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ON SOME NEW DEVELOPMENTS IN THE THEORY OF SUBGROUP LATTICES OF GROUPS

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ABSTRACT. A rather natural way for trying to obtain a lattice-theoretic characterization of a class of groups $\mathcal{X}$ is to replace the concepts appearing in the definition of $\mathcal{X}$ by lattice-theoretic concepts. The first to use this idea were Kontorovič and Plotkin who in 1954 introduced the notion of modular chain in a lattice, as translation of a central series of a group, to determine a lattice-theoretic characterization of the class of torsion-free nilpotent groups. The aim of this paper is to present a recent application of this translation method to some generalized nilpotency properties.

If G is any group, the set $\mathcal{L}(G)$ of all subgroups of G is a complete lattice with respect to the ordinary set-theoretic inclusion. In this lattice, the operations $\vee$ and $\wedge$ are given by the rules

$$X \wedge Y = X \cap Y$$

and

$$X \vee Y = \langle X, Y \rangle$$

for each pair (X, Y) of subgroups of G.

The central theme in the Theory of Subgroup Lattices of Groups is the relation between the structure of a group and that of its subgroup lattice. The oldest result of this type is a beautiful theorem due to O. Ore, showing that a group has distributive subgroup lattice if and only if it is locally cyclic (see [10]).

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Let G and $\bar{G}$ be groups. A *projectivity* from G onto $\bar{G}$ is an isomorphism from the lattice $\mathcal{L}(G)$ of all subgroups of G onto the subgroup lattice $\mathcal{L}(\bar{G})$; if there exists such a map, $\bar{G}$ is said to be a projective image of G. A group class $\mathcal{X}$ is invariant under projectivities if all projective images of groups in $\mathcal{X}$ are likewise $\mathcal{X}$-groups. Relevant examples of group classes invariant under projectivities are the class of all finite groups, the class of all periodic groups, the class of all soluble groups (as we will point out in the following).

Moreover, it is clear that any group class defined by a lattice-theoretic property is invariant under projectivities; in particular, it follows from Ore's theorem quoted above that the class of locally cyclic groups is invariant under projectivities.

On the other hand, the class of all abelian groups does not have such property. In fact, the elementary abelian group of order 9 and the symmetric group of degree 3 have isomorphic subgroup lattices. Clearly, this example shows that also the class of nilpotent groups is not invariant under projectivities.

There is a rather natural way for trying to obtain a lattice-theoretic characterization of a class $\mathcal{X}$ of groups. We take a suitable definition of the groups in $\mathcal{X}$ and replace concepts appearing in it by lattice-theoretic concepts that are equivalent to them or nearly so, for instance: finite group by finite lattice, since a group G is finite if and only if $\mathcal{L}(G)$ is finite.

Many classes of groups are defined via properties of normal subgroups. To decide wheter such a class is invariant under projectivities, the study of projective images of normal subgroups is crucial.

Let L be any lattice. An element m of L is said to be *modular* if

$$m \vee (x \wedge y) = (m \vee x) \wedge y$$

for all $x, y \in L$ such that $m \leq y$ and

$$x \vee (m \wedge y) = (x \vee m) \wedge y$$

for all $x, y \in L$ such that $x \leq y$. The lattice L is *modular* if all its elements are modular, i.e. if the identity

$$(x \vee y) \wedge z = x \vee (y \wedge z)$$

holds in L, whenever x, y, z are elements such that $x \leq z$.

A subgroup M of a group G is called *modular* if M is a modular element in $\mathcal{L}(G)$; obviously, every normal subgroup N of G is modular in G and for every projectivity ϕ of G, N^{ϕ} is a modular subgroup of G^{ϕ}. In particular, any projective image of an abelian group has modular subgroup lattice, and groups with modular subgroup lattice (the so called *M-groups*) can be considered as suitable lattice approximations of abelian groups. Locally finite groups with modular subgroup lattice have been completely classified by K. Iwasawa [8] almost eighty years ago, while a full characterization of periodic groups with modular subgroup lattice has been obtained by R. Schmidt [15] in 1986, after that Olshanskii in 1979, using a complicated inductive construction, produced the first examples of Tarski groups and extended Tarski groups. A Tarski group is an infinite simple group in which every

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proper non-trivial subgroup has prime order and an extended Tarski group is a group G containing a cyclic normal subgroup N of prime-power order such that G/N is a Tarski group and each subgroup of G either contains or is contained in N; such groups have modular subgroup lattice and they were used by R. Schimdt in the final description of periodic M-groups.

Tarski groups show that the approximation of normal subgroups with modular subgroups cannot be satisfactory in infinite groups; in the following we will see that this gap has been filled with the introduction, due to G. Zacher, of the concept of permodular subgroup. Anyway, for finite groups, the situation is quite different, since modularity seems to be strong enough to translate normality. For instance, applying the translation method described above to one of the equivalent definitions of solubility, R. Schmidt obtained the following lattice-theoretic characterization (see [12]).

Theorem 0.1. *(R. Schmidt, 1968) A finite group G is soluble if and only if there exists a chain*

$$\{1\} = M_0 \leq M_1 \leq \cdots \leq M_n = G$$

of modular subgroups of G such that $[M_{i+1}/M_i]$ is modular, for every $i < n$.

The proof of Theorem 0.1, as well as a lattice characterization of the class of finite simple groups, is based on the following basic result (see [13]).

Theorem 0.2. *(R. Schimdt, 1969) If M is a maximal proper modular subgroup of a finite group G, then either M is normal in G or G/M_G is nonabelian of order pq for primes p and q.*

As we said before, in order to give lattice translations of results involving normality in the universe of infinite groups, a turning point has been reached when G. Zacher introduced permodularity, which is a condition intermediate between modularity and normality (see [21]). A modular subgroup M of a group G is *permodular* if it satisfies the following condition:

($\star$) for every cyclic subgroup X in G and for every $H \leq G$ such that $M \leq H \leq \langle M, X\rangle$ and the interval $[\langle M, X\rangle/H]$ is finite, then the index $|\langle M, X\rangle : H|$ is finite.

Since cyclic subgroups of G (by Ore's theorem) and the finiteness of the index of a subgroup (by a theorem of Zacher-Rips) can be recognized in the subgroup lattice, then ($\star$) is a lattice-theoretic property.

Every normal subgroup, but even every permutable subgroup X of a group G (i.e. a subgroup such that $XH = HX$ for every subgroup H of G) is permodular.

A lattice L is permodular if every its element is modular and

(*) for all a, b in L such that $b \leq a$, [a/b] is a finite lattice whenever it has finite length.

Actually, it is known that $\mathcal{L}(G)$ is permodular if and only if every subgroup of G is permodular (see [16, Theorem 6.4.3])

Using permodularity, E. Previato, in 1978, obtained a lattice characterization of finitely generated soluble groups (see [11]), while G. Busetto and G. Zacher described supersoluble groups (see [2] and [21]):

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Theorem 0.3. *(E. Previato, 1978) A finitely generated group G is soluble if and only if there exists a chain*

$$\{1\} = M_0 \leq M_1 \leq \cdots \leq M_n = G$$

of permodular subgroups of G such that $[M_{i+1}/M_i]$ is a permodular lattice, for every $i < n$.

Theorem 0.4. *(G. Busetto, 1980, G. Zacher, 1982) A group G is supersoluble if and only if there exists a chain*

$$\{1\} = M_0 \leq M_1 \leq \cdots \leq M_n = G$$

of permodular subgroups of G such that the lattice $[M_{i+1}/M_i]$ is distributive and satisfies the maximal condition, for every $i < n$.

In 1999, G. Zacher [22] obtained the following description of soluble groups, which involves only lattice-theoretic properties:

Theorem 0.5. *(G. Zacher, 1999) A group G is soluble if and only if there exist a group lattice L and elements a, b of L such that the following conditions hold:*

(1) *a and b are permodular in L, $a \wedge b = 0$, $a \vee b = I$, $[a/0] \simeq \mathcal{L}(G)$, $[b/0] \simeq \mathcal{L}(\mathbb{Q})$ (where 0 is the minimum of L, I is the maximum of L and $\mathbb{Q}$ is the additive group of the rational numbers);*

(2) *$[a/0]$ contains a chain $0 = s_0 \leq s_1 \leq \cdots \leq s_n = a$ where s_i is permodular in $[s_{i+1} \vee b/0]$, and $[s_{i+1}/s_i]$ is a permodular lattice, for every $i < n$.*

Since there exists a non-soluble group containing a permutable subgroup M such that $[M/\{1\}]$ and $[G/M]$ are permodular (see [20]), in Theorem 0.3 the hypothesis that the group is finitely generated cannot be dropped. Nevertheless, Theorem 0.5 does not put an end to the investigation in that direction: while emphasizing its nature purely lattice-theoretic, R. Schmidt in [18] argues that "it would be nice to find lattice-theoretic approximations of *normal subgroup* and *abelian group* for which the translation method yields a lattice-theoretic characterization of the class of infinite soluble groups".

In order to move from solubility to nilpotency, we have to consider the problem of the lattice translation of centrality; clearly, a subgroup C of a group G is contained in $Z(G)$ if and only if $\langle x, C\rangle$ is abelian for all $x \in G$.

Replacing commutativity by modularity, or permodularity, in this property, a subgroup M of a group G is said to be *(per)modularly embedded* in G if the subgroup lattice of $\langle x, M\rangle$ is (per)modular for each element x of G.

The first to use this idea, in 1954, were Kontorovič and Plotkin, introducing the concept of a permodular chain as translation of central series of groups.

The definition of (per)modular chain is done in a general (complete) lattice and it is quite technical (see for instance [4]). For groups, substantially we have:

a (per)modular chain of a group G is a chain

$$\{1\} = M_0 \leq M_1 \leq \cdots \leq M_n = G$$

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of (per)modular subgroups of G such that for any $x \in G$ the interval $[\langle M_{i+1}, x\rangle / M_i]$ is (per)modular for every $i < n$ (M_{i+1} is (per)modularly embedded in $[G/M_i]$, for every $i < n$).

P. G. Kontorovič and B. I. Plotkin used this notion to obtain the following lattice-theoretic characterization of the class of torsion-free nilpotent groups (see [9]):

Theorem 0.6. *(P.G. Kontorovič and B.I. Plotkin, 1954) A torsion-free group is nilpotent if and only if there exists a modular chain*

$$\{1\} = M_0 \leq M_1 \leq \cdots \leq M_n = G$$

such that for every $i = 0, \ldots, n$, the index $|\langle x\rangle : \langle x\rangle \cap M_i|$ is infinite for each element of $G \setminus M_i$.

Recall that a group G is said to be a P*-group* if it is a semidirect product of an abelian group A of prime exponent and of a group of prime order which induces a non-trivial power automorphism on A. We have already noted that the symmetric group of degree 3 is lattice-isomorphic to the elementary abelian group of order 9. More in general, if G is a P-group, the subgroup lattice of G is isomorphic to the subgroup lattice of an abelian group (see [16, Theorem 2.2.3]). The situation is not better in the mixed case; in fact, we can consider that the semidirect product $\bar{G}$ of an abelian Prüfer group T and an infinite cyclic group $\langle z\rangle$ with respect to the automorphism $t^\phi = t^{1+p}$ ($t \in T$) is lattice-isomorphic to the direct product $G = T \times \langle z\rangle$. Clearly, G is abelian and $|\zeta_n(\bar{G})| = p^n$ for all n, so that $\bar{G}$ is not nilpotent.

For a finite group G, there are some properties that are equivalent to the fact that G is nilpotent; this fact suggests a different approach to obtain group classes which are interesting from a lattice point of view and are related with nilpotency. More precisely, we can turn our attention to some given characterizations of nilpotency, applying the usual process of translation to the property stated in it, and then we can examine the lattice theoretic condition that emerges. This was the approach applied by R. Schmidt to the classical Wielandt's Theorem, stating that a finite group is nilpotent if and only if every maximal subgroup is normal: translating this latter property in a lattice-theoretic condition, he obtained the following description of the class of finite groups in which every maximal subgroup is modular (see [14]).

Theorem 0.7. *(R. Schmidt, 1970) Every maximal subgroup of a finite group G is modular if and only if G is supersoluble and induces an automorphism group of order* 1 *or prime in every complemented chief factor of G.*

Clearly, for a finite group, also the condition that in each subgroup every maximal subgroup is normal, is equivalent to nilpotency. If we translate this, we obtain the condition for a group G that

($\cdot$) every maximal subgroup of every subgroup H of G is modular in H.

Moreover, since a subgroup M is modular in G if and only if for every subgroup K of G, the lattices $[(M \vee K)/M]$ and $[K/K \wedge M]$ are isomorphic, the condition ($\cdot$) is equivalent to the following:

if $K, M \leq G$ and M is maximal in $\langle M, K\rangle$, then $M \cap K$ is maximal in K

The groups with that property are called *lower semimodular* and were characterized by Ito (see [7]).

Theorem 0.8. *(N. Ito, 1951) A finite group G is lower semimodular if and only if G is supersoluble and induces an automorphism group of order 1 or prime in every chief factor of G.*

The relation with groups with a (per)modular chain is established in the following result (see [17]):

Theorem 0.9. *(R. Schimdt, 2013) If G is a finite group with a modular chain, then G is lower semimodular.*

The converse is not true and R. Schmidt in [17] shows that the structure of finite groups with a modular chain is much more restricted than the structure of lower semimodular groups.

In 2011 a pupil of R. Schmidt, S. Andreeva, developed in her PhD dissertation a new method consisting in the translation of well-known group-theoretical theorems into the theory of subgroup lattices of groups and applied it on results based on finite minimal non-nilpotent groups and on finite p-nilpotent groups.

Andreeva traced back her point of view to the paper [3], which contains a lattice translation of the famous theorem of I. Schur stating that in a group whose centre has finite index the commutator subgroup is finite (see [19]).

Theorem 0.10. *(M. De Falco, F. de Giovanni, C. Musella, 2008) Let G be a group containing a modularly embedded subgroup of finite index. Then G has a finite normal subgroup N such that the subgroup lattice $\mathcal{L}(G/N)$ is modular.*

Schur's theorem was generalized by R. Baer [1], by proving that if G is a group whose k-th centre $\zeta_k(G)$ has finite index for some positive integer k, then the corresponding term $\gamma_{k+1}(G)$ of the lower central series of G is finite.

If a group G has the k-th centre of finite index, then from a lattice point of view, it satisfies the property that there exists a finite chain

$$\{1\} \leq \cdots \leq M_i \leq M_{i+1} \leq \cdots \leq M_k$$

of permodular subgroups of G such that each M_{i+1} is permodularly embedded in $[G/M_i]$ and M_k has finite index in G; in this case, M_k is k-permodularly embedded in G, and G has a k-permodularly embedded subgroup of finite index (see [4]). A partial converse of Baer's Theorem was obtained by P. Hall [5], while it is known that for a finitely generated group the condition that $G/\zeta_k(G)$ is finite is equivalent to the fact that $\gamma_{k+1}(G)$ is finite (see for instance [6, Theorem 2.10]). For this latter result, the following translation has been recently obtained (see [4]):

Theorem 0.11. *Let G be a finitely generated group and let k be a positive integer. Then G contains a k-permodularly embedded subgroup of finite index if and only if there exist a finite permodular subgroup*

L and a finite chain $L = L_0 \leq \cdots \leq L_i \leq L_{i+1} \leq \cdots \leq L_k = G$ of permodular subgroups of G such that each L_{i+1} is permodularly embedded in $[G/L_i]$.

Let's put our attention again on Wielandt's Theorem, which, in the finite case, characterizes nilpotent groups as the groups in which every maximal subgroup is normal. This latter condition, in the general case, is equivalent to the fact that the factor group $G/\Phi(G)$ is abelian, where $\Phi(G)$ is the Frattini subgroup of G (i.e. the intersection of all maximal subgroups of G, with the assumption $\Phi(G) = G$ if G has no maximal subgroups).

We have seen that for finite groups, the descritpion of groups in which every maximal subgroup is modular gives rise to an interesting class of groups, in which the behaviour of complemented chief factors plays a crucial role. A similar phenomenon appears in infinite groups, as the following theorem shows (see [4]):

Theorem 0.12. *Let G be a group. Then every maximal subgroup of G is permodular if and only if $G/\Phi(G)$ is metabelian and for every complemented chief factor X/Y of G with $\Phi(G) \leq X$, X/Y has prime order and $G/C_G(X/Y)$ is either trivial or of prime order.*

Proof. Assume first that every maximal subgroup of G is permodular, so that it follows from [4, Lemma 4.3] that $G/\Phi(G)$ is metabelian; therefore if X/Y is a chief factor of G such that $\Phi(G) \leq X$, then X/Y is abelian, and hence the statement follows from [4, Lemma 4.4].

Conversely, if $G/\Phi(G)$ has the structure described in the statement, then [4, Lemma 4.6] yields that every maximal subgroup of $G/\Phi(G)$ is permodular, and hence the same happens to G. □

A natural generalization, for an infinite group G, of the condition that every maximal subgroup is normal, is the property that G has finitely many non-normal maximal subgroups. Let $\delta(G)$ be the intersection of all non-normal maximal subgroups of G (with the assumption $\delta(G) = G$ if every maximal subgroup is normal in G). When all but finitely many maximal subgroups of the group G are normal, then each non-normal maximal subgroup of G has finite index; in fact, notice that all conjugates of non-normal maximal subgroups have the same property, and so they have finite index, since they are self-normalizing. It follows that a group G has finitely many non-normal maximal subgroups if and only if $\delta(G)$ has finite index in G.

Proposition 0.13. *A group G has finitely many non-normal maximal subgroups if and only if the Frattini factor group $G/\Phi(G)$ is central-by-finite.*

Proof. The subgroup $[G, \delta(G)]$ is contained in $\delta(G)$, and hence in every non-normal maximal subgroups; moreover, every maximal normal subgroups of G contains G', so that $[G, \delta(G)] \leq \Phi(G)$, i.e. $\delta(G)/\Phi(G) \leq \zeta(G/\Phi(G))$.

On the other hand, if M is any maximal non-normal subgroup of G, then $M\zeta(G) < G$, so that $\zeta(G) \leq M$; therefore $\zeta(G) \leq \delta(G)$.

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It follows that $\delta(G)/\Phi(G) = \zeta(G/\Phi(G))$, and hence G has finitely many non-normal maximal subgroups if and only if $G/\Phi(G)$ is central-by-finite. □

In [4] groups G with finitely many non-permodular maximal subgroups have been characterized in terms of the existence in $G/\Phi(G)$ of a subgroup of finite index with an embedding property that generalizes the property of being central. More precisely, a normal subgroup N of a group G is θ-*embedded* in G if every abelian complemented chief factor X/Y of G with $X \leq N$ has prime order and $G/C_G(X/Y)$ is either trivial or of prime order. Let now $\theta(G)$ be the intersection of all maximal subgroups of a group G which are not permodular (again, we put $\theta(G) = G$ if all maximal subgroups of G are permodular). Then $(\theta(G))'' \leq \Phi(G) \leq \theta(G)$ by [4, Lemma 4.3] and $\theta(G)$ is θ-embedded in G by [4, Lemma 4.4]. Moreover, keeping in mind that all conjugates of non-permodular maximal subgroups have the same property, it follows that a group G has finitely many non-permodular maximal subgroups if and only if $\theta(G)$ has finite index in G.

Theorem 0.14. *A group G has finitely many non-permodular maximal sugroups if and only if $G/\Phi(G)$ contains a normal subgroup $L/\Phi(G)$ of finite index such that $L/\Phi(G)$ is metabelian and θ-embedded in $G/\Phi(G)$.*

If a group G is finite over its hypercentre, then there are only finitely many non-normal maximal subgroups. The proof of the lattice translation of this result, stated in the following Theorem 0.16 (see [4]), naturally involves the consideration of k-permodularly embedded subgroups of finite groups; in particular, it is relevant the fact that any k-permodularly embedded normal subgroup of a finite group G is contained in $\theta(G)$ (see [4, Lemma 4.8]). A crucial step in the proof of this latter property is the following Lemma, which slightly generalizes the statement and the proof of Theorem 2.11 in [17]. Here, we present it in details, for the convenience of the reader.

Proposition 0.15. *Let G be a finite group and let X be a normal subgroup which is k-permodularly embedded in G for some positive integer k. Then there exists a normal subgroup N of G of prime order such that $N \leq X$ and the factor group $G/C_G(N)$ is either trivial or of prime order.*

Proof. Clearly, we can assume that $X \cap Z(G) = \{1\}$.

Let $\{1\} = X_0 < X_1 \leq \cdots \leq X_i \leq X_{i+1} \leq \cdots \leq X_k = X$ be a finite chain of permodular subgroups of G such that each X_{i+1} is permodularly embedded in $[G/X_i]$, and let A be a subgroup of X_1 of prime order p.

Suppose first that A is normal in G, so that $G/C_G(A)$ is cyclic, and there exists an element $x \in G$ such that $G = C_G(A)\langle x\rangle$. Since X_1 is permodularly embedded in G, the group $A\langle x\rangle$ has modular subgroup lattice, and hence $A\langle x\rangle/C_{\langle x\rangle}(A)$ is a nonabelian group of order pq for some prime q (see [16, Theorem 2.4.4]); therefore $G/C_G(A)$ has order q.

Suppose now that A is not normal in G. Since $A^G \leq X$, we have that $A^G \cap Z(G) = \{1\}$, so that A is not permutable in G (see [16, Theorem 5.2.9]). Since A is permodular in G (see [4, Corollary 2.7]), it

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follows from [16, Lemma 5.1.9] that A^G is a nonabelian P-group and $G = A^G \times K$ with $(|A^G|, |K|) = 1$. Let N be a normal subgroup of prime order of A^G; then $A^G/C_{A^G}(N)$ has prime order, so that N is normal in G and $G/C_G(N)$ has prime order. □

Theorem 0.16. *Let G be a group containing a k-permodularly embedded subgroup of finite index for some positive integer k. Then G has only finitely many maximal subgroups which are not permodular.*

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