = \{0\}$, but $\mathcal{C}^{k-1}\ell \neq \{0\}$ for some $k \in \mathbb{N}$. If an n-dimensional Lie algebra ℓ is $(n-1)$-step nilpotent then it is called *filiform*. A filiform Lie algebra ℓ is *standard filiform*, if it contains a basis $\{G_1, \dots, G_n\}$ such that the nontrivial Lie bracket relations are given by $[G_1, G_j] = G_{j+1}, j = 2, \dots, n-1$.

In this paper we investigate 6-dimensional filiform Lie algebras. They are defined by the following non-vanishing commutators (see [5, pp. 646-647]),

$$\ell_{6,14} : [G_1, G_2] = G_3,\ [G_1, G_3] = G_4,\ [G_1, G_4] = G_5,\ [G_2, G_3] = G_5,\ [G_2, G_5] = G_6,\ [G_4, G_3] = G_6,$$

$$\ell_{6,15} : [G_1, G_2] = G_3,\ [G_1, G_3] = G_4,\ [G_1, G_4] = G_5,\ [G_2, G_3] = G_5,\ [G_1, G_5] = G_6,\ [G_2, G_4] = G_6,$$

$$\ell_{6,16} : [G_1, G_2] = G_3,\ [G_1, G_3] = G_4,\ [G_1, G_4] = G_5,\ [G_2, G_5] = G_6,\ [G_4, G_3] = G_6,$$

$$\ell_{6,17} : [G_1, G_2] = G_3,\ [G_1, G_3] = G_4,\ [G_1, G_4] = G_5,\ [G_1, G_5] = G_6,\ [G_2, G_3] = G_6,$$

$$\ell_{6,18} : [G_1, G_2] = G_3,\ [G_1, G_3] = G_4,\ [G_1, G_4] = G_5,\ [G_1, G_5] = G_6,$$

with respect to a distinguished basis $\{G_i\}_{i=1}^6$ which is called the *canonical basis* of the corresponding Lie algebra. In this list of Lie algebras, $\ell_{6,14}$, $\ell_{6,15}$, $\ell_{6,16}$ and $\ell_{6,17}$ are non-standard filiform Lie algebras, $\ell_{6,18}$ is standard filiform. The *metric Lie algebra* is a Lie algebra equipped with an inner product, the automorphisms preserving the inner product are called *orthogonal automorphisms*.

In the following $\mathbb{E}^6$ denotes a 6-dimensional Euclidean vector space with a distinguished orthonormal basis $\mathcal{E} = \{E_1, E_2, E_3, E_4, E_5, E_6\}$. Taking into account the procedure given by [4, pp. 371-372], for the classification of metric Lie algebras up to isometric isomorphisms, first we use the Gram-Schmidt process to the ordered basis $\{G_6, G_5, G_4, G_3, G_2, G_1\}$ in the metric Lie algebra $(\ell, \langle.,.\rangle)$ to obtain an orthonormal basis $\{F_1, F_2, F_3, F_4, F_5, F_6\}$ expressed by $F_i = \sum_{k=i}^n a_{ik} G_k, a_{ik} \in \mathbb{R}$, such that $a_{ii} \geq 0$.

After this, we define a Lie bracket on $\mathbb{E}^6$ with the same structure coefficients with respect to its distinguished basis $\mathcal{E}$ as the metric Lie algebra $(\ell, \langle .,. \rangle)$ has with respect to its basis F. The obtained metric Lie algebra on $\mathbb{E}^6$ is isometrically isomorphic to $(\ell, \langle .,. \rangle)$. Finally, we have to find conditions on the real parameters of metric Lie algebras on $\mathbb{E}^6$ to get a one-to-one correspondence between the equivalence classes of isometrically isomorphic metric Lie algebras and a family of metric Lie algebras on $\mathbb{E}^6$.

An orthogonal direct sum decomposition $\mathfrak{n} = V_1 \oplus \cdots \oplus V_n$ on one-dimensional subspaces $V_1, \ldots, V_n$ of a metric Lie algebra $(\mathfrak{n}, \langle .,. \rangle)$ is called a *framing*, if any orthogonal automorphism of $(\mathfrak{n}, \langle .,. \rangle)$ preserves this decomposition. An orthonormal basis $\{G_1, G_2, \ldots, G_n\}$ of $(\mathfrak{n}, \langle .,. \rangle)$ is *adapted* to the framing $\mathfrak{n} = V_1 \oplus \cdots \oplus V_n$ if $V_i = \mathbb{R}\, G_i$ for $i = 1, \ldots, n$. The metric Lie algebra $(\mathfrak{n}, \langle .,. \rangle)$ is called *framed*, if it has a framing.

We often use the following (see [4, Lemma 1]).

Lemma 2.1. *Let $(\mathfrak{n}, \langle .,. \rangle)$ and $(\mathfrak{n}^*, \langle .,. \rangle^*)$ be isometrically isomorphic framed metric Lie algebras of dimension n with framings $\mathfrak{n} = \mathbb{R}\, G_1 \oplus \cdots \oplus \mathbb{R}\, G_n$ and $\mathfrak{n}^* = \mathbb{R}\, G_1^* \oplus \cdots \oplus \mathbb{R}\, G_n^*$, where $(G_1, \ldots, G_n)$, respectively $(G_1^*, \ldots, G_n^*)$ are orthonormal bases. If the commutators $[.,.]$ of $\mathfrak{n}$ and $[.,.]^*$ of $\mathfrak{n}^*$ are of the form*

$$[G_i, G_j] = \sum_{k=1}^{n} a_{i,j}^k G_k \quad \textit{and} \quad [G_i^*, G_j^*]^* = \sum_{k=1}^{n} {a^*}_{i,j}^k G_k^*, \quad i, j, k = 1, \ldots, n,$$

then $a_{i,j}^k = \pm {a^*}_{i,j}^k$ for all $i, j, k = 1, \ldots, n$. Particularly, if $a_{i,j}^k$, ${a^*}_{i,j}^k \geq 0$ then $a_{i,j}^k = {a^*}_{i,j}^k$.

3. Filiform metric Lie algebras of dimension 6

We consider the 6-dimensional standard filiform Lie algebra $\ell_{6,18}$.

According [4, Theorem 7] let $\mathfrak{n}_{6,18}(a, b, c, d, e_j), j = 1, \ldots, 6$ be a filiform Lie algebra defined by the following non-vanishing commutators with respect to the distinguished orthonormal basis $\{E_1, \ldots, E_6\}$ of the Euclidean vector space $\mathbb{E}^6$,

$$[E_1, E_2] = aE_3 + e_1E_4 + e_2E_5 + e_3E_6, \qquad [E_1, E_3] = bE_4 + e_4E_5 + e_5E_6,$$
$$[E_1, E_4] = cE_5 + e_6E_6, \qquad [E_1, E_5] = dE_6,$$

where $a, b, c, d, e_j \in \mathbb{R}$, $a, b, c, d \neq 0$. The metric Lie algebra $(\ell_{6,18}, \langle .,. \rangle)$ is isometrically isomorphic to a unique metric Lie algebra $\mathfrak{n}_{6,18}(a, b, c, d, e_j)$ such that $a, b, c, d > 0$ and if the set $J = \{j \in \{1, 3, 4, 6\} : e_j \neq 0\} \neq \emptyset$, then $e_{j_\circ} > 0$ for the minimal element $j_\circ \in J$. The group $\mathcal{OA}(\mathfrak{n}_{6,18}(a, b, c, d, e_j))$ of orthogonal automorphisms of $\mathfrak{n}_{6,18}(a, b, c, d, e_j)$ with respect to the basis $\{E_1, E_2, E_3, E_4, E_5, E_6\}$ is the group

(1) if $J = \emptyset$, then $\mathcal{OA}(\mathfrak{n}_{6,18}(a, b, c, d, e_j)) = \{TE_1 = \varepsilon_1 E_1, TE_i = \varepsilon_2 E_i, i = 2, 4, 6, TE_j = \varepsilon_1\varepsilon_2 E_j, j = 3, 5, \varepsilon_1, \varepsilon_2 = \pm 1\} \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$.
(2) if $J \neq \emptyset$, then $\mathcal{OA}(\mathfrak{n}_{6,18}(a, b, c, d, e_j)) = \{TE_1 = E_1, TE_i = \varepsilon E_i, i = 2, \ldots, 6, \varepsilon = \pm 1\} \simeq \mathbb{Z}_2$.

Now we consider the Lie algebra $\ell_{6,14}$.

Definition 3.1. *Let $\mathfrak{n}_{6,14}(\alpha_i, \beta_j)$ be a metric Lie algebra defined on $\mathbb{E}^6$ by the non-vanishing commutators:*

$$[E_1, E_2] = \alpha_1 E_3 + \beta_1 E_4 + \beta_2 E_5 + \beta_3 E_6, \qquad [E_1, E_3] = \alpha_2 E_4 - \left(\frac{\alpha_5\beta_1 + \alpha_2\beta_7}{\alpha_4}\right) E_5 + \beta_4 E_6,$$
$$[E_1, E_4] = \frac{\alpha_1\alpha_5}{\alpha_4} E_5 + \beta_5 E_6, \qquad [E_2, E_3] = \alpha_3 E_5 + \beta_6 E_6,$$
$$[E_2, E_4] = \beta_7 E_6, \qquad [E_2, E_5] = \alpha_4 E_6,$$
$$[E_4, E_3] = \alpha_5 E_6,$$

where $\alpha_i \neq 0, i = 1, \ldots, 5$ and $\beta_j \in \mathbb{R}, j = 1, \ldots, 7$.

Theorem 3.2. *Let $\langle ., . \rangle$ be an inner product on the 6-dimensional filiform Lie algebra $\ell_{6,14}$.*

(1) *There is a unique metric Lie algebra $(\mathfrak{n}_{6,14}(\alpha_i, \beta_j), \langle ., . \rangle)$ with $\alpha_i, \beta_j \in \mathbb{R}$ such that $\alpha_i > 0$ satisfying*
   (a) *$(\mathfrak{n}_{6,14}(\alpha_i, \beta_j), \langle ., . \rangle)$ is isometrically isomorphic to the metric Lie algebra $(\ell_{6,14}, \langle ., . \rangle)$,*
   (b) *if the set $J = \{j \in \{1, 4, 7\} : \beta_j \neq 0\} \neq \emptyset$, then $\beta_{j_\circ} > 0$ for the minimal element $j_\circ \in J$.*

(2) *The group of orthogonal automorphisms of $\mathfrak{n}_{6,14}(\alpha_i, \beta_j)$ with the respect to $\{E_1, \ldots, E_6\}$ is the group:*
   (a) *if the set $J = \emptyset$, then one has $\mathcal{OA}(\mathfrak{n}_{6,14}(\alpha_i, \beta_j)) = \{TE_i = \varepsilon E_i, i = 1, 3, 5, 6, TE_2 = E_2, TE_4 = E_4, \varepsilon = \pm 1\} \simeq \mathbb{Z}_2$.*
   (b) *if the set $J \neq \emptyset$, then $\mathcal{OA}(\mathfrak{n}_{6,14}(\alpha_i, \beta_j))$ is trivial.*

Proof. According to the proof of [4, Theorem 4], the subspaces span $(G_i, \ldots, G_6), i = 1, \ldots, 6$ of $\ell_{6,14}$ form a descending series of ideals which are invariant under all automorphisms of $\ell_{6,14}$. The Gram-Schmidt process applied to the ordered basis $\{G_6, G_5, G_4, G_3, G_2, G_1\}$ yields an orthonormal basis $\{F_1, F_2, F_3, F_4, F_5, F_6\}$ of $\ell_{6,14}$ such that the vector F_i is a positive multiple of G_i modulo the subspace span $(G_j; j > i)$ and orthogonal to span $(G_j; j > i)$. The orthogonal direct sum $\mathbb{R} F_1 \oplus \cdots \oplus \mathbb{R} F_6$ is a framing of $(\ell_{6,14}, \langle ., . \rangle)$. Hence one has $F_i = \sum_{k=i}^{6} a_{ik} G_k, a_{ii} > 0$ and the commutators can be expressed as

$$(3.1) \qquad [F_1, F_2] = \alpha_1 F_3 + \beta_1 F_4 + \beta_2 F_5 + \beta_3 F_6, \qquad [F_1, F_3] = \alpha_2 F_4 - \left(\frac{\alpha_5\beta_1 + \alpha_2\beta_7}{\alpha_4}\right) F_5 + \beta_4 F_6,$$
$$[F_1, F_4] = \frac{\alpha_1\alpha_5}{\alpha_4} F_5 + \beta_5 F_6, \qquad [F_2, F_3] = \alpha_3 F_5 + \beta_6 F_6,$$
$$[F_2, F_4] = \beta_7 F_6, \qquad [F_2, F_5] = \alpha_4 F_6,$$
$$[F_4, F_3] = \alpha_5 F_6,$$

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with $\alpha_i > 0, i = 1, \dots, 5$ and $\{\alpha_i, \beta_j\} \in \mathbb{R}$. Changing the orthonormal basis: $\tilde{F}_1 = -F_1, \tilde{F}_2 = F_2, \tilde{F}_3 = -F_3, \tilde{F}_4 = F_4, \tilde{F}_5 = -F_5, \tilde{F}_6 = -F_6$ yields

$$
\begin{aligned}
&[\tilde{F}_1, \tilde{F}_2] = \alpha_1 \tilde{F}_3 - \beta_1 \tilde{F}_4 + \beta_2 \tilde{F}_5 + \beta_3 \tilde{F}_6, && [\tilde{F}_1, \tilde{F}_3] = \alpha_2 \tilde{F}_4 + \left(\frac{\alpha_5\beta_1 + \alpha_2\beta_7}{\alpha_4}\right) \tilde{F}_5 - \beta_4 \tilde{F}_6,\\
&[\tilde{F}_1, \tilde{F}_4] = \frac{\alpha_1\alpha_5}{\alpha_4} \tilde{F}_5 + \beta_5 \tilde{F}_6, && [\tilde{F}_2, \tilde{F}_3] = \alpha_3 \tilde{F}_5 + \beta_6 \tilde{F}_6,\\
&[\tilde{F}_2, \tilde{F}_4] = -\beta_7 \tilde{F}_6, && [\tilde{F}_2, \tilde{F}_5] = \alpha_4 \tilde{F}_6,\\
&[\tilde{F}_4, \tilde{F}_3] = \alpha_5 \tilde{F}_6.
\end{aligned}
$$

Hence there is an orthonormal basis satisfying (3.1) such that if the set $J = \{j \in \{1, 4, 7\}, \beta_j \neq 0\} \neq \emptyset$, then we may assume that $\beta_{j_\circ} > 0$ for the minimal element $j_\circ \in J$. Hence the existence of metric Lie algebra $\mathfrak{n}_{6,14}(\alpha_i, \beta_j)$ satisfying Theorem 3.2 (1) is proved.

Let the linear map $T : \mathfrak{n}_{6,14}(\alpha_i, \beta_j) \to \mathfrak{n}_{6,14}(\alpha'_i, \beta'_j)$ be an isometric isomorphism. The decomposition $\mathbb{R} E_1 \oplus \mathbb{R} E_2 \oplus \mathbb{R} E_3 \oplus \mathbb{R} E_4 \oplus \mathbb{R} E_5 \oplus \mathbb{R} E_6$ is a framing of both Lie algebras, where $\alpha_i, \alpha'_i > 0$. Hence by Lemma 2.1 we have $\alpha_i = \alpha'_i$, moreover $|\beta'_j| = \beta_j$ for all $j = 1, \dots, 7$. Let be $T(E_i) = \varepsilon_i E_i$, $\varepsilon_i = \pm 1$, $i = 1, \dots, 6$. Then we obtain from $[TE_i, TE_j]' = T[E_i, E_j]$, $i, j = 1, \dots, 6$, using the commutation relations (3.1), the equations

$$
\begin{aligned}
\varepsilon_1\varepsilon_2 \left(\alpha_1 E_3 + \beta'_1 E_4 + \beta'_2 E_5 + \beta'_3 E_6\right) &= \alpha_1 \varepsilon_3 E_3 + \beta_1 \varepsilon_4 E_4 + \beta_2 \varepsilon_5 E_5 + \beta_3 \varepsilon_6 E_6,\\
\varepsilon_1\varepsilon_3 \left(\alpha_2 E_4 - \left(\frac{\alpha_5\beta'_1 + \alpha_2\beta'_7}{\alpha_4}\right) E_5 + \beta'_4 E_6\right) &= \alpha_2 \varepsilon_4 E_4 - \left(\frac{\alpha_5\beta_1 + \alpha_2\beta_7}{\alpha_4}\right) \varepsilon_5 E_5 + \beta_4 \varepsilon_6 E_6,\\
\varepsilon_1\varepsilon_4 \left(\frac{\alpha_1\alpha_5}{\alpha_4} E_5 + \beta'_5 E_6\right) &= \frac{\alpha_1\alpha_5}{\alpha_4} \varepsilon_5 E_5 + \beta_5 \varepsilon_6 E_6,\\
\varepsilon_2\varepsilon_3 \left(\alpha_3 E_5 + \beta'_6 E_6\right) &= \alpha_3 \varepsilon_5 E_5 + \beta_6 \varepsilon_6 E_6,\\
\varepsilon_2\varepsilon_4 \left(\beta'_7 E_6\right) &= \beta_7 \varepsilon_6 E_6,\\
\varepsilon_2\varepsilon_5 \left(\alpha_4 E_6\right) &= \alpha_4 \varepsilon_6 E_6,\\
\varepsilon_4\varepsilon_3 \left(\alpha_5 E_6\right) &= \alpha_5 \varepsilon_6 E_6.
\end{aligned}
$$

It follows $\varepsilon_1\varepsilon_2 = \varepsilon_3$, $\varepsilon_1\varepsilon_3 = \varepsilon_4$, $\varepsilon_1\varepsilon_4 = \varepsilon_5 = \varepsilon_2\varepsilon_3$, and $\varepsilon_2\varepsilon_5 = \varepsilon_6 = \varepsilon_4\varepsilon_3$. Then one has $\varepsilon_2 = \varepsilon_4 = 1$, $\varepsilon_1 = \varepsilon_3 = \varepsilon_5 = \varepsilon_6$. Using these relations we have $\varepsilon_1\varepsilon_2 = \varepsilon_1 = \varepsilon_5$, $\varepsilon_1\varepsilon_2 = \varepsilon_1 = \varepsilon_6$, $\varepsilon_1\varepsilon_4 = \varepsilon_1 = \varepsilon_6$, $\varepsilon_2\varepsilon_3 = \varepsilon_3 = \varepsilon_6$. Therefore one has $\beta'_2 = \beta_2$, $\beta'_3 = \beta_3$, $\beta'_5 = \beta_5$, $\beta'_6 = \beta_6$. Assume that $J \neq \emptyset$. If $\beta_1 = \beta'_1 > 0$, then one has the additional condition $\varepsilon_1\varepsilon_2 = \varepsilon_4$. If $\beta_4 = \beta'_4 > 0$, then we get $\varepsilon_1\varepsilon_3 = \varepsilon_6$. If $\beta_7 = \beta'_7 > 0$, then we have $\varepsilon_2\varepsilon_4 = \varepsilon_6$. In all these cases we obtain $\varepsilon_i = 1, i = 1, \dots, 6$. This proves that Lie algebra $\mathfrak{n}_{6,14}(\alpha_i, \beta_j)$ is uniquely determined.

If the map $E_i \longmapsto \varepsilon_i E_i$ is an orthogonal automorphism of $\mathfrak{n}_{6,14}(\alpha_i, \beta_j)$, then

$$\varepsilon_1\varepsilon_2 (\alpha_1 E_3 + \beta_1 E_4 + \beta_2 E_5 + \beta_3 E_6) = \alpha_1 \varepsilon_3 E_3 + \beta_1 \varepsilon_4 E_4 + \beta_2 \varepsilon_5 E_5 + \beta_3 \varepsilon_6 E_6,$$

$$\varepsilon_1\varepsilon_3 \left(\alpha_2 E_4 - \left(\frac{\alpha_5\beta_1 + \alpha_2\beta_7}{\alpha_4}\right) E_5 + \beta_4 E_6\right) = \alpha_2\varepsilon_4 E_4 - \left(\frac{\alpha_5\beta_1 + \alpha_2\beta_7}{\alpha_4}\right)\varepsilon_5 E_5 + \beta_4\varepsilon_6 E_6,$$

$$\varepsilon_1\varepsilon_4 \left(\frac{\alpha_1\alpha_5}{\alpha_4} E_5 + \beta_5 E_6\right) = \frac{\alpha_1\alpha_5}{\alpha_4}\varepsilon_5 E_5 + \beta_5\varepsilon_6 E_6,$$

$$\varepsilon_2\varepsilon_3 (\alpha_3 E_5 + \beta_6 E_6) = \alpha_3\varepsilon_5 E_5 + \beta_6\varepsilon_6 E_6,$$

$$\varepsilon_2\varepsilon_4 (\beta_7 E_6) = \beta_7\varepsilon_6 E_6,$$

$$\varepsilon_2\varepsilon_5 (\alpha_4 E_6) = \alpha_4\varepsilon_6 E_6,$$

$$\varepsilon_4\varepsilon_3 (\alpha_5 E_6) = \alpha_5\varepsilon_6 E_6.$$

If $J = \emptyset$, then one has $\varepsilon_2 = \varepsilon_4 = 1$, $\varepsilon_1 = \varepsilon_3 = \varepsilon_5 = \varepsilon_6$. Hence the group of orthogonal automorphisms of $\mathfrak{n}_{6,14}(\alpha_i, \beta_j)$ is isomorphic to the group given by (2a) of Theorem 3.2.

If $J \neq \emptyset$, then we get $\varepsilon_i = 1, i = 1, \ldots, 6$, i.e. the group of orthogonal automorphisms of $\mathfrak{n}_{6,14}(\alpha_i, \beta_j)$ is trivial. This yields the second assertion. □

Corollary 3.3. *Let $(\aleph_{6,14}(\alpha_i, \beta_j), \langle ., . \rangle)$ be the connected and simply connected Riemannian nilmanifold corresponding to the metric Lie algebra $(\mathfrak{n}_{6,14}(\alpha_i, \beta_j), \langle ., . \rangle)$. The isometry group of $(\aleph_{6,14}(\alpha_i, \beta_j), \langle ., . \rangle)$ is*

$$\mathcal{I}((\aleph_{6,14}(\alpha_i, \beta_j)) = \begin{cases} \mathbb{Z}_2 \ltimes \aleph_{6,14}(\alpha_i, \beta_j) & if\, J = \emptyset, \\ \aleph_{6,14}(\alpha_i, \beta_j) & if\, J \neq \emptyset. \end{cases}$$

Now we consider the Lie algebra $\ell_{6,15}$.

Definition 3.4. *Let $\{E_1, E_2, E_3, E_4, E_5, E_6\}$ be an orthonormal basis in the Euclidean vector space $\mathbb{E}^6$. Denote by $\mathfrak{n}_{6,15}(\alpha_i, \beta_j), \alpha_i, \beta_j \in \mathbb{R}, i = 1, \ldots, 5, j = 1, \ldots, 7$ with $\alpha_i \neq 0$ the metric Lie algebra defined on $\mathbb{E}^6$ given by the non-vanishing commutators*

$$[E_1, E_2] = \alpha_1 E_3 + \beta_1 E_4 + \beta_2 E_5 + \beta_3 E_6, \qquad [E_1, E_3] = \alpha_2 E_4 + \beta_4 E_5 + \beta_5 E_6,$$

$$[E_1, E_4] = \alpha_3 E_5 + \beta_6 E_6, \qquad [E_2, E_3] = \frac{\alpha_2\alpha_5}{\alpha_4} E_5 + \beta_7 E_6,$$

$$[E_1, E_5] = \alpha_4 E_6, \qquad [E_2, E_4] = \alpha_5 E_6.$$

Theorem 3.5. *Let $\langle ., . \rangle$ be an inner product on the 6-dimensional filiform Lie algebra $\ell_{6,15}$.*

(1) *There is a unique metric Lie algebra $(\mathfrak{n}_{6,15}(\alpha_i, \beta_j), \langle ., . \rangle)$ with $\alpha_i, \beta_j \in \mathbb{R}$ such that $\alpha_i > 0$ satisfying*
   (a) *$(\mathfrak{n}_{6,15}(\alpha_i, \beta_j), \langle ., . \rangle)$ is isometrically isomorphic to the metric Lie algebra $(\ell_{6,15}, \langle ., . \rangle)$,*
   (b) *if the set $J = \{j \in \{1, 3, 4, 6, 7\} : \beta_j \neq 0\} \neq \emptyset$, then $\beta_{j_\circ} > 0$ for the minimal element $j_\circ \in J$.*

(2) *The group of orthogonal automorphisms of $\mathfrak{n}_{6,15}(\alpha_i, \beta_j)$ with the respect to $\{E_1, \ldots, E_6\}$ is the group:*

(a) *if the set* $J = \emptyset$, *then one has* $\mathcal{OA}(\mathfrak{n}_{6,15}(\alpha_i, \beta_j)) = \{TE_i = \varepsilon E_i, i = 1,3,5, TE_2 = E_2, TE_4 = E_4, TE_6 = E_6, \varepsilon = \pm 1\} \simeq \mathbb{Z}_2$.

(b) *if the set* $J \neq \emptyset$, *then* $\mathcal{OA}(\mathfrak{n}_{6,15}(\alpha_i, \beta_j))$ *is trivial.*

Proof. According to the proof of [4, Theorem 4], the subspaces $\text{span}(G_i, \dots, G_6), i = 1, \dots, 6$ of $\ell_{6,15}$ form a descending series of ideals which are invariant under all automorphisms of $\ell_{6,15}$. Using Gram-Schmidt process to the ordered basis $\{G_6, G_5, G_4, G_3, G_2, G_1\}$ we obtain the orthonormal basis $\{F_6, F_5, F_4, F_3, F_2, F_1\}$ where the vector F_i is a positive multiple of G_i modulo $\text{span}(G_j; j > i)$ and orthogonal to span $(G_j; j > i)$. Expressing the vectors of the new basis in the form $F_i = \sum_{k=1}^{6} a_{ik} G_k, i = 1, \dots, 6$ of $\ell_{6,15}$ with $a_{ii} > 0$ we get

(3.2)

$$[F_1, F_2] = \alpha_1 F_3 + \beta_1 F_4 + \beta_2 F_5 + \beta_3 F_6, \quad [F_1, F_3] = \alpha_2 F_4 + \beta_4 F_5 + \beta_5 F_6, \quad [F_1, F_4] = \alpha_3 F_5 + \beta_6 F_6,$$
$$[F_2, F_3] = \frac{\alpha_2 \alpha_5}{\alpha_4} F_5 + \beta_7 F_6, \quad [F_1, F_5] = \alpha_4 F_6, \quad [F_2, F_4] = \alpha_5 F_6,$$

where $\alpha_i > 0$ and $\{\alpha_i, \beta_j\} \in \mathbb{R}$. Changing the orthonormal basis, $F_1 \mapsto -F_1$, $F_2 \mapsto F_2$, $F_3 \mapsto -F_3$, $F_4 \mapsto F_4$, $F_5 \mapsto -F_5$, $F_6 \mapsto F_6$, we obtain

$$[F_1, F_2] = \alpha_1 F_3 - \beta_1 F_4 + \beta_2 F_5 - \beta_3 F_6, \quad [F_1, F_3] = \alpha_2 F_4 - \beta_4 F_5 + \beta_5 F_6, \quad [F_1, F_4] = \alpha_3 F_5 - \beta_6 F_6,$$
$$[F_2, F_3] = \frac{\alpha_2 \alpha_5}{\alpha_4} F_5 - \beta_7 F_6, \quad [F_1, F_5] = \alpha_4 F_6, \quad [F_2, F_4] = \alpha_5 F_6.$$

There is an orthonormal basis satisfying (3.2) such that if the set $J = \{j \in \{1,3,4,6,7\}, \beta_j \neq 0\} \neq \emptyset$, then we may assume that $\beta_{j_\circ} > 0$ for the minimal element $j_\circ \in J$. Hence the existence of metric Lie algebra $\mathfrak{n}_{6,15}(\alpha_i, \beta_j)$ satisfying Theorem 3.5 (1) is proved.

Let the linear map $T : \mathfrak{n}_{6,15}(\alpha_i, \beta_j) \to \mathfrak{n}_{6,15}(\alpha'_i, \beta'_j)$ be an isometric isomorphism. The decomposition $\mathbb{R} E_1 \oplus \mathbb{R} E_2 \oplus \mathbb{R} E_3 \oplus \mathbb{R} E_4 \oplus \mathbb{R} E_5 \oplus \mathbb{R} E_6$ is a framing of both Lie algebras, where $\alpha_i, \alpha'_i > 0$. Hence by Lemma 2.1 we have $\alpha_i = \alpha'_i$, moreover $|\beta'_j| = \beta_j$ for $j = 2, 5$. Let $T(E_i) = \varepsilon_i E_i$, $\varepsilon_i = \pm 1$, $i = 1, \dots, 6$, then we obtain from $[TE_i, TE_j]' = T[E_i, E_j]$, $i, j = 1, \dots, 6$, using the commutation relations (3.2) the equations

$$\varepsilon_1 \varepsilon_2 \left(\alpha_1 E_3 + \beta'_1 E_4 + \beta'_2 E_5 + \beta'_3 E_6 \right) = \alpha_1 \varepsilon_3 E_3 + \beta_1 \varepsilon_4 E_4 + \beta_2 \varepsilon_5 E_5 + \beta_3 \varepsilon_6 E_6,$$
$$\varepsilon_1 \varepsilon_3 \left(\alpha_2 E_4 + \beta'_4 E_5 + \beta'_5 E_6 \right) = \alpha_2 \varepsilon_4 E_4 + \beta_4 \varepsilon_5 E_5 + \beta_5 \varepsilon_6 E_6,$$
$$\varepsilon_1 \varepsilon_4 \left(\alpha_3 E_5 + \beta'_6 E_6 \right) = \alpha_3 \varepsilon_5 E_5 + \beta_6 \varepsilon_6 E_6,$$
$$\varepsilon_2 \varepsilon_3 \left(\frac{\alpha_2 \alpha_5}{\alpha_4} E_5 + \beta'_7 E_6 \right) = \frac{\alpha_2 \alpha_5}{\alpha_4} \varepsilon_5 E_5 + \beta_7 \varepsilon_6 E_6,$$
$$\varepsilon_1 \varepsilon_5 \left(\alpha_4 E_6 \right) = \alpha_4 \varepsilon_6 E_6,$$
$$\varepsilon_2 \varepsilon_4 \left(\alpha_5 E_6 \right) = \alpha_5 \varepsilon_6 E_6.$$

It follows $\varepsilon_1 \varepsilon_2 = \varepsilon_3$, $\varepsilon_1 \varepsilon_3 = \varepsilon_4$, $\varepsilon_1 \varepsilon_4 = \varepsilon_5 = \varepsilon_2 \varepsilon_3$ and $\varepsilon_1 \varepsilon_5 = \varepsilon_6 = \varepsilon_2 \varepsilon_4$ hence $\varepsilon_2 = \varepsilon_4 = \varepsilon_6 = 1$, $\varepsilon_1 = \varepsilon_3 = \varepsilon_5$. Using these relations we have $\varepsilon_1 \varepsilon_2 = \varepsilon_1 = \varepsilon_5$, $\varepsilon_1 \varepsilon_3 = \varepsilon_6 = 1$. Therefore one has $\beta'_2 = \beta_2$, $\beta'_5 = \beta_5$. Assume that $J \neq \emptyset$. If $\beta_1 = \beta'_1 > 0$, then one has the addition condition $\varepsilon_1 \varepsilon_2 = \varepsilon_4$. If $\beta_3 = \beta'_3 > 0$, then we obtain $\varepsilon_1 \varepsilon_2 = \varepsilon_6$. If $\beta_4 = \beta'_4 > 0$, then we get $\varepsilon_1 \varepsilon_3 = \varepsilon_5$. If $\beta_6 = \beta'_6 > 0$, then one has $\varepsilon_1\varepsilon_4 = \varepsilon_6$. If $\beta_7 = \beta_7' > 0$, then we have $\varepsilon_2\varepsilon_3 = \varepsilon_6$. In all these cases we obtain $\varepsilon_i = 1, i = 1, \dots, 6$. Hence we obtain that the metric Lie algebra $\mathfrak{n}_{6,15}(\alpha_i, \beta_j)$ is uniquely determined.

If the map $E_i \longmapsto \varepsilon_i E_i$ is an orthogonal automorphism of $\mathfrak{n}_{6,15}(\alpha_i, \beta_j)$, then

$$\begin{aligned}
\varepsilon_1\varepsilon_2 (\alpha_1 E_3 + \beta_1 E_4 + \beta_2 E_5 + \beta_3 E_6) &= \alpha_1\varepsilon_3 E_3 + \beta_1\varepsilon_4 E_4 + \beta_2\varepsilon_5 E_5 + \beta_3\varepsilon_6 E_6,\\
\varepsilon_1\varepsilon_3 (\alpha_2 E_4 + \beta_4 E_5 + \beta_5 E_6) &= \alpha_2\varepsilon_4 E_4 + \beta_4\varepsilon_5 E_5 + \beta_5\varepsilon_6 E_6,\\
\varepsilon_1\varepsilon_4 (\alpha_3 E_5 + \beta_6 E_6) &= \alpha_3\varepsilon_5 E_5 + \beta_6\varepsilon_6 E_6,\\
\varepsilon_2\varepsilon_3 \left(\frac{\alpha_2\alpha_5}{\alpha_4} E_5 + \beta_7 E_6\right) &= \frac{\alpha_2\alpha_5}{\alpha_4}\varepsilon_5 E_5 + \beta_7\varepsilon_6 E_6,\\
\varepsilon_1\varepsilon_5 (\alpha_4 E_6) &= \alpha_4\varepsilon_6 E_6,\\
\varepsilon_2\varepsilon_4 (\alpha_5 E_6) &= \alpha_5\varepsilon_6 E_6.
\end{aligned}$$

If $J = \emptyset$, then we obtain $\varepsilon_2 = \varepsilon_4 = \varepsilon_6 = 1$, $\varepsilon_1 = \varepsilon_3 = \varepsilon_5$. It follows that the group of orthogonal automorphisms of $\mathfrak{n}_{6,15}(\alpha_i, \beta_j)$ is isomorphic to the group given by (2a) of Theorem 3.5.

If $J \neq \emptyset$, then $\varepsilon_i = 1, i = 1, \dots, 6$. The group of orthogonal automorphisms of $\mathfrak{n}_{6,15}(\alpha_i, \beta_j)$ is trivial. Hence assertion (2) is proved. □

Corollary 3.6. *Let $(\aleph_{6,15}(\alpha_i, \beta_j), \langle ., . \rangle)$ be the connected and simply connected Riemannian nilmanifold corresponding to the metric Lie algebra $(\mathfrak{n}_{6,15}(\alpha_i, \beta_j), \langle ., . \rangle)$. The isometry group of $(\aleph_{6,15}(\alpha_i, \beta_j), \langle ., . \rangle)$ is*

$$\mathcal{I}((\aleph_{6,15}(\alpha_i, \beta_j)) = \begin{cases} \mathbb{Z}_2 \ltimes \aleph_{6,15}(\alpha_i, \beta_j) & if\, J = \emptyset, \\ \aleph_{6,15}(\alpha_i, \beta_j) & if\, J \neq \emptyset. \end{cases}$$

Now we consider the Lie algebra $\ell_{6,16}$.

Definition 3.7. *Let $\{E_1, E_2, E_3, E_4, E_5, E_6\}$ be an orthonormal basis in the Euclidean vector space $\mathbb{E}^6$. Denote by $\mathfrak{n}_{6,16}(a, b, c, d, \beta_j)$, $a, b, c, d, \beta_j \in \mathbb{R}, j = 1, \dots, 8$ with $a, b, c, d \neq 0$ the metric Lie algebra defined on $\mathbb{E}^6$ by the non-vanishing commutators*

$$\begin{aligned}
&[E_1, E_2] = aE_3 + \beta_1 E_4 + \beta_2 E_5 + \beta_3 E_6, && [E_1, E_3] = bE_4 - \left(\frac{c}{a} + \frac{b\beta_8}{d}\right) E_5 + \beta_4 E_6,\\
&[E_1, E_4] = cE_5 + \beta_5 E_6, && [E_1, E_5] = \beta_6 E_6,\\
&[E_2, E_3] = \beta_7 E_6, && [E_2, E_4] = \beta_8 E_6,\\
&[E_2, E_5] = dE_6, && [E_4, E_3] = \frac{cd}{a} E_6.
\end{aligned}$$

Theorem 3.8. *Let $\langle ., . \rangle$ be an inner product on the 6-dimensional filiform Lie algebra $\ell_{6,16}$.*

(1) *There is a unique metric Lie algebra $\mathfrak{n}_{6,16}(a, b, c, d, \beta_j)$ which is isometrically isomorphic to the metric Lie algebra $(\ell_{6,16}, \langle ., . \rangle)$ with the properties that $a, b, c, d > 0$ and one of the following cases is satisfied* 1. $\beta_1 > 0, \beta_3 > 0$, 2. $\beta_1 = 0$, $\beta_3 > 0$, $\beta_4 > 0$, 3. $\beta_3 = 0$, $\beta_1 > 0$, $\beta_4 > 0$, 4. $\beta_1 = \beta_3 = 0$, $\beta_4 > 0, \beta_5 > 0$, 5. $\beta_1 = \beta_3 = \beta_4 = 0$, $\beta_5 > 0, \beta_8 > 0$, 6. $\beta_1 = \beta_3 = \beta_5 = 0$, $\beta_4 > 0, \beta_8 > 0$, 7. $\beta_1 = \beta_3 = \beta_4 = \beta_5 = 0$, $\beta_8 \geq 0$, $\beta_6 \geq 0$.

(2) *The group of orthogonal automorphisms of $\mathfrak{n}_{6,16}(a, b, c, d, \beta_j)$ is the group:*

(a) *for* $\beta_1 = \beta_3 = \beta_4 = \beta_5 = \beta_6 = \beta_8 = 0$: $\mathcal{OA}(\mathfrak{n}_{6,16}(a,b,c,d,\beta_j)) = \{TE_i = \varepsilon_1 E_i, i = 1,6, TE_i = \varepsilon_2 E_i, i = 2,4, TE_i = \varepsilon_1\varepsilon_2 E_i, i = 3,5, \varepsilon_1, \varepsilon_2 = \pm 1\} \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$.

(b) *for* $\beta_1 = \beta_3 = \beta_4 = \beta_5 = \beta_8 = 0, \beta_6 > 0$:
$\mathcal{OA}(\mathfrak{n}_{6,16}(a,b,c,d,\beta_j)) = \{TE_3 = E_3, TE_5 = E_5, TE_i = \varepsilon E_i, i = 1,2,4,6, \varepsilon = \pm 1\} \simeq \mathbb{Z}_2$.

(c) *for* $\beta_1 = \beta_3 = \beta_4 = \beta_5 = \beta_6 = 0, \beta_8 > 0$:
$\mathcal{OA}(\mathfrak{n}_{6,16}(a,b,c,d,\beta_j)) = \{TE_1 = E_1, TE_6 = E_6, TE_i = \varepsilon E_i, i = 2,3,4,5, \varepsilon = \pm 1\} \simeq \mathbb{Z}_2$.

(d) *if one of the following cases holds* 1. $\beta_1 > 0, \beta_3 > 0$, 2. $\beta_1 = 0$, $\beta_3 > 0$, $\beta_4 > 0$, 3. $\beta_3 = 0$, $\beta_1 > 0$, $\beta_4 > 0$, 4. $\beta_1 = \beta_3 = 0$, $\beta_4 > 0, \beta_5 > 0$, 5. $\beta_1 = \beta_3 = \beta_4 = 0$, $\beta_5 > 0, \beta_8 > 0$, 6. $\beta_1 = \beta_3 = \beta_5 = 0$, $\beta_4 > 0, \beta_8 > 0$, 7. $\beta_1 = \beta_3 = \beta_4 = \beta_5 = 0$, $\beta_8 > 0$, $\beta_6 > 0$, *then one has* $\mathcal{OA}(\mathfrak{n}_{6,16}(a,b,c,d,\beta_j))$ *is trivial.*

Proof. The subspaces $\mathrm{span}(G_i, \ldots, G_6), i = 1, \ldots, 6$ of $\ell_{6,16}$ form a descending series of ideals which are invariant under all automorphisms of $\ell_{6,16}$ (see [4, the proof of Theorem 4]). The Gram-Schmidt process applied to the ordered basis $\{G_6, G_5, G_4, G_3, G_2, G_1\}$ yields an orthonormal basis $\{F_1, F_2, F_3, F_4, F_5, F_6\}$ of $\ell_{6,16}$ such that the vector F_i is a positive multiple of G_i modulo the subspace span $(G_j; j > i)$ and orthogonal to span $(G_j; j > i)$. The orthogonal direct sum $\mathbb{R}F_1 \oplus \cdots \oplus \mathbb{R}F_6$ is a framing of $(\ell_{6,16}, \langle .,. \rangle)$. Expressing the vectors of the new basis in the form $F_i = \sum_{k=i}^{6} a_{ik} G_k$ with $a_{ii} > 0$ we get

$$
\begin{aligned}
&[F_1, F_2] = aF_3 + \beta_1 F_4 + \beta_2 F_5 + \beta_3 F_6, && [F_1, F_3] = bF_4 - \left(\frac{c}{a} + \frac{b\beta_8}{d}\right) F_5 + \beta_4 F_6, \\
&[F_1, F_4] = cF_5 + \beta_5 F_6, && [F_1, F_5] = \beta_6 F_6, \\
&[F_2, F_3] = \beta_7 F_6, && [F_2, F_4] = \beta_8 F_6, \\
&[F_2, F_5] = dF_6, && [F_4, F_3] = \frac{cd}{a} F_6,
\end{aligned} \tag{3.3}
$$

with $a,b,c,d > 0$ and $\{a,b,c,d,\beta_j\} \in \mathbb{R}$. Changing the orthonormal basis: $\tilde{F_1} = -F_1, \tilde{F_2} = F_2, \tilde{F_3} = -F_3, \tilde{F_4} = F_4, \tilde{F_5} = -F_5, \tilde{F_6} = -F_6$ we obtain

$$
\begin{aligned}
&[\tilde{F_1}, \tilde{F_2}] = a\tilde{F_3} - \beta_1 \tilde{F_4} + \beta_2 \tilde{F_5} + \beta_3 \tilde{F_6}, && [\tilde{F_1}, \tilde{F_3}] = b\tilde{F_4} + \left(\frac{c}{a} + \frac{b\beta_8}{d}\right) \tilde{F_5} - \beta_4 \tilde{F_6}, \\
&[\tilde{F_1}, \tilde{F_4}] = c\tilde{F_5} + \beta_5 \tilde{F_6}, && [\tilde{F_1}, \tilde{F_5}] = -\beta_6 \tilde{F_6}, \\
&[\tilde{F_2}, \tilde{F_3}] = \beta_7 \tilde{F_6}, && [\tilde{F_2}, \tilde{F_4}] = -\beta_8 \tilde{F_6}, \\
&[\tilde{F_2}, \tilde{F_5}] = d\tilde{F_6}, && [\tilde{F_4}, \tilde{F_3}] = \frac{cd}{a} \tilde{F_6}.
\end{aligned}
$$

Similarly, the change of the basis: $\tilde{F_1} = F_1, \tilde{F_2} = -F_2, \tilde{F_3} = -F_3, \tilde{F_4} = -F_4, \tilde{F_5} = -F_5, \tilde{F_6} = F_6$ yields

$$
\begin{aligned}
&[\tilde{F_1}, \tilde{F_2}] = a\tilde{F_3} + \beta_1 \tilde{F_4} + \beta_2 \tilde{F_5} - \beta_3 \tilde{F_6}, && [\tilde{F_1}, \tilde{F_3}] = b\tilde{F_4} - \left(\frac{c}{a} + \frac{b\beta_8}{d}\right) \tilde{F_5} - \beta_4 \tilde{F_6}, \\
&[\tilde{F_1}, \tilde{F_4}] = c\tilde{F_5} - \beta_5 \tilde{F_6}, && [\tilde{F_1}, \tilde{F_5}] = -\beta_6 \tilde{F_6}, \\
&[\tilde{F_2}, \tilde{F_3}] = \beta_7 \tilde{F_6}, && [\tilde{F_2}, \tilde{F_4}] = \beta_8 \tilde{F_6}, \\
&[\tilde{F_2}, \tilde{F_5}] = d\tilde{F_6}, && [\tilde{F_4}, \tilde{F_3}] = \frac{cd}{a} \tilde{F_6}.
\end{aligned}
$$

Hence there is an orthonormal basis such that in commutators (3.3) for the coefficients one of the following cases holds: 1. $\beta_1 > 0, \beta_3 > 0$, 2. $\beta_1 = 0$, $\beta_3 > 0$, $\beta_4 > 0$, 3. $\beta_3 = 0$, $\beta_1 > 0$, $\beta_4 > 0$, 4. $\beta_1 = \beta_3 = 0$, $\beta_4 > 0, \beta_5 > 0$, 5. $\beta_1 = \beta_3 = \beta_4 = 0$, $\beta_5 > 0, \beta_8 > 0$, 6. $\beta_1 = \beta_3 = \beta_5 = 0$, $\beta_4 > 0, \beta_8 > 0$, 7. $\beta_1 = \beta_3 = \beta_4 = \beta_5 = 0$, $\beta_8 \geq 0$, $\beta_6 \geq 0$. This proves the existence of a metric Lie algebra $\mathfrak{n}_{6,16}(a, b, c, d, \beta_j)$ satisfying Theorem 3.8 (1).

Let the linear map $T : \mathfrak{n}_{6,16}(a, b, c, d, \beta_j) \to \mathfrak{n}_{6,16}(a', b', c', d', \beta'_j)$ be an isometric isomorphism. The decomposition $\mathbb{R}\, E_1 \oplus \mathbb{R}\, E_2 \oplus \mathbb{R}\, E_3 \oplus \mathbb{R}\, E_4 \oplus \mathbb{R}\, E_5 \oplus \mathbb{R}\, E_6$ is a framing of both Lie algebras, where $a, a', b, b', c, c', d, d' > 0$. Hence by Lemma 2.1 we have $a = a'$, $b = b'$, $c = c'$, $d = d'$ and $|\beta'_j| = \beta_j$ for all $j = 1, \ldots, 8$. Let be $T(E_i) = \varepsilon_i E_i$, $\varepsilon_i = \pm 1$, $i = 1, \ldots, 6$, then we obtain from $[TE_i, TE_j]' = T[E_i, E_j]$, $i, j = 1, \ldots, 6$, using the commutation relations (3.3) the equations

$$\begin{aligned}
\varepsilon_1\varepsilon_2 \left(aE_3 + \beta'_1 E_4 + \beta'_2 E_5 + \beta'_3 E_6\right) &= a\varepsilon_3 E_3 + \beta_1\varepsilon_4 E_4 + \beta_2\varepsilon_5 E_5 + \beta_3\varepsilon_6 E_6,\\
\varepsilon_1\varepsilon_3 \left(bE_4 - \left(\frac{c}{d} + \frac{b\beta'_8}{d}\right) E_5 + \beta'_4 E_6\right) &= b\varepsilon_4 E_4 - \left(\frac{c}{d} + \frac{b\beta_8}{d}\right)\varepsilon_5 E_5 + \beta_4\varepsilon_6 E_6,\\
\varepsilon_1\varepsilon_4 \left(cE_5 + \beta'_5 E_6\right) &= c\varepsilon_5 E_5 + \beta_5\varepsilon_6 E_6,\\
\varepsilon_1\varepsilon_5 \left(\beta'_6 E_6\right) &= \beta_6\varepsilon_6 E_6,\\
\varepsilon_2\varepsilon_3 \left(\beta'_7 E_6\right) &= \beta_7\varepsilon_6 E_6,\\
\varepsilon_2\varepsilon_4 \left(\beta'_8 E_6\right) &= \beta_8\varepsilon_6 E_6,\\
\varepsilon_2\varepsilon_5 \left(dE_6\right) &= d\varepsilon_6 F_6,\\
\varepsilon_4\varepsilon_3 \left(\frac{cd}{a} E_6\right) &= \frac{cd}{a}\varepsilon_6 E_6.
\end{aligned}$$

It follows $\varepsilon_1\varepsilon_2 = \varepsilon_3$, $\varepsilon_1\varepsilon_3 = \varepsilon_4$, $\varepsilon_1\varepsilon_4 = \varepsilon_5$, $\varepsilon_4\varepsilon_3 = \varepsilon_6 = \varepsilon_2\varepsilon_5$, then one has $\varepsilon_4 = \varepsilon_2, \varepsilon_1\varepsilon_2 = \varepsilon_3 = \varepsilon_5$, $\varepsilon_6 = \varepsilon_1$. Using these relations we have $\varepsilon_1\varepsilon_2 = \varepsilon_5$, $\varepsilon_2\varepsilon_3 = \varepsilon_1 = \varepsilon_6$. Therefore one has $\beta'_2 = \beta_2$, $\beta'_7 = \beta_7$. If $\beta_1 = \beta'_1 > 0$, $\beta_3 = \beta'_3 > 0$, we get additionally $\varepsilon_1\varepsilon_2 = \varepsilon_4$ and $\varepsilon_1\varepsilon_2 = \varepsilon_6$, hence $\varepsilon_i = 1, i = 1, \ldots, 6$.

If $\beta_1 = \beta'_1 = 0$, $\beta_3 = \beta'_3 > 0$, $\beta_4 = \beta'_4 > 0$, then we obtain in addition $\varepsilon_1\varepsilon_2 = \varepsilon_6 = \varepsilon_1\varepsilon_3$, hence $\varepsilon_i = 1, i = 1, \ldots, 6$.

If $\beta_3 = \beta'_3 = 0$, $\beta_1 = \beta'_1 > 0$, $\beta_4 = \beta'_4 > 0$, then we have in addition $\varepsilon_1\varepsilon_2 = \varepsilon_4$ and $\varepsilon_1\varepsilon_3 = \varepsilon_6$, hence $\varepsilon_i = 1, i = 1, \ldots, 6$.

If $\beta_1 = \beta'_1 = \beta_3 = \beta'_3 = 0$, $\beta_4 = \beta'_4 > 0, \beta_5 = \beta'_5 > 0$, then one has in addition $\varepsilon_1\varepsilon_3 = \varepsilon_6 = \varepsilon_1\varepsilon_4$, hence $\varepsilon_i = 1, i = 1, \ldots, 6$.

If $\beta_1 = \beta'_1 = \beta_3 = \beta'_3 = \beta_4 = \beta'_4 = 0$, $\beta_5 = \beta'_5 > 0, \beta_8 = \beta'_8 > 0$, we get additionally $\varepsilon_1\varepsilon_4 = \varepsilon_6 = \varepsilon_2\varepsilon_4$, hence $\varepsilon_i = 1, i = 1, \ldots, 6$.

If $\beta_1 = \beta'_1 = \beta_3 = \beta'_3 = \beta_5 = \beta'_5 = 0$, $\beta_4 = \beta'_4 > 0, \beta_8 = \beta'_8 > 0$, then we obtain in addition $\varepsilon_1\varepsilon_3 = \varepsilon_6 = \varepsilon_2\varepsilon_4$, hence $\varepsilon_i = 1, i = 1, \ldots, 6$.

If $\beta_1 = \beta'_1 = \beta_3 = \beta'_3 = \beta_4 = \beta'_4 = \beta_5 = \beta'_5 = 0$, $\beta_8 \geq 0$, $\beta_6 \geq 0$, then one has $\beta'_8 \geq 0$, $\beta'_6 \geq 0$ and the uniqueness of the Lie algebra $\mathfrak{n}_{6,16}(a, b, c, d, \beta_j)$ follows from Lemma 2.1.

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If the map $E_i \longmapsto \epsilon_i E_i$ is an orthogonal automorphism of $\mathfrak{n}_{6,16}(a,b,c,d,\beta_j)$, then

$$\begin{aligned}
\varepsilon_1\varepsilon_2\left(aE_3+\beta_1E_4+\beta_2E_5+\beta_3E_6\right) &= a\varepsilon_3E_3+\beta_1\varepsilon_4E_4+\beta_2\varepsilon_5E_5+\beta_3\varepsilon_6E_6,\\
\varepsilon_1\varepsilon_3\left(bE_4-\left(\frac{c}{d}+\frac{b\beta_8}{d}\right)E_5+\beta_4E_6\right) &= b\varepsilon_4E_4-(\frac{c}{d}+\frac{b\beta_8}{d})\varepsilon_5E_5+\beta_4\varepsilon_6E_6,\\
\varepsilon_1\varepsilon_4\left(cE_5+\beta_5E_6\right) &= c\varepsilon_5E_5+\beta_5\varepsilon_6E_6,\\
\varepsilon_1\varepsilon_5\left(\beta_6E_6\right) &= \beta_6\varepsilon_6E_6,\\
\varepsilon_2\varepsilon_3\left(\beta_7E_6\right) &= \beta_7\varepsilon_6E_6,\\
\varepsilon_2\varepsilon_4\left(\beta_8E_6\right) &= \beta_8\varepsilon_6E_6,\\
\varepsilon_2\varepsilon_5\left(dE_6\right) &= d\varepsilon_6F_6,\\
\varepsilon_4\varepsilon_3\left(\frac{cd}{a}E_6\right) &= \frac{cd}{a}\varepsilon_6E_6.
\end{aligned}$$

If $\beta_1=\beta_3=\beta_4=\beta_5=\beta_6=\beta_8=0$, then one has $\varepsilon_4=\varepsilon_2$, $\varepsilon_1\varepsilon_2=\varepsilon_3=\varepsilon_5$ and $\varepsilon_6=\varepsilon_1$. It follows that the group of orthogonal automorphisms of $\mathfrak{n}_{6,16}(a,b,c,d,\beta_j)$ with $\beta_1=\beta_3=\beta_4=\beta_5=\beta_6=\beta_8=0$ is isomorphic to the group given by (2a) of Theorem 3.8.

If $\beta_1=\beta_3=\beta_4=\beta_5=0$, $\beta_8=0, \beta_6>0$, then one has in addition $\varepsilon_1\varepsilon_5=\varepsilon_6$. Hence $\varepsilon_5=1=\varepsilon_3$, $\varepsilon_1=\varepsilon_2=\varepsilon_4=\varepsilon_6$. The group of orthogonal automorphisms of $\mathfrak{n}_{6,16}(a,b,c,d,\beta_j)$ with $\beta_1=\beta_3=\beta_4=\beta_5=0$, $\beta_8=0, \beta_6>0$ is isomorphic to the group given by (2b) of Theorem 3.8.

If $\beta_1=\beta_3=\beta_4=\beta_5=0$, $\beta_6=0, \beta_8>0$, then one has in addition $\varepsilon_2\varepsilon_4=\varepsilon_6$. Hence $\varepsilon_6=1=\varepsilon_1$, $\varepsilon_2=\varepsilon_3=\varepsilon_4=\varepsilon_5$. It follows that the group of orthogonal automorphisms of $\mathfrak{n}_{6,16}(a,b,c,d,\beta_j)$ with $\beta_1=\beta_3=\beta_4=\beta_5=\beta_6=0, \beta_8>0$ is isomorphic to the group given by (2c) of Theorem 3.8.

Finally, assume that one of the following cases is satisfied: 1. $\beta_1>0, \beta_3>0$, 2. $\beta_1=0$, $\beta_3>0$, $\beta_4>0$, 3. $\beta_3=0$, $\beta_1>0$, $\beta_4>0$, 4. $\beta_1=\beta_3=0$, $\beta_4>0, \beta_5>0$, 5. $\beta_1=\beta_3=\beta_4=0$, $\beta_5>0, \beta_8>0$, 6. $\beta_1=\beta_3=\beta_5=0$, $\beta_4>0, \beta_8>0$, 7. $\beta_1=\beta_3=\beta_4=\beta_5=0$, $\beta_8>0$, $\beta_6>0$. In all these cases we have $\varepsilon_i=1, i=1,\dots,6$. Hence the group of orthogonal automorphisms is trivial. This proves assertions (2). □

Corollary 3.9. *Let $(\aleph_{6,16}(a,b,c,d,\beta_j),\langle .,.\rangle)$ be the connected and simply connected Riemannian nil-manifold corresponding to the metric Lie algebra $(\mathfrak{n}_{6,16}(a,b,c,d,\beta_j),\langle .,.\rangle)$. The isometry group of $(\aleph_{6,16}(a,b,c,d,\beta_j),\langle .,.\rangle)$ is*

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$$\mathcal{I}((\aleph_{6,16}(a,b,c,d,\beta_j)) = \begin{cases} \mathbb{Z}_2 \times \mathbb{Z}_2 \ltimes \aleph_{6,16}(a,b,c,d,\beta_j) & if\ \beta_j = 0, j = 1,3,4,5,6,8, \\ \mathbb{Z}_2 \ltimes \aleph_{6,14}(a,b,c,d,\beta_j) & if\ \beta_6 > 0, \beta_j = 0, j = 1,3,4,5,8, \\ & or\ \beta_8 > 0, \beta_j = 0, j = 1,3,4,5,6, \\ \aleph_{6,16}(a,b,c,d,\beta_j) & if\ \beta_1 > 0, \beta_3 > 0, \\ & or\ \beta_1 = 0, \beta_3 > 0, \beta_4 > 0, \\ & or\ \beta_3 = 0, \beta_1 > 0, \beta_4 > 0, \\ & or\ \beta_1 = \beta_3 = 0, \beta_4 > 0, \beta_5 > 0, \\ & or\ \beta_1 = \beta_3 = \beta_4 = 0, \beta_5 > 0, \beta_8 > 0, \\ & or\ \beta_1 = \beta_3 = \beta_5 = 0, \beta_4 > 0, \beta_8 > 0, \\ & or\ \beta_j = 0, j = 1,3,4,5, \beta_6 > 0, \beta_8 > 0. \end{cases}$$

Now consider the Lie algebra $\ell_{6,17}$ with its canonical basis $\{G_1, G_2, G_3, G_4, G_5, G_6\}$.

Definition 3.10. *Let $\{E_1, E_2, E_3, E_4, E_5, E_6\}$ be an orthonormal basis in the Euclidean vector space $\mathbb{E}^6$. Denote by $\mathfrak{n}_{6,17}(\alpha_i, \beta_j), \alpha_i, \beta_j \in \mathbb{R}, i = 1, \ldots, 5, j = 1, \ldots, 6$ with $\alpha_i \neq 0$ the metric Lie algebra defined on $\mathbb{E}^6$ by the non-vanishing commutators*

$$[E_1, E_2] = \alpha_1 E_3 + \beta_1 E_4 + \beta_2 E_5 + \beta_3 E_6, \qquad [E_1, E_3] = \alpha_2 E_4 + \beta_4 E_5 + \beta_5 E_6,$$
$$[E_1, E_4] = \alpha_3 E_5 + \beta_6 E_6, \qquad [E_1, E_5] = \alpha_4 E_6,$$
$$[E_2, E_3] = \alpha_5 E_6.$$

Theorem 3.11. *Let $\langle ., . \rangle$ be an inner product on the 6-dimensional filiform Lie algebra $\ell_{6,17}$.*

(1) *There is a unique metric Lie algebra $(\mathfrak{n}_{6,17}(\alpha_i, \beta_j), \langle ., . \rangle)$ with $\alpha_i, \beta_j \in \mathbb{R}$ such that $\alpha_i > 0$ satisfying*
   (a) *$(\mathfrak{n}_{6,17}(\alpha_i, \beta_j), \langle ., . \rangle)$ is isometrically isomorphic to the metric Lie algebra $(\ell_{6,17}, \langle ., . \rangle)$,*
   (b) *if the set $J = \{j \in \{1,3,4,6\} : \beta_j \neq 0\} \neq \emptyset$, then $\beta_{j_\circ} > 0$ for the minimal element $j_\circ \in J$.*

(2) *The group of orthogonal automorphisms of $\mathfrak{n}_{6,17}(\alpha_i, \beta_j)$ with the respect to $\{E_1, \ldots, E_6\}$ is the group:*
   (a) *if the set $J = \emptyset$, then one has $\mathcal{OA}(\mathfrak{n}_{6,17}(\alpha_i, \beta_j)) = \{TE_i = \varepsilon E_i, i = 1,2,4,6, TE_3 = E_3, TE_5 = E_5, \varepsilon = \pm 1\} \simeq \mathbb{Z}_2$.*
   (b) *if the set $J \neq \emptyset$, then $\mathcal{OA}(\mathfrak{n}_{6,17}(\alpha_i, \beta_j))$ is trivial.*

Proof. From the proof of [4, Theorem 4] the subspaces $\text{span}(G_i, \ldots, G_6), i = 1, \ldots, 6$ of $\ell_{6,17}$ form a descending series of ideals which are invariant under all automorphisms of $\ell_{6,17}$. Using Gram-Schmidt process applied to the ordered basis $\{G_6, G_5, G_4, G_3, G_2, G_1\}$ yields an orthonormal basis $\{F_1, F_2, F_3, F_4, F_5, F_6\}$ of $\ell_{6,17}$ such that the vector F_i is a positive multiple of G_i modulo the subspace span $(G_j; j > i)$ and orthogonal to span $(G_j; j > i)$. The orthogonal direct sum $\mathbb{R}F_1 \oplus \cdots \oplus \mathbb{R}F_6$ is a framing of $(\ell_{6,17}, \langle ., . \rangle)$. Expressing the vectors of the new basis in the form $F_i = \sum_{k=i}^{6} a_{ik} G_k$ with

$a_{ii} > 0$ we get

$$(3.4)\qquad \begin{aligned} &[F_1, F_2] = \alpha_1 F_3 + \beta_1 F_4 + \beta_2 F_5 + \beta_3 F_6, && [F_1, F_3] = \alpha_2 F_4 + \beta_4 F_5 + \beta_5 F_6,\\ &[F_1, F_4] = \alpha_3 F_5 + \beta_6 F_6, && [F_1, F_5] = \alpha_4 F_6,\\ &[F_2, F_3] = \alpha_5 F_6, \end{aligned}$$

where $\alpha_i > 0$ $i = 1, \dots, 5$, and $\{\alpha_i, \beta_j\} \in \mathbb{R}$. Changing the orthonormal basis, $F_1 \mapsto -F_1$, $F_2 \mapsto -F_2$, $F_3 \mapsto F_3$, $F_4 \mapsto -F_4$, $F_5 \mapsto F_5$, $F_6 \mapsto -F_6$, yields

$$\begin{aligned} &[F_1, F_2] = \alpha_1 F_3 - \beta_1 F_4 + \beta_2 F_5 - \beta_3 F_6, && [F_1, F_3] = \alpha_2 F_4 - \beta_4 F_5 + \beta_5 F_6,\\ &[F_1, F_4] = \alpha_3 F_5 - \beta_6 F_6, && [F_1, F_5] = \alpha_4 F_6,\\ &[F_2, F_3] = \alpha_5 F_6, \end{aligned}$$

Hence there is an orthonormal basis satisfying (3.4) such that if the set $J = \{j \in \{1, 3, 4, 6\}, \beta_j \neq 0\} \neq \emptyset$, then we may assume that $\beta_{j_\circ} > 0$ for the minimal element $j_\circ \in J$. Hence the existence of metric Lie algebra $\mathfrak{n}_{6,17}(\alpha_i, \beta_j)$ satisfying Theorem 3.11 (1) is proved.

Let the linear map $T : \mathfrak{n}_{6,17}(\alpha_i, \beta_j) \to \mathfrak{n}_{6,17}(\alpha_i', \beta_j')$ be an isometric isomorphism. The decomposition $\mathbb{R}\,E_1 \oplus \mathbb{R}\,E_2 \oplus \mathbb{R}\,E_3 \oplus \mathbb{R}\,E_4 \oplus \mathbb{R}\,E_5 \oplus \mathbb{R}\,E_6$ is a framing of both Lie algebras, where $\alpha_i, \alpha_i' > 0$. Hence by Lemma 2.1 we have $\alpha_i = \alpha_i'$, moreover $|\beta_j'| = \beta_j$ for all $j = 1, \dots, 6$. Let be $T(E_i) = \varepsilon_i E_i$, $\varepsilon_i = \pm 1$, $i = 1, \dots, 6$, then we obtain from $[TE_i, TE_j]' = T[E_i, E_j]$, $i, j = 1, \dots, 6$, using the commutation relations (3.4) the equations

$$\begin{aligned} \varepsilon_1\varepsilon_2(\alpha_1 E_3 + \beta_1' E_4 + \beta_2' E_5 + \beta_3' E_6) &= \alpha_1\varepsilon_3 E_3 + \beta_1\varepsilon_4 E_4 + \beta_2\varepsilon_5 E_5 + \beta_3\varepsilon_6 E_6,\\ \varepsilon_1\varepsilon_3(\alpha_2 E_4 + \beta_4' E_5 + \beta_5' E_6) &= \alpha_2\varepsilon_4 E_4 + \beta_4\varepsilon_5 E_5 + \beta_5\varepsilon_6 E_6,\\ \varepsilon_1\varepsilon_4(\alpha_3 E_5 + \beta_6' E_6) &= \alpha_3\varepsilon_5 E_5 + \beta_6\varepsilon_6 E_6,\\ \varepsilon_1\varepsilon_5(\alpha_4 E_6) &= \alpha_4\varepsilon_6 E_6,\\ \varepsilon_2\varepsilon_3(\alpha_5 E_6) &= \alpha_5\varepsilon_6 E_6. \end{aligned}$$

It follows $\varepsilon_1\varepsilon_2 = \varepsilon_3$, $\varepsilon_1\varepsilon_3 = \varepsilon_4$, $\varepsilon_1\varepsilon_4 = \varepsilon_5$, and $\varepsilon_1\varepsilon_5 = \varepsilon_6 = \varepsilon_2\varepsilon_3$ hence $\varepsilon_3 = \varepsilon_5 = 1$ $\varepsilon_1 = \varepsilon_2 = \varepsilon_4 = \varepsilon_6$. Using these relations we have $\varepsilon_1\varepsilon_2 = \varepsilon_5 = 1$, $\varepsilon_1\varepsilon_3 = \varepsilon_1 = \varepsilon_6$. Therefore one has $\beta_2' = \beta_2$, $\beta_5' = \beta_5$. Assume that $J \neq \emptyset$. If $\beta_1 = \beta_1' > 0$, then one has the additional condition $\varepsilon_1\varepsilon_2 = \varepsilon_4$. If $\beta_3 = \beta_3' > 0$, then we obtain $\varepsilon_1\varepsilon_2 = \varepsilon_6$. If $\beta_4 = \beta_4' > 0$, then we get $\varepsilon_1\varepsilon_3 = \varepsilon_5$. If $\beta_6 = \beta_6' > 0$, then $\varepsilon_1\varepsilon_4 = \varepsilon_6$. In all these cases we obtain $\varepsilon_i = 1, i = 1, \dots, 6$. This proves that the Lie algebra $\mathfrak{n}_{6,17}(\alpha_i, \beta_j)$ is uniquely determined.

The map $E_i \longmapsto \epsilon_i E_i$ is an orthogonal automorphism of $\mathfrak{n}_{6,17}(\alpha_i, \beta_j)$. Hence one has

$$\begin{aligned} \varepsilon_1\varepsilon_2(\alpha_1 E_3 + \beta_1 E_4 + \beta_2 E_5 + \beta_3 E_6) &= \alpha_1\varepsilon_3 E_3 + \beta_1\varepsilon_4 E_4 + \beta_2\varepsilon_5 E_5 + \beta_3\varepsilon_6 E_6,\\ \varepsilon_1\varepsilon_3(\alpha_2 E_4 + \beta_4 E_5 + \beta_5 E_6) &= \alpha_2\varepsilon_4 E_4 + \beta_4\varepsilon_5 E_5 + \beta_5\varepsilon_6 E_6,\\ \varepsilon_1\varepsilon_4(\alpha_3 E_5 + \beta_6 E_6) &= \alpha_3\varepsilon_5 E_5 + \beta_6\varepsilon_6 E_6,\\ \varepsilon_1\varepsilon_5(\alpha_4 E_6) &= \alpha_4\varepsilon_6 E_6,\\ \varepsilon_2\varepsilon_3(\alpha_5 E_6) &= \alpha_5\varepsilon_6 E_6. \end{aligned}$$

If $J = \emptyset$, then we obtain $\varepsilon_3 = \varepsilon_5 = 1$, $\varepsilon_1 = \varepsilon_2 = \varepsilon_4 = \varepsilon_6$. It follows that the group of orthogonal automorphisms of $\mathfrak{n}_{6,17}(\alpha_i, \beta_j)$ is isomorphic to the group given by (2a) of Theorem 3.11.

If $J \neq \emptyset$, then $\varepsilon_i = 1, i = 1, \dots, 6$, i.e. the group of orthogonal automorphisms of $\mathfrak{n}_{6,17}(\alpha_i, \beta_j)$ is trivial. Hence assertion (2) is proved. □

Corollary 3.12. *Let $(\aleph_{6,17}(\alpha_i, \beta_j), \langle ., . \rangle)$ be the connected and simply connected Riemannian nilmanifold belonging to the metric Lie algebra $(\mathfrak{n}_{6,17}(\alpha_i, \beta_j), \langle ., . \rangle)$. The isometry group of $(\aleph_{6,17}(\alpha_i, \beta_j), \langle ., . \rangle)$ is*

$$\mathcal{I}((\aleph_{6,17}(\alpha_i, \beta_j)) = \begin{cases} \mathbb{Z}_2 \ltimes \aleph_{6,17}(\alpha_i, \beta_j) & if J = \emptyset, \\ \aleph_{6,17}(\alpha_i, \beta_j) & if J \neq \emptyset. \end{cases}$$

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