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# IRREDUNDANT FAMILIES OF MAXIMAL SUBGROUPS OF FINITE SOLVABLE GROUPS

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Abstract. Let $\mathcal{M}$ be a family of maximal subgroups of a group $G$. We say that $\mathcal{M}$ is irredundant if its intersection is not equal to the intersection of any proper subfamily of $\mathcal{M}$. The maximal dimension of $G$ is the maximal size of an irredundant family of maximal subgroups of G. In this paper we study a class of solvable groups, called $\mathcal{M}$-groups, in which the maximal dimension has properties analogous to that of the dimension of a vector space such as the span property, the extension property and the basis exchange property.

## 1. Introduction

All groups considered in this paper are finite. If $G$ is a finite group, then we denote by $\Phi(G)$ the Frattini subgroup of $G$ and by $F(G)$ the Fitting subgroup of $G$. If $H$ and $K$ are subgroups of $G$, then $H \vee H = \langle H, K\rangle$ is the subgroup generated by $H$ and $K$. For other notation, terminology and results one can consult for example [6, 12, 13].

Let $I = \{1, 2, \ldots\}$. Following [5, 7] we say that a family $\mathcal{H} = \{H_i : i \in I\}$ of subgroups of a group $G$ is *irredundant*, if for any $i \in I$

$$\bigcap_{j\in I} H_j < \bigcap_{i\neq j\in I} H_j. \tag{1.1}$$

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It is easy to see that if $\mathcal{H}$ is a family of maximal subgroups of a group $G$, then $\mathcal{H}$ is irredundant if and only if, for any $i \in I$

$$\left(\bigcap_{i\neq j\in I} H_j\right) \vee H_i = G. \tag{1.2}$$

The *maximal dimension* of a group $G$, denoted by $Maxdim(G)$, is the maximal size of an irredundant family of maximal subgroups of $G$ (see [7]). The *minimal dimension* of a group $G$, denoted by $Mindim(G)$, is the minimal size of a maximal irredundant family of maximal subgroups of $G$ (see [8]). These invariants were investigated for example in [3, 5, 7, 8, 16, 17].

A family $\mathcal{H} = \{H_1, \dots, H_n\}$ of subgroups of a group $G$ is said to be *sequentially irredundant* (*s-irredundant* for short) if for any $2 \le i \le n$,

$$\bigcap_{j=1}^{i} H_j < \bigcap_{j=1}^{i-1} H_j. \tag{1.3}$$

If all subgroups of $\mathcal{H}$ are maximal, then the condition (1.3) is equivalent to the following one:

$$\left(\bigcap_{j=1}^{i-1} H_j\right) \vee H_i = G \tag{1.4}$$

for all $2 \le i \le n$.

Conditions (1.2) and (1.4) arise from the study of the dual Goldie dimension of dually balanced lattices. The notion of the dual Goldie (uniform) dimension was introduced in [9] for modular lattices. In [11] it was proved that the Goldie dimension has the same property in a larger class of lattices, called permutable in [11] or balanced in [10]. So we can consider the dual Goldie dimension in dually balanced lattices (see [2]). If $G$ is a group, $\mathcal{H} = \{H_1, \dots, H_n\}$ is a family of subgroups of $G$ and $I = \{1, 2, \dots, n\}$, then the subgroup lattice $L(G)$ of $G$ is *dually balanced* if the following implication holds

$$\left(\bigcap_{j=1}^{i-1} H_j\right) \vee H_i = G, \quad i \in I \setminus \{1\} \implies \left(\bigcap_{i\neq j\in I} H_j\right) \vee H_i = G, \quad i \in I.$$

In the paper we restrict our attention to maximal subgroups. We say that a group $G$ has *property* $\mathcal{M}$ (is an $\mathcal{M}$-*group* for short) if every s-irredundant family of maximal subgroups of $G$ is irredundant. It turns out that the maximal dimension of $\mathcal{M}$-groups and the dual Goldie dimension of groups with a dually balanced subgroup lattice have properties analogous to that of the dimension of vector spaces. That is why we borrow the terminology from linear algebra. We say that an irredundant family of maximal subgroups of $G$ whose intersection is the Frattini subgroup of $G$ is an *m-base* of $G$.

The aim of this paper is to initiate studies of $\mathcal{M}$-groups. In every finite group there exist maximal irredundant sets, but the intersections of these sets may properly contain the Frattini subgroup. We begin Section 2 by proving the following theorem.

**Theorem 1.1.** *If $G$ is an $\mathcal{M}$-group then every maximal irredundant set of maximal subgroups of $G$ is an m-base of $G$.*

Next we show that every $\mathcal{M}$-group is a minmax group. A minmax group, a notion introduced in [3], is a group with $Mindim(G) = Maxdim(G)$. We give examples of minmax groups which are not $\mathcal{M}$-groups. Thus the class of minmax groups is essentially larger then the class of $\mathcal{M}$-groups. However in $\mathcal{M}$-groups the maximal dimension satisfies additionally properties as described in the following.

**Theorem 1.2.** *Let $G$ be an $\mathcal{M}$-group, $\mathcal{M}(G)$ be a set of all maximal subgroups of $G$ and $\mathcal{I}$ be a family of all irredundant subsets of $\mathcal{M}(G)$. Then the pair $(\mathcal{M}(G), \mathcal{I})$ forms a matroid. In particular, m-bases of $G$ are bases of $(\mathcal{M}(G), \mathcal{I})$ and $Maxdim(G)$ is equal to the rank of $(\mathcal{M}(G), \mathcal{I})$.*

Section 3 is devoted to study the structure of solvable $\mathcal{M}$-groups. We characterize $\mathcal{M}$-groups using the notion of the Fitting series. It is the sequence of subgroups $F_i(G)$ of $G$ defined inductively (see [13]), by $F_0(G) = 1$ and

$$F_{i+1}(G)/F_i(G) = F(G/F_i(G))$$

for $i \geq 0$. The main results of this section are the following.

**Theorem 1.3.** *If $G$ is a Frattini-free solvable $\mathcal{M}$-group, then there exists a subgroup $H$ of $G$ such that $G = F(G) \rtimes H$, where $H \cap H^a$ is a maximal subgroup of $H$ or $H = H^a$, for all $a \in F(G)$. Moreover, $\Phi(G/F_i(G)) = 1$, for all $i \geq 0$.*

**Theorem 1.4.** *Let $G$ be a Frattini-free solvable group. If one of the following holds:*

(1) *$G$ is an elementary abelian p-group;*
(2) *$G = F(G) \rtimes H$, where $H$ is a group of prime order;*
(3) *$G$ is a direct product of groups given in (1) or in (2) with pair-wise coprime orders,*

*then $G$ is an $\mathcal{M}$-group.*

In Section 4 we focus on invariants introduced in [1] and studied in [1, 4], which are related to irredundant sets of maximal subgroups. For a non-trivial group $G$, the *intersection number* of $G$, denoted $\iota(G)$, is the minimum number of maximal subgroups whose intersection equals $\Phi(G)$ and the *inconjugate intersection number* of $G$, denoted $\hat{\iota}(G)$, is the minimum number of inconjugate maximal subgroups whose intersection equals $\Phi(G)$. If G has no inconjugate maximal subgroups whose intersection is equal to $\Phi(G)$, then we set $\hat{\iota}(G) = \infty$. Moreover, it is proved in [1] that if G is a non-trivial nilpotent group, then $\iota(G) = \hat{\iota}(G)$. The authors also posed there the following question.

**Question 1.5.** *What structural characteristics of $G$ imply that $\iota(G) = \hat{\iota}(G)$?*

In the next theorem we give a partial answer to the above question.

**Theorem 1.6.** *If $G$ is a solvable $\mathcal{M}$-group, then $\hat{\iota}(G) < \infty$ and $\iota(G) = \hat{\iota}(G)$.*

## 2. Irredundant sets of $\mathcal{M}$-groups

Let us start with two examples. They illustrate that Theorems 1.2 and Theorem 1.6 do not hold for general groups which are not $\mathcal{M}$-groups.

**Example 2.1.** Let $p, q$ be primes and $k > 1$ be an integer such that $q^k$ divides $p-1$. Let also $s$ be a primitive root of 1 modulo $p$ of degree $q^k$. Then in the group

$$G_1 = \langle a, x \mid a^p = x^{q^k} = 1, a^x = a^s \rangle$$

$M = \langle a, x^q \rangle$ is the unique maximal subgroup containing $a$; all other maximal subgroups are cyclic, generated by conjugates of $x$, i.e. they are of the form $M_i = \langle a^i x \rangle$, $i = 0, 1, \ldots, p-1$. Hence:

- $M_i \cap M_j = 1 = \Phi(G_1)$ for $i \neq j$, and
- $M \cap M_i = \langle b_i x^q \rangle$ for some $b_i \in \langle a \rangle$.

So $\{M_i, M_j\}$ is an m-base for all $i \neq j$ and $\{M, M_i\}$ is a maximal irredundant set of $G_1$, which is not contained in any m-base. So $\mathcal{M}(G_1)$ is not a matroid. However, $G_1$ is a minmax group and $Mindim = Maxdim(G_1) = 2$. Additionally, we obtain that $\iota(G_1) = 2$ and $\hat{\iota}(G_1) = \infty$, because the intersection of any two inconjugate maximal subgroups properly contains $\Phi(G_1)$.

On the other hand for $i = 0, 1, \ldots, p-1$,

$$M \vee M_i = (M \cap M_i) \vee M_j = G_1, \quad \text{but} \quad (M_i \cap M_j) \vee M = M.$$

So $G_1$ is not an $\mathcal{M}$-group.

**Example 2.2.** Let $p, q, r$ be different primes such that $qr \mid (p-1)$ and let $s$ and $t$ be positive integers which are primitive roots of unity modulo $p$ of degree $q$ and $r$ respectively. Let

$$G_2 = \langle a, x, y \mid a^p = x^q = y^r = 1, a^x = a^s, a^y = a^t, xy = yx \rangle.$$

Then $G_2$ has the following maximal subgroups: $M_x = \langle a, x \rangle$, $M_y = \langle a, y \rangle$, $M_i = \langle x, y \rangle^{a^i}$, $i = 0, 1, \ldots, p-1$. Since

$$M_x \vee M_0 = (M_x \cap M_0) \vee M_1 = G_2, \quad (M_0 \cap M_1) \vee M_x = M_x,$$

it follows that $G_2$ is not an $\mathcal{M}$-group.

Moreover, $\{M_x, M_y, M_0\}$ and $\{M_0, M_1\}$ are m-bases such that $M_x \cap M_y \cap M_0 = \Phi(G_2) = 1$, $M_0 \cap M_1 = 1$. So $G_2$ is not a minmax group and $\mathcal{M}(G_2)$ is not a matroid. However, it is easy to calculate that every maximal irredundant set of $G_2$ is an m-base of $G_2$. Additionally, we obtain that $\iota(G_2) = 2$ and $\hat{\iota}(G_2) = 3$.

*Proof. of Theorem 1.1.* Let $\mathcal{M} = \{M_1, \ldots, M_n\}$ be a maximal irredundant family of $G$ such that $M_1 \cap \cdots \cap M_n > \Phi(G)$. If for any maximal subgroup $M$ of $G$, $M_1 \cap M_2 \cap \cdots \cap M_n < M$, then

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$M_1 \cap M_2 \cap \cdots \cap M_n < \bigcap_{M \subseteq G} M = \Phi(G)$, a contradiction. Hence there exists a maximal subgroup $M_{n+1}$ of $G$ such that

$$(M_1 \cap \cdots \cap M_n) \vee M_{n+1} = G. \tag{2.1}$$

Since $\mathcal{M}$ is an irredundant family of $G$, it follows by (2.1) that $\mathcal{M} \cup \{M_{n+1}\}$ is s-irredundant. However, $\mathcal{M}$ is a maximal irredundant family of $G$, so $\mathcal{M} \cup \{M_{n+1}\}$ is not irredundant. This implies that $G$ is not an $\mathcal{M}$-group, which contradicts our assumption. □

**Lemma 2.3.** *Let $G$ be an $\mathcal{M}$-group and $\mathcal{M}$ be an m-base of $G$. If $\mathcal{N}$ is an irredundant family of $G$ then $|\mathcal{N}| \leq |\mathcal{M}|$.*

*Proof.* Assume that $\mathcal{M} = \{M_1, \ldots, M_n\}$ is an m-base of $G$, $\mathcal{N} = \{N_1, \ldots, N_k\}$ is an irredundant family of maximal subgroups of $G$ and $k > n$. Suppose, by induction, that $\mathcal{M}_i := \{M_1, \ldots, M_i, N_{i+1}, \ldots, N_k\}$ is irredundant. We want to show that there exists $M \in \mathcal{M}$ such that $\mathcal{M}_{i+1} = \{M_1, \ldots, M_i, M, N_{i+2}, \ldots, N_k\}$ is irredundant.

For this purpose set $H = M_1 \cap \cdots \cap M_i \cap N_{i+2} \cap \cdots \cap N_k$. Then, by assumption, we deduce $H \vee N_{i+1} = G$. Hence $H$ is not contained in $\Phi(G)$.

Suppose that $H \vee M_j = M_j$, for $j = 1, \ldots, n$. Then, by Theorem 1.1, $H \leq \bigcap_{j=1}^{n} M_j = \Phi(G)$, a contradiction. Thus there exists $j \in \{i+1, \ldots, n\}$ such that

$$H \vee M_j = (M_1 \cap \cdots \cap M_i \cap N_{i+2} \cap \cdots \cap N_k) \vee M_j = G. \tag{2.2}$$

Set $\mathcal{M}_{i+1} = \{M_1, \ldots, M_i, M_j, N_{i+2}, \ldots, N_k\}$. By induction hypothesis, we know that $\{M_1, \ldots, M_i, N_{i+2}, \ldots, N_k\}$ is irredundant. This and the equation (2.2) imply that $\mathcal{M}_{i+1}$ is s-irredundant. Since $G$ is an $\mathcal{M}$-group, $\mathcal{M}_{i+1}$ is irredundant.

Finally, we obtain an irredundant set $\mathcal{M}_n$ such that $\mathcal{M} \subset \mathcal{M}_n$. This contradicts the assumption that $\mathcal{M}$ is an m-base of $G$. The proof is complete. □

**Corollary 2.4.** *Every $\mathcal{M}$-group is a minmax group. In particular, the size of any m-base of an $\mathcal{M}$-group is equal to $Maxdim(G)$.*

The above facts show that the maximal dimension of $\mathcal{M}$-groups shares some properties with the dimension of vector spaces. Further we show that we can associate a matroid with a family of maximal subgroups such that $Maxdim(G)$ is a rank of this matroid. A matroid is a structure that generalizes the properties of independence of sets. We recall the definition of a matroid from [15, Chapter 7]. Although this definition can be given in a number of equivalent ways, in this text we concentrate on the one which characterizes a matroid in terms of bases.

Let $\mathcal{S}$ be a set and $\mathcal{B}$ be a non-empty family of subsets of $\mathcal{S}$ called bases, satisfying the axioms

- $(B_1)$ No proper subset of a base is a base.
- $(B_2)$ If $B_1, B_2 \in \mathcal{B}$, then for any $b \in B_1 \setminus B_2$, there exists $c \in B_2 \setminus B_1$, such that $(B_1 \setminus \{b\}) \cup \{c\} \in \mathcal{B}$. ( this condition is known as the *exchange bases property*).

*Proof of Theorem 1.2.* Let $\mathcal{M}(G)$ be a set of all maximal subgroups of a group $G$ and $\mathcal{B}$ be a family of all m-bases of $G$. Then the set $\mathcal{B}$ obviously satisfies the axiom $(B_1)$.

Now let $\mathcal{B}_1$ and $\mathcal{B}_2$ be different m-bases of $G$. By Corollary 2.4, $|\mathcal{B}_1| = |\mathcal{B}_2|$. So we may assume that $\mathcal{B}_1 = \{X_1, \ldots, X_n\}$ and $\mathcal{B}_2 = \{Y_1, \ldots, Y_n\}$. Moreover, suppose that for some $X_i \in \mathcal{B}_1$,

$$\bigcap(\mathcal{B}_1 \setminus \{X_i\}) \vee Y_j = Y_j,$$

for all $j = 1, \ldots, n$. Then

$$\bigcap(\mathcal{B}_1 \setminus \{X_i\}) \leq \bigcap \mathcal{B}_2 = \Phi(G),$$

a contradiction. Hence there exists $j \in \{1, \ldots, k\}$ such that $\bigcap(\mathcal{B}_1 \setminus \{X_i\}) \vee Y_j = G$. It follows that $(\mathcal{B}_1 \setminus \{X_i\}) \cup Y_j$ is s-irredundant, as $(\mathcal{B}_1 \setminus \{X_i\})$ is an irredundant family of $G$. Since $G$ is an $\mathcal{M}$-group, $(\mathcal{B}_1 \setminus \{X_i\}) \cup Y_j$ is irredundant so is an m-base of $G$. Hence the axiom $(B_2)$ holds. Thus the proof is complete. □

A matroid in which $\mathcal{S}$ is the set of all maximal subgroups of a group was considered for example in [14].

## 3. The structure of $\mathcal{M}$-groups

We start this section with proving that the class of $\mathcal{M}$-groups is closed due to taking homomorphic images and direct products of groups of coprime orders.

**Theorem 3.1.** *If $G$ is an $\mathcal{M}$-group then every its homomorphic image is an $\mathcal{M}$-group. Moreover, if $H$ is a normal subgroup of $G$, then $Maxdim(G/H) \leq Maxdim(G)$.*

*Proof.* Let $H$ be a normal subgroup of $G$ and set $\bar{G} = G/H$. Then, by the Correspondence Theorem (see [12, 1.4.6]), $\bar{M}$ is a maximal subgroup of $\bar{G}$ if and only if $M$ is a maximal subgroup of $G$ that contain $H$. Moreover, $\bar{\mathcal{X}}$ is an irredundant (an s-irredundant) family of maximal subgroups of $\bar{G}$ if and only if $\mathcal{X}$ is an irredundant (an s-irredundant) family of maximal subgroups of $G$ containing $H$. So $\bar{G}$ is an $\mathcal{M}$-group.

Furthermore, if $\bar{\mathcal{X}}$ is an m-base of $\bar{G}$, then $\mathcal{X}$ is an irredundant family of $G$. By Theorem 2.3, $|\mathcal{X}| \leq Maxdim(G)$. Hence $Maxdim(G/H) \leq Maxdim(G)$. □

Since the Frattini subgroup of a group $G$ is properly contained in every maximal subgroup of $G$, using the Correspondence Theorem we can formulate the following.

**Corollary 3.2.** *Let $G$ be a group. $G$ is an $\mathcal{M}$-group if and only if $G/\Phi(G)$ is an $\mathcal{M}$-group. In particular, in this case $Maxdim(G) = Maxdim(G/\Phi(G))$.*

**Theorem 3.3.** *Let $G_1$, $G_2$ be groups with coprime orders. Then $G_1$, $G_2$ are $\mathcal{M}$-groups if and only if $G_1 \times G_2$ is an $\mathcal{M}$-group. In particular*

$$Maxdim(G_1 \times G_2) = Maxdim(G_1) + Maxdim(G_2).$$

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*Proof.* By the Goursat Lemma, $M$ is a maximal subgroup of $G_1 \times G_2$ if and only if $M = M_1 \times G_2$ or $M = G_1 \times M_2$, where $M_1, M_2$ are maximal subgroups of $G_1, G_2$ respectively. Assume that $\mathcal{M} = \{M_1 \times G_2, \dots, M_r \times G_2, G_1 \times N_1, \dots, G_1 \times N_s\}$ is an s-irredundant family of maximal subgroups of $G_1 \times G_2$. Let $I = \{1, \dots, r\}$, $J = \{1, \dots, s\}$. Then

$$\bigcap_{l\in I}(M_l \times G_2) \cap \bigcap_{l\in J}(G_1 \times N_l) = \bigcap_{l\in I} M_l \times \bigcap_{l\in J} N_l.$$

Since $G_1, G_2$ are $\mathcal{M}$-groups, for all $i \in I = \{1, \dots, r\}$ and $j \in J = \{1, \dots, s\}$

$$\bigcap_{l\in I} M_l < \bigcap_{l\in I\setminus\{i\}} M_l \quad \text{and} \quad \bigcap_{l\in J} N_l < \bigcap_{l\in J\setminus\{j\}} N_l.$$

Hence for all $i \in I$ and for all $j \in J$

$$\bigcap_{l\in I} M_l \times \bigcap_{l\in J} N_l < \bigcap_{l\in I\setminus\{i\}} M_l \times \bigcap_{l\in J} N_l, \quad \bigcap_{l\in I} M_l \times \bigcap_{l\in J} N_l < \bigcap_{l\in I} M_l \times \bigcap_{l\in J\setminus\{j\}} N_l.$$

And further

$$\bigcap_{l\in I}(M_l \times G_2) \cap \bigcap_{l\in J}(G_1 \times N_l) < \bigcap_{l\in I\setminus\{i\}}(M_l \times G_2) \cap \bigcap_{l\in J}(G_1 \times N_l),$$

$$\bigcap_{l\in I}(M_l \times G_2) \cap \bigcap_{l\in J}(G_1 \times N_l) < \bigcap_{l\in I}(M_l \times G_2) \cap \bigcap_{l\in J\setminus\{j\}}(G_1 \times N_l).$$

It follows that $\mathcal{M}$ is irredundant so $G_1 \times G_2$ is an $\mathcal{M}$-group.

By Theorem 3.1 the proof is complete. □

**Lemma 3.4.** *Let $G$ be an abelian group and $\mathcal{H} = \{H_1, \dots, H_k\}$ be a set of subgroups of $G$. If for $i \in \{2, \dots, k\}$, $\left(\bigcap_{j=1}^{i-1} H_j\right) \vee H_i = G$ then $\left(\bigcap_{i\neq j} H_j\right) \vee H_i = G$, for all $i \in \{1, \dots, k\}$. In particular, $G$ is an $\mathcal{M}$-group.*

*Proof.* Let $I = \{1, \dots, l\}$. Suppose that $\left(\bigcap_{l=1}^{i-1} H_l\right) \vee H_i = G$, for $i \in \{2, \dots, k\}$. We show that $\left(\bigcap_{l\neq j} H_l\right) \vee H_j = G$ for $i \in I$, using induction on $k$ and Dedekind's Modular Law (see [12, 1.3.14.])

$$\left(\bigcap_{l\in I\setminus\{i\}} H_l\right) \vee H_i = \left(\bigcap_{l\in I\setminus\{i\}} H_l\right) \vee \left(\bigcap_{l\in I\setminus\{k\}} H_l\right) \vee H_i = \left[\left(\bigcap_{l\in I\setminus\{i,k\}} H_l\right) \cap \left(H_k \vee \bigcap_{l\in I\setminus\{k\}} H_l\right)\right] \vee H_i =$$

$$= \left[\left(\bigcap_{l\in I\setminus\{i,k\}} H_l\right) \cap G\right] \vee H_i = \left(\bigcap_{l\in I\setminus\{i,k\}} H_l\right) \vee H_i = G.$$

Moreover, we assume that $\mathcal{H}$ is an s-irredundant family of maximal subgroups of $G$. Then by (1.2) and (1.4), $\mathcal{H}$ is irredundant. So $G$ is an $\mathcal{M}$-group. □

The following theorem is proved by applying Theorem 3.2 and Lemma 3.4.

**Theorem 3.5.** *Every nilpotent group is an $\mathcal{M}$-group.*

*Proof of Theorem 1.3.* If $G$ is an abelian group, then we have $F_0(G) = 1$ and $F_i(G) = G$ for $i \in \{1, 2, \dots\}$. So $\Phi(G/F_i(G)) = 1$ for all $i \in \{0, 1, 2, \dots\}$.

Assume that $G$ is a non-abelian $\mathcal{M}$-group. By [6, Theorem A.10.6], $F(G) = F_1 \times \cdots \times F_n$, where $F_i$ is a minimal normal subgroup of $G$ for $i \in \{1, \dots, n\}$. Moreover, there exists a non-trivial subgroup $H$ of $G$ which is a complement of $F(G)$ in $G$. Hence $G$ is a semidirect product $G = F(G) \rtimes H$, that is each $F_i$ is a simple $F_p[H]$-module. In particular, $H$ acts faithfully on each $F_i$.

For arbitrary $i \in I$ we consider a group $G_i = F_i \rtimes H$. From the above paragraph, it follows that $H$ is a maximal subgroup of $G_i$. By Theorem 3.1, $G_i$ is an $\mathcal{M}$-group. Let $a \in F_i \setminus C_{F_i}(H)$ and $K$ be a maximal subgroup of $H$ properly containing $H \cap H^a$. By [6, Lemma A.16.3] we know that $H \cap H^a = C_H(a)$. Set $M_1 = F_i \rtimes K$, $\quad M_2 = H$, $\quad M_3 = H^a$. Since $H > K > H \cap H^a$ and $H^a \neq H \cap H^a \neq H$, we obtain that $(K \vee H^a) \cap F_i \neq 1$. It implies $M_1 \vee M_2 = (M_1 \cap M_2) \vee M_3 = G_i$, but $(M_2 \cap M_3) \vee M_1 = M_1$. Hence $\{M_1, M_2, M_3\}$ is s-irredundant but is not irredundant in $G_i$, which contradicts our assumption.

It follows that for all $a \in F_i \setminus C_{F_i}(H)$, $H \cap H^a = C_H(a)$ is a maximal subgroup of $H$. Hence $C_H(a)$ is a maximal subgroup of $H$ or $C_H(a) = H$, for all $a \in F_i$. Thus for any $i \in I$ $C_H(F_i) = \bigcap_{a \in F_i} C_H(a) \geq \Phi(H)$. By [6, Theorem A.13.8] we conclude

$$F(G) = \bigcap_{i=1}^{n} C_G(F_i) \geq \bigcap_{i=1}^{n} C_H(F_i) \geq \Phi(H).$$

Hence $\Phi(H) = 1$. So the statement $\Phi(G/F_i(G)) = 1$ is true for $i = 1$. Let $i > 1$, and suppose inductively that it is true for all $t < i + 1$. Hence $G/F_i(G)$ is Frattini-free. By Theorem 3.1, $G/F_i(G)$ is an $\mathcal{M}$-group. Since $G/F_i(G)$ is solvable, by induction assumption

$$1 = \Phi((G/F_i(G))/F(G/F_i(G))) = \Phi(G/F_{i+1}(G)).$$

The proof is complete. □

**Lemma 3.6.** *Let $G = A \rtimes H$, where $A, H$ are subgroups of $G$. If $A_1, A_2$ are subgroups of $A$ and $K$ is a conjugate of $H$, then*

$$A_1K \cap A_2K = (A_1 \cap A_2)K.$$

*Proof.* Let $g \in A_1K \cap A_2K$. Then $g = a_1k_1 = a_2k_2$ for some $a_1 \in A_1$, $a_2 \in A_2$ and $k_1, k_2 \in K$. Observe that $a_2^{-1}a_1 = k_2k_1^{-1} \in A \cap K = 1$. Hence $a_1 = a_2 \in A_1 \cap A_2$. So $g \in (A_1 \cap A_2)K$. It implies that $A_1K \cap A_2K \subseteq (A_1 \cap A_2)K$.

The inclusion $(A_1 \cap A_2)K \subseteq A_1K \cap A_2K$ is obvious. So the proof is complete. □

**Lemma 3.7.** *Let $G$ be a Frattini-free solvable group and $H$ be a complement of $F(G)$ in $G$. Assume further that $\{M_1, \dots, M_k\}$ is a set of maximal subgroups of $G$ supplementing $F(G)$. If $M_i \cap F(G) = N_i$ and $\mathcal{N} = \{N_1, \dots, N_k\}$ is an s-irredundant family of $F(G)$, then*

(1) *$\mathcal{N}$ is an irredundant family of $F(G)$;*
(2) *there exists a conjugate $K$ of $H$ such that $K \leq \bigcap_{i=1}^{k} M_i$;*
(3) *$\{M_1, \dots, M_k\}$ is an irredundant family of $G$.*

*Proof.* Let $\mathcal{N} = \{N_1, \dots, N_k\}$ and $I = \{1, \dots, k\}$. By [6, Theorem A.10.6], $G = F(G) \rtimes H$ and every $N_i$ is a maximal $H$-invariant subgroup of $F(G)$.

(1) By assumption we have $\bigcap_{j=1}^{i} N_j < \bigcap_{j=1}^{i-1} N_j$. Since $N_i$ is a maximal H-invariant subgroup of $F(G)$ and $\left(\bigcap_{j=1}^{i-1} N_j\right) \vee N_i$ is also $H$-invariant in $F(G)$, we conclude that $\bigcap_{j=1}^{i-1} N_j \vee N_i = F(G)$, for all $i \in I$. By Lemma 3.4

$$\left(\bigcap_{i \neq j} N_j\right) \vee N_i = F(G), \tag{3.1}$$

as $F(G)$ is abelian. So $\mathcal{N}$ is irredundant in $F(G)$.

(2) Clearly, $M_i = N_i H^{a_i}$ for some $a_i \in F(G)$. In view of (3.1), we may suppose that $a_i \in \bigcap_{j \neq i} N_j$. This means that $a_j \in N_i$ for all $j \in I$ and $j \neq i$. From this we conclude that $H^{a_1 \cdots a_k} \leq M_j$ for $j \in I$. Hence $H^{a_1 \cdots a_k} \leq \bigcap_{j \in I} M_j$. Set $K = H^{a_1 \cdots a_k}$.

(3) From the above consideration, it follows that $M_i = N_i K$, for $i \in I$. Then

$$\left(\bigcap_{j \neq i} M_j\right) \vee M_i = \left(\bigcap_{j \neq i} N_j K\right) \vee N_i K = \left(\bigcap_{j \neq i} N_j \vee N_i\right) K = G.$$

This completes the proof. □

*Proof of Theorem 1.4* If $G$ is an elementary abelian $p$-group, the proof follows from Theorem 3.5.

Assume that $G$ is non-abelian, that is $H \neq 1$. Let $\mathcal{M} = \{M_1, \dots, M_k\}$ be an s-irredundant family of maximal subgroups of $G$ and $I = \{1, \dots, k\}$. Suppose first that $M_j \in \mathcal{M}$ is a supplement of $F(G)$ and $N_j = M_j \cap F(G)$, for all $j \in I$. Then $N_j$ is a maximal $H$-invariant subgroup of $F(G)$ and $M_j = N_j H^{a_j}$, where $a_j \in F(G)$, for $j \in I$. Set $\mathcal{N} = \{N_1, \dots, N_k\}$. If $\mathcal{N}$ is s-irredundant in $F(G)$, then by Lemma 3.7, $\mathcal{M} = \{M_1, \dots, M_k\}$ is an irredundant family of $G$.

So we assume that $\mathcal{N}$ is not s-irredundant. This means that there exists $i \in I$, such that $\bigcap_{l=1}^{i-1} N_l = \bigcap_{l=1}^{i} N_l$. Suppose that $i$ is minimal in $I$ with this property. Since $\bigcap_{l=1}^{i} M_l > \bigcap_{l=1}^{i-1} M_l$, we conclude that $\bigcap_{l=1}^{i} M_l \leq F(G)$ and $\bigcap_{l=1}^{i-1} M_l$ contains a conjugate of $H$. If $\bigcap_{l=1}^{j-1} N_l = \bigcap_{l=1}^{j} N_l$ for some $j \in I$, $j > i$ then $F(G) \geq \bigcap_{l=1}^{j} M_l = \bigcap_{l=1}^{j-1} M_l$, a contradiction. So there exists at most one index $i$ such that $\bigcap_{l=1}^{i-1} N_l = \bigcap_{l=1}^{i} N_l$. It follows that $\mathcal{N} \setminus \{N_i\}$ is an s-irredundant family of subgroups of $F(G)$. So again by Lemma 3.7, $\mathcal{N} \setminus \{N_i\}$ is an irredundant family of H-invariant subgroups of $F(G)$ and $\mathcal{M} \setminus \{M_i\}$ is an irredundant family of maximal subgroups of $G$.

We have to show that $\mathcal{M}$ is irredundant. By Lemma 3.7, there exists $b \in F(G)$ satisfying $H^b \leq \bigcap_{l \in I \setminus \{i\}} M_l$. Since $\bigcap_{l=1}^{i} M_l \leq F(G)$, $H^b \nleq M_i$. This and Lemma 3.6 yield

$$\left(\bigcap_{l \neq i} M_l\right) \vee M_i = \left(\bigcap_{l \neq i} N_l H^b\right) \vee M_i = \left(\bigcap_{l \neq i} N_l\right) \vee H^b \vee M_i = G.$$

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Let $j \in I$ and $j \neq i$. If $j > i$, then

$$\left(\bigcap_{l\neq j} M_l\right) \vee M_j \geq \left(\bigcap_{l\neq j} N_l\right) \vee N_j \vee H^{a_j} = F(G) \vee H^{a_j} = G.$$

So suppose that $j < i$. If $\left(\bigcap_{l\neq j} N_l\right) \vee N_j = F(G)$, then

$$\left(\bigcap_{l\neq j} M_l\right) \vee M_j \geq \left(\bigcap_{l\neq j} N_l\right) \vee N_j \vee H^{a_j} = F(G) \vee H^{a_i} = G.$$

Hence we assume that $\bigcap_{l\neq j} N_l \leq N_j$. If $\left(\bigcap_{l\neq j,i} N_l\right) \leq N_i$, then $F(G) = \left(\bigcap_{l\neq j,i} N_l\right) \vee N_j = \left(\bigcap_{l\neq j} N_l\right) \vee N_j$, a contradiction. Thus we may further assume that

$$\left(\bigcap_{l\neq j,i} N_l\right) \vee N_i = F(G). \tag{3.2}$$

It follows that $a_i = c_1c_2$, $b = b_1b_2$, where $c_1, b_1 \in \bigcap_{l\neq j,i} N_l$ and $c_2, b_2 \in N_i$. If $h^{b_1}(h^{c_1})^{-1} \in N_i$ for all $h \in H$, then $h^{b_1} = h^{b_1}(h^{c_1})^{-1}h^{c_1}$, where $h^{c_1} \in M_i$. Hence $h^{b_1} \in M_i$ for all $h \in H$. So $H^{b_1} \subseteq M_i$ and further $H^b \subseteq M_i$, which contradicts the choice of $H^b$. So $h^{b_1}(h^{c_1})^{-1} \notin N_i$. It implies that $h^{b_1}(h^{c_1})^{-1} \notin N_j$. Otherwise $h^{b_1}(h^{c_1})^{-1} = b_1^{-1}c_1(b_1c_1^{-1})^{h^{-1}} \in \bigcap_{l\neq i} N_l < N_i$, a contradiction. Since $h^{b_1}(h^{c_1})^{-1} \in H^{c_1b_2} \vee H^{b_1b_2}$, we obtain

$$\left(\bigcap_{l\neq j} M_l\right) \vee M_j = \left[\left(\bigcap_{l\neq j,i} N_lH^b\right) \cap N_iH^{a_i}\right] \vee N_jH^b = \left[\left(\bigcap_{l\neq j,i} N_l\right) H^{c_1b_2} \cap N_iH^{c_1b_2}\right] \vee N_jH^{b_1b_2} =$$

$$= \left(\bigcap_{l\neq j} N_l\right) H^{c_1b_2} \vee N_jH^{b_1b_2} = H^{c_1b_2} \vee H^{b_1b_2} \vee N_j = G.$$

We proved that $\mathcal{M}$ is irredundant.

Assume now that $\mathcal{M}\cup\{F(G)\}$ is s-irredundant in $G$. First suppose that $\mathcal{M}\cup\{F(G)\} = \{M_1,\ldots,M_k,F(G)\}$ is s-irredundant. Thus $\mathcal{M}$ is s-irredundant and $\left(\bigcap_{j\in I} M_j\right) \vee F(G) = G$. Then, there exists $K$ a conjugate of $H$ such that $K \leq \bigcap_{j\in I} M_j$. By a first part of the proof we know that $\mathcal{M}$ is irredundant. It implies, by the Dedekind Modular Law that

$$\left(\bigcap_{l\neq j} M_l \cap F(G)\right) \vee M_j = \left(\bigcap_{l\neq j} (N_lK) \cap F(G)\right) \vee N_jK =$$

$$= \left[\left((\bigcap_{l\neq j} N_l)K\right) \cap F(G)\right] \vee N_jK = \left[\left(\bigcap_{l\neq j} N_l\right) \vee (K \cap F(G))\right] \vee N_jK =$$

$$= \left(\bigcap_{l\neq j} N_l\right) \vee N_jK = \left(\bigcap_{l\neq j}(N_lK)\right) \vee N_jK = \left(\bigcap_{l\neq j} M_l\right) \vee M_j = G.$$

This completes the proof in this case.

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Suppose now that $\mathcal{M} \cup \{F(G)\} = \{M_1, \ldots, M_{i-1}, F(G), M_i, \ldots, M_k\}$ is s-irredundant. Then $\mathcal{M}$ is s-irredundant and so irredundant.

Since $\left(\bigcap_{l=1}^{i-1} M_l\right) \vee F(G) = G$, there exists a conjugate $K$ of $H$ such that $K \leq \bigcap_{l=1}^{i-1} M_l$. Let $j \in I$. If $j < i$, then

$$G = \left(\bigcap_{l=1}^{j-1} M_l\right) \vee M_j = \left(\bigcap_{l=1}^{j-1} N_l K\right) \vee N_j K = \left(\bigcap_{l=1}^{j-1} N_l \vee N_j\right) K.$$

Hence $\left(\bigcap_{l=1}^{j-1} N_l\right) \vee N_j = F(G)$.

If $j \geq i$, then

$$G = \left(\bigcap_{l=1}^{j-1} M_l \cap F(G)\right) \vee M_j = \left(\bigcap_{l=1}^{j-1} N_l\right) \vee N_j \vee H^{a_j}.$$

Hence $\left(\bigcap_{l=1}^{j-1} N_l\right) \vee N_j = F(G)$.

It follows that $\{N_1, \ldots, N_k\}$ is s-irredundant in $F(G)$ and again, by Lemma 3.7, there exists a conjugate $R$ of H such that $R \leq \bigcap_{j \in I} M_j$. Thus $\left(\bigcap_{j \in I} M_j\right) \vee F(G) = G$. Furthermore, by the Dedekind Modular Law we have

$$\left(F(G) \cap \bigcap_{l \neq j} M_l\right) \vee M_j = \left(F(G) \cap \bigcap_{l \neq j} N_l R\right) \vee N_j R =$$

$$= \left[F(G) \cap \left(\left(\bigcap_{l \neq j} N_l\right) R\right)\right] \vee N_j R = \left[\bigcap_{l \neq j} N_l \vee (F(G) \cap R)\right] \vee N_j R =$$

$$\left(\bigcap_{l \neq j} N_l\right) \vee N_j R = \left(\bigcap_{l \neq j} N_l R\right) \vee N_j R = G.$$

It follows that $\{M_1, \ldots, M_{i-1}, F(G), M_i, \ldots, M_k\}$ is irredundant in $G$. Thus the statement (2) follows.

The statement (3) is an immediate consequence of Theorem 3.3. □

## 4. Inconjugate Intersection Number

The intersection number and the inconjugate intersection number correspond to the notion of an m-base.

**Lemma 4.1.** *Let $G$ be a group.*

(1) *If $\iota(G) = n$ for some positive integer $n$ then there exists an m-base $\mathcal{B}$ of $G$ with $|\mathcal{B}| = n$.*
(2) *If $\hat{\iota}(G) = n$ for some positive integer $n$ then there exists an m-base $\mathcal{B}$ of $G$ with $|\mathcal{B}| = n$ and elements of $\mathcal{B}$ are pairwise inconjugate.*

*Proof.* (1) Let $\iota(G) = n$. Then, by definition, there exists a family $\mathcal{B}$ of maximal subgroups of $G$ for which $\bigcap \mathcal{B} = \Phi(G)$ and $\bigcap \mathcal{N} > \Phi(G)$ for any proper subset $\mathcal{N}$ of $\mathcal{B}$. Moreover $|\mathcal{B}| = n$. Hence $\mathcal{B}$ is a maximal irredundant set of maximal subgroups of $G$ which intersect on the Frattini subgroup. So $\mathcal{B}$ is an m-base.

(2) If $\hat{\iota}(G) = n$, we can repeat the argument applied in (1). Moreover, we know that any two of elements of $\mathcal{B}$ are inconjugate. □

In [1] it was proved the following

**Proposition 4.2.** *Let $N$ be a normal subgroup of a non-trivial group $G$. If $N \subseteq \Phi(G)$, then $\iota(G/N) = \iota(G)$.*

Analogous property holds in the case of the inconjugate intersection number.

**Proposition 4.3.** *Let $N$ be a normal subgroup of the non-trivial group $G$. If $N \subseteq \Phi(G)$, then $\hat{\iota}(G/N) = \hat{\iota}(G)$.*

*Proof.* First we observe that $M^g = N$ if and only if $(M/\Phi(G))^{g\Phi(G)} = N\Phi(G)$, where $M, N$ are maximal subgroups of $G$ and $g \in G \setminus M$. Hence $M$ and $N$ are inconjugate if and only if $M/\Phi(G)$ and $N/\Phi(G)$ are inconjugate.

If $\hat{\iota}(G) = n$, then, by Lemma 4.1, there exists an m-base of $G$ with $n$ elements. Assume that $\{M_1, \dots, M_n\}$ is such an m-base. Additionally subgroups of this m-base are inconjugate. Then by the Correspondence Theorem (see [12, 1.4.6]), $\{M_1/\Phi(G), \dots, M_n/\Phi(G)\}$ is an m-base of $G/\Phi(G)$ and its subgroups are inconjugate. So $\hat{\iota}(G/\Phi(G)) \leq \hat{\iota}(G)$. Conversely if $\hat{\iota}(G/\Phi(G)) = n$, then again by Lemma 4.1, there exists an m-base of $G/\Phi(G)$ with $n$ inconjugate elements. Let $\{M_1/\Phi(G), \dots, M_n/\Phi(G)\}$ be this m-base. By the Correspondence Theorem, $\{M_1, \dots, M_n\}$ is an m-base of $G$ with inconjugate elements. So $\hat{\iota}(G) \leq \hat{\iota}(G/\Phi(G))$. Thus we conclude that $\hat{\iota}(G) = \hat{\iota}(G/\Phi(G))$. □

*Proof of Theorem 1.6* If $G$ is a nilpotent group, then by [1, Proposition 5.1], $\iota(G) = \hat{\iota}(G)$.

So assume that $G$ is a non-nilpotent solvable $\mathcal{M}$-group. By Propositions 4.2 and 4.3, we may assume that $\Phi(G) = 1$. Hence by [6, Theorem A.10.6], $F(G) = F_1 \times \cdots \times F_k$, where $F_i$ is a minimal normal subgroup of $G$. Moreover, there exists a non-trivial subgroup $H$ of $G$ such that $G = F(G) \rtimes H$.

We use induction on $|G|$. From Theorem 3.1 and 1.3 we know that $H$ is a Frattini-free solvable $\mathcal{M}$-group. Since $|H| < |G|$, by induction $\iota(H) = \hat{\iota}(H)$. Let $\iota(H) = \hat{\iota}(H) = n$, for some positive integer $n$. Suppose that $\mathcal{H} = \{H_1, \dots, H_n\}$ is an irredundant family of inconjugate maximal subgroups of $H$ such that for any $i \in I = \{1, \dots, n\}$, $\bigcap_{l \in I} H_l = 1$ and $\bigcap_{l \in I \setminus \{i\}} H_l > 1$.

We show that $M_1 = F(G)H_1, \dots, M_n = F(G)H_n$ are inconjugate maximal subgroups of $G$. To this purpose assume that for some $i, j \in I$ there exists $g \in G$ such that $(F(G)H_i)^g = F(G)H_j$. Let $g = ha$, $h \in H$, $a \in F(G)$. Then $F(G)H_j = (F(G)H_i)^g = F(G)H_i^{ha} = F(G)H_i^h$ and further by the Dedekind's Modular Law

$$H_j = (F(G) \cap H)H_j = F(G)H_j \cap H = F(G)H_i^h \cap H = (F(G) \cap H)H_i^h = H_i^h.$$

which contradicts the choice of $H_i$ and $H_j$. It follows that subgroups $M_1, \dots, M_n$ are inconjugate.

Let $g \in \bigcap_{l \in I} M_l$. Then $g = a_i h_i$, where $a_i \in F(G)$ and $h_i \in H_i$, for $i \in I$. Hence for $i, j \in I$, $i \neq j$, $a_j^{-1} a_i = h_j h_i^{-1} \in F(G) \cap H = 1$. So $h_i = h_j$ and $h_i \in \bigcap_{l \in I} H_l = 1$. Thus $g \in F(G)$ and

$\bigcap_{l\in I} M_l = F(G)$. Since $\bigcap_{l\in I\setminus\{i\}} H_l > 1$, for any $i \in I$

$$\bigcap_{l\in I\setminus\{i\}} M_l > F(G). \tag{4.1}$$

Let $J = \{1, \dots, k\}$ and $N_i = F_1 \times \cdots \times F_{i-1} \times F_{i+1} \times \cdots \times F_k$ for $i \in J$. Then $N_1, \dots, N_k$ are maximal $H$-invariant subgroups of $F(G)$ such that $\bigcap_{l\in I} N_l = 1$ and

$$\bigcap_{l\in I\setminus\{i\}} N_l > 1. \tag{4.2}$$

Suppose that $M_{n+1} = N_1 \rtimes H, \dots, M_{n+k} = N_k \rtimes H$. It is clear that subgroups $M_{n+1}, \dots, M_{n+k}$ are inconjugate and by Lemma 3.6, $M_{n+1} \cap \cdots \cap M_{n+k} = (N_1 \cap \cdots \cap N_k)H = H$.

Set $\mathcal{M} = \{M_1, \dots, M_n, M_{n+1}, \dots, M_{n+k}\}$. Obviously for any $i \in I$ and $j \in J$, $M_i$ and $M_j$ are inconjugate. From this and the above paragraphs we conclude that $\mathcal{M}$ is a set of inconjugate maximal subgroups of $G$ and $M_1 \cap \cdots \cap M_n \cap M_{n+1} \cap \cdots \cap M_{n+k} = 1$. Moreover, for $i = 1, \dots, k$ by (4.1) and for $i = n+1, \dots, n+k$ by (4.2) follows $M_1 \cap \cdots \cap M_{i-1} \cap M_{i+1} \cap \cdots \cap M_{n+k} > 1$.

So $\hat{\imath}(G) \leq \infty$. Since $G$ is an $\mathcal{M}$-group, all m-bases have the same size by Theorem 1.2. Hence $\iota(G) = \hat{\imath}(G) = n + k$. The proof is complete. □


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## References


[1] K. Archer, H. B. Serrano, K. Cook, L. K. Lauderdale, Y. Perez and V. Villalobos, On the intersection numbers of finite groups, preprint, arXiv:1907.02898, 2019.

[2] C. Bagiński and A. Stocka, Finite groups with $L$-free lattices of subgroups, *Illinois J. Math.*, **52** no. 3 (2008) 887–900.

[3] T. C. Burness, M. Garonzi and A. Lucchini, On the minimal dimension of a finite simple group, *J. Combin. Theory Ser. A*, **171** (2020) 32 pp.

[4] T. C. Burness, M. Garonzi and A. Lucchini, Finite groups, minimal bases and the intersection numbers, preprint, arXiv:2009.10137v1, 2020.

[5] E. Detomi and A. Lucchini, Maximal subgroups of finite soluble groups in general position, *Ann. Mat. Pura Appl. (4)*, **195** no. 4 (2016) 1177–1183.

[6] K. Doerk and T. Hawkes, *Finite soluble groups*, De Gruyter Expositions in Mathematics, **4**, Walter de Gruyter & Co., Berlin, 1992.

[7] R. Fernando, On an inequality of dimension-like invariants for finite groups, preprint, arXiv:1502.00360, 2015.

[8] M. Garonzi and A. Lucchini, Maximal irredundant families of minimal size in the alternating group, *Arch. Math. (Basel)*, **113** (2019) 119–126.

[9] P. Grzeszczuk and E. R. Puczyowski, On Goldie and dual Goldie dimensions, *J. Pure Appl. Algebra*, **31** (1984) 47–54.

[10] J. Krempa and A. Sakowicz, On uniform dimension of finite groups, *Colloq. Math.*, **89** (2001) 223–231.

[11] J. Krempa and B. Terlikowska-Osłowska, On uniform dimension of lattices, Contributions to general algebra, **9**, Hölder-Pichler-Tempsky, Vienna, 1995 219-230.

[12] D. J. S. Robinson, *A course in the theory of groups*, Second edition. Graduate Texts in Mathematics, **8**, Springer-Verlag, New York, 1996.

[13] R. Schmidt, *Subgroup lattices of groups*, De Gruyter Expositions in Mathematics, **14**, Walter de Gruyter & Co., Berlin, 1994.

[14] M. Weidner, Independence and maximal subgroups, *Illinois J. Math.*, **40** no. 1 (1996) 47–76.

[15] H. Whitney, On the abstract properties of linear dependence, *Amer. J. Math.*, **57** no. 3 (1935) 509–533.

[16] T. Liu and R. Keith Dennis, MaxDim of some simple groups, preprint, arXiv:1712.04553, 2017.

[17] W. Niu, On Behaviors of Maximal Dimension, preprint, arXiv:1801.08327, 2018.

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