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RESIDUALLY SOLVABLE JUST INFINITE PROFINITE LIE ALGEBRAS

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Abstract. We provide some characterization theorems about just infinite profinite residually solvable Lie algebras, similarly to what C. Reid has done for just infinite profinite groups. In particular, we prove that a profinite residually solvable Lie algebra is just infinite if and only if its obliquity subalgebra has finite codimension in the Lie algebra, and we establish a criterion for a profinite residually solvable Lie algebra to be just infinite, looking at the finite Lie algebras occurring in the inverse system.

1. Introduction

In 2010s C. Reid proved some characterization theorems about profinite just infinite groups. In particular, in [1, Theorem A], provides an equivalence among the just infiniteness of a profinite group and the finiteness of the index of the oblique core of an open subgroup of the group. Furthermore, in [2, Theorem 4.1] establishes a criterion for the just infiniteness of the inverse limit of an inverse system of finite discrete groups.

In this paper we ask if similar results hold for profinite Lie algebras, where a finite Lie algebra is intended as a finite-dimensional Lie algebra over a finite field. Unfortunately, we are not able to provide a complete characterization for profinite Lie algebras in general, hence we will focus on a certain class of profinite Lie algebras, namely residually solvable Lie algebras, for which we can state results similar to those established by Reid for profinite groups.

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A *profinite Lie algebra* L is the inverse limit of a surjective inverse system (indexed over a directed set $(\Omega, \preceq)$) consisting of a pair of families of discrete finite Lie algebras $(L_i)_{i\in\Omega}$ and continuous maps

$$(\phi_{i,j} : L_j \to L_i)_{\substack{i,j\in\Omega \\ i\preceq j}}$$

We remark that the profinite limit arises as the limit of an inverse system of finite objects, in order to preserve some important properties that characterizes profinite groups, above all the compactness of the inverse limit.

As usual, we endow such an inverse limit with the subspace topology inherited by the product topology on $\mathrm{Car}_{i\in\Omega} L_i$.

A profinite Lie algebra is *just infinite* if each non-trivial closed ideal is open or, equivalently, if it has finite codimension.

Recall that a Lie algebra L is *residually solvable* if the family of ideals such that the corresponding quotient is solvable has trivial intersection. In other words, L is residually solvable if the derived series has trivial intersection, that is if

$$L^{(\omega)} = \bigcap_{n\in\mathbb{N}^+} L^{(n)}$$

is trivial.

2. **Subcartesian products**

In this preliminary section we prove a technical result about subcartesian products, that will be useful in the following. We remind the reader that a set X is a *subcartesian product* of a family of sets $(X_i)_{i\in I}$ if $X \subseteq \mathrm{Car}_{i\in I} X_i$ and if, for each $i \in I$, the projection map on the i-th component $\pi_i : X \to X_i$ is surjective. This condition does not ensure on its own that X is itself a Cartesian product (and this is not even true for direct products, think, for instance, about diagonal maps), but we can investigate additional conditions to make this true. We state the following lemma in the case of finite Lie algebras, but similar results can be proved for other algebraic structures by referring to their simple objects.

Lemma 2.1. *Let S be a closed subcartesian product of pairwise non-isomorphic finite simple Lie algebras $\mathrm{Car}_{i\in I} S_i$. Then S coincides with the entire Cartesian product.*

Proof. For each finite subset $J = \{j_1, \ldots, j_n\}$ contained in I we denote by $\pi_J : S \to \prod_{h=1}^n S_{j_h}$ the map defined by

$$s \mapsto (\pi_{j_1}(s), \ldots, \pi_{j_n}(s))$$

First, we prove the surjectivity of each π_J. The composition map

$$\pi_{j_k} \circ \pi_J : S \to \prod_{h=1}^n S_{j_h} \to S_{j_k}$$

is surjective for each $k = 1, \ldots, n$, by definition of subcartesian product, so for all k there exists a composition factor of $\pi_J(S)$ isomorphic to S_{j_k}; in other words, each S_{j_k} occurs, up to isomorphism,

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among the composition factors of $\pi_J(S)$. Hence, all the simple Lie algebras $S_{j_1}, \ldots, S_{j_n}$ must occur in the composition series of $\pi_J(S)$, since they are pairwise non-isomorphic by hypothesis. Therefore, as the cardinality of $\pi_J(S)$ equals to the product of the cardinalities of its composition factors, we have

$$|\pi_J(S)| \geq \prod_{h=1}^{n} |S_{j_h}|$$

But $\pi_J(S)$ is also the image of S in $\prod_{h=1}^{n} S_{j_h}$, thus also the opposite inequality holds, so $\pi_J(S) = \prod_{h=1}^{n} S_{j_h}$ and each map π_J, with J finite, is surjective.

We now focus on the general case. Suppose that there exists $(s_i)_{i\in I}$ belonging to the Cartesian product but not to S. As S is closed, $(s_i)_{i\in I}$ has an open neighbourhood disjoint from S, say, by definition of product topology,

$$\pi_{j_1}^{-1}(U_1) \cap \cdots \cap \pi_{j_r}^{-1}(U_r) \cap S = \emptyset$$

for some positive natural number r and some $j_1, \ldots, j_r \in I$, where $s_{j_h} \in U_h$ for every $h = 1, \ldots, r$. So, the element $(s_{j_1}, \ldots, s_{j_r})$ belongs to $\prod_{h=1}^{r} S_{j_h}$ but it is not in $\pi_{\{j_1,\ldots,j_r\}}(S)$; this contradicts the surjectivity of the maps π_J with J finite. Hence $(s_i)_{i\in I} \in S$ and $S = \mathrm{Car}_{i\in I} S_i$. □

A more general property holds, as we prove in the following lemma. Simple Lie algebras are assumed, as usual, non-abelian.

Lemma 2.2. *Let S be a closed subcartesian product of finite simple Lie algebras $Car_{i\in I} S_i$. Then S is itself a Cartesian product.*

Proof. If the simple algebras are pairwise non-isomorphic, we can apply Lemma 2.1.
Otherwise, we partition the set I through the following equivalence relation: two indices i and j are equivalent if and only if $\ker \pi_i = \ker \pi_j$. Consider the quotient set of indices $I/\sim$ corresponding to this equivalence relation and let T be a transversal set for this relation, namely a set consisting of one class representative $t \in I$ for each class in $I/\sim$. Then, we denote by

$$\pi_T : S \to \operatorname*{Car}_{t\in T} S_t$$

the map defined as $s \mapsto (\pi_t(s))_{t\in T}$.
Such a function is clearly injective: for, if $x \in \ker \pi_T$, then $x \in \ker \pi_t$ for every $t \in T$, thus $x \in \ker \pi_i$ for every $i \in I$ by definition of our equivalence relation; hence $x = 0$ since $S \leq \mathrm{Car}_{i\in I} S_i$.

We now look at surjectivity. We first assume that the transversal set is finite. For all $s \in S$ consider the support of $\pi_T(s)$:

$$\mathrm{supp}\,(\pi_T(s)) = \{t \in T \mid s_t \neq 0\}$$

where s_t denotes the t-th component of $\pi_T(s)$. Clearly the support is non-empty for every $s \in S$, since otherwise the map π_T would not be injective.
By finiteness of T we can pick $x \in S$ such that $\pi_T(x)$ has minimal support; namely, such that there

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are no elements with fewer non-trivial components in the projection on $\mathrm{Car}_{t\in T} S_t$; we want to prove that the support of such x is a singleton.

Suppose by contradiction that there exist two distinct indices t and u in T such that both x_t and x_u are non-trivial. Since S_t is simple non-abelian, there exists $a \in S_t$ such that a does not commute with $\pi_t(x)$, namely such that $[a, \pi_t(x)] \neq 0$. Certainly, by surjectivity of π_t, there exists $y \in S$ such that $\pi_t(y) = a$; furthermore, such a y can be chosen in $\ker \pi_u$, since, by maximality of the kernels, $S = \ker \pi_u + \ker \pi_t$. Thus, $\pi_t([x, y]) \neq 0$ while $\pi_u([x, y]) = 0$, as $\pi_u(y) = 0$.

Moreover, if v does not belong to the support of $\pi_T(x)$, then v does not belong to the support of $\pi_T([x, y])$, since $\pi_v(x) = 0$.

Therefore, we can conclude that the support of $[x, y]$ is strictly contained in the support of x, contradicting the minimality of x. Thus, the support of a minimal element x is a singleton.

Now, we prove that π_T is surjective by induction on $|T|$. If T is a singleton, the thesis holds by injectivity of π_T.

If $|T| > 1$ we proceed as follows: for all $t \in T$ we say that an element $x \in S$ is t-minimal if it has minimal support and if its unique non-trivial component is the t-th component. If for all $t \in T$ there exists a t-minimal element, then each S_t is contained in the image of S under π_T, hence π_T is surjective. Otherwise, we consider the proper subset $T' \subseteq T$ whose elements are the indices $t \in T$ such that there is not a t-minimal element. We then consider the map

$$\pi_{T'} : S \to \operatorname*{Car}_{t\in T'} S_t$$

Such a map is not injective, since $\pi_{T'}(y) = 0$ for every y which is t-minimal for some $t \in T \setminus T'$, thus we quotient S by $\ker(\pi_{T'})$ in order to have injectivity. We can prove as above that, if the support of an element in $S/\ker(\pi_{T'})$ is minimal, then it is a singleton. If for all $t \in T'$ the simple Lie algebra S_t is contained in the image of $S/\ker(\pi_{T'})$, then the map

$$S/\ker(\pi_{T'}) \to \operatorname*{Car}_{t\in T'} S_t$$

is surjective, hence also the map $\pi_{T'}$ is surjective. Otherwise, we repeat the same argument, through a new proper subset $T'' \subseteq T' \subseteq T$. Since $|T|$ is finite, this argument must ends in a finite number of steps, showing that each S_t is contained in the image of S. Hence, we have proved the surjectivity of the map π_T if T is finite.

We have now to prove the surjectivity of π_T in the case T is not finite, and we proceed similarly to Lemma 2.1. Suppose by contradiction that there exists $(s_t)_{t\in T}$ belonging to the Cartesian product $\mathrm{Car}_{t\in T} S_t$ but not to S. As S is closed, we can find an open neighbourhood of $(s_t)_{t\in T}$ disjoint from S, say

$$\pi_{t_1}^{-1}(U_1) \cap \cdots \cap \pi_{t_r}^{-1}(U_r) \cap S = \emptyset$$

for some $r \in \mathbb{N}^+$, where each U_i is an open subset of S_{t_i}. Thus, the element $(s_{t_1}, \ldots, s_{t_r})$ belongs to $S_{t_1} \times \cdots \times S_{t_r}$ while it does not belong to $\pi_{\{t_1,\ldots,t_r\}}(S)$. Note that the support of $\pi_{\{t_1,\ldots,t_r\}}(s)$ is

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not empty, since, otherwise, $s \in \ker \pi_{t_j}$ for each $j = 1, \dots, r$, hence $\pi_{t_1}^{-1}(U_1) \cap \cdots \cap \pi_{t_r}^{-1}(U_r) \cap S$ would contains 0 and would be not empty. Therefore, we can repeat the argument used in the case with T finite to prove that $\pi_{\{t_1,\dots,t_r\}}$ is surjective, contradicting the fact that $(s_{t_1}, \dots, s_{t_r})$ is not in $\pi_{\{t_1,\dots,t_r\}}(s)$. Hence, $S = \mathrm{Car}_{t\in T} S_t$. □

Lemma 2.3. *Let L be an abstract Lie algebra and let S be an ideal of L. Suppose that S decomposes as a Cartesian product of finite simple Lie algebras, say $S = Car_{\lambda\in\Lambda} T_\lambda$. Then, each T_λ is an ideal in L.*

Proof. Let $x \in L$ and $\alpha \in \Lambda$. Consider the subset $[T_\alpha, x] + T_\alpha$ in S. We claim that it is an ideal of S. Clearly, it is an additive subgroup.
If $b \in S$ we have

$$[[T_\alpha, x] + T_\alpha, b] = [[T_\alpha, x], b] + [T_\alpha, b]$$

where the second summand is contained in T_α since that is an ideal of S; on the other hand, for the first summand we have, by Jacobi identity,

$$[[T_\alpha, x], b] \subseteq [[b, T_\alpha], x] + [[x, b], T_\alpha] \subseteq [T_\alpha, x] + T_\alpha$$

Thus, for all $b \in S$ we have

$$[[T_\alpha, x] + T_\alpha, b] \subseteq [T_\alpha, x] + T_\alpha$$

as desired.
Let now $C := C_S(T_\alpha)$ be the centralizer of T_α in S; because of the decomposition of S, C certainly contains $\mathrm{Car}_{\lambda\neq\alpha} T_\lambda$; besides, if $C = S$ we would have T_α abelian, contradicting the hypothesis; thus, $C = \mathrm{Car}_{\lambda\neq\alpha} T_\lambda$.
Now, if $y \in C$, we have for every $a \in T_\alpha$ and $x \in L$

$$[[a, x], y] = -[[x, y], a] - [[y, a], x] \in T_\alpha$$

hence $[[T_\alpha, x], C] \leq T_\alpha$, from which $[[T_\alpha, x], C, C] = 0$ follows by definition of C.
Now, if $[T_\alpha, x] \nleq T_\alpha$, then there exists, for some $\lambda \neq \alpha$, a non-trivial surjective map from $[T_\alpha, x]$ onto T_λ, so we can pick $z \in T_\lambda$ which is not centralized by $C_S(T_\alpha)$, being $C_S(C_S(T_\alpha)) = T_\alpha$. Thus, the condition $[[T_\alpha, x], C, C] = 0$ implies $[T_\alpha, x] \leq T_\alpha$ (these conditions are indeed equivalent), and our claim is proved. □

3. Narrow ideals

Now we introduce the notion of narrow ideal of a Lie algebra L. Such ideals will be useful in the following sections.
First, let A and B be two ideals of L such that B is properly contained in A. The factor algebra A/B is called a *chief factor* of the Lie algebra L if A/B is a minimal ideal of L/B. Two chief factors I/J and A/B are *associated* if and only if $I + B = A + J$ and $J + B < I + A$.

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Let now K be a non-trivial ideal of L and let $\mathcal{I}_L(K)$ be the family of proper subalgebras of finite index of K which are ideals in L. We denote by $M_L(K)$ the intersection of the maximal elements in $\mathcal{I}_L(K)$. The ideal K of L is *narrow* if $M_L(K)$ is the unique maximal element in $\mathcal{I}_L(K)$. A narrow ideal K of L is *associated* to a chief factor I/J if $K/M_L(K)$ is associated to I/J.

Lemma 3.1. *Let I and J be closed ideals of a profinite Lie algebra L. Then $I + M_L(J) \geq J$ if and only if $I \geq J$.*

Proof. Suppose that $I + M_L(J) \geq J$ and assume, by contradiction, that $I \ngeq J$. Then $I \cap J$ is an ideal in L which is strictly contained in J, thus there exists a maximal element K in $\mathcal{I}_L(J)$ such that $I \cap J \leq K$ (possibly $I \cap J = K$). Now, since $M_L(J) < J$, we have

$$(I + M_L(J)) \cap J = I \cap J + M_L(J) \leq K + M_L(J) \leq K < J$$

where the first relation holds by Dedekind's law. Therefore, $I + M_L(J) < J$, contradicting the hypothesis.
The converse is clear. □

Now we want to prove the existence of narrow ideals associated to chief factors and we characterize them. Before that, we state a preliminary Lemma.

Lemma 3.2. *Let L be a profinite Lie algebra and let K be an open ideal of L. Let moreover $(S_i)_{i\in\Omega}$ be a descending chain of closed subalgebras such that each one is not contained in K and let S be the intersection of these subalgebras. Then also S is not contained in K.*

Proof. Consider the closed subset $C_i = S_i \cap (L \smallsetminus K)$. Since the intersection of finitely many C_i is not empty (as the chain $(C_i)_{i\in\Omega}$ is descending), then also the intersection of all the C_i is not empty, by the finite intersection property. □

Lemma 3.3. *Let I/J be a chief factor of a profinite Lie algebra L. Then, there exists a narrow ideal K of L associated to such a chief factor.*
The narrow ideals associated to I/J are exactly the narrow ideals of L contained in I but not in J; moreover, in this case $M_L(K) = K \cap J$.
In particular, every non-trivial ideal of L contains a narrow ideal of L.

Proof. First, we prove that there exists a narrow ideal of L associated to a chief factor I/J. Let

$$\mathcal{K}_L(I) = \{H \lhd_c L \mid H \leq I\}$$

and let

$$\mathcal{D}_L(I, J) = \mathcal{K}_L(I) \setminus \mathcal{K}_L(J)$$

be the set of closed ideals of L which are contained in I but not in J.
Let $(D_i)_{i\in I}$ be a descending chain in $\mathcal{D}_L(I, J)$; since, by Lemma 3.2, the intersection $\bigcap_{i\in I} D_i$ is not

contained in J, we have $\bigcap_{i\in I} D_i \in \mathcal{D}_L(I,J)$, hence $\mathcal{D}_L(I,J)$ admits, by Zorn's lemma, a minimal element, say K. Now, let $H \triangleleft_c L$ properly contained in K; by minimality of K, we have $H \leq J$, hence $K \cap J$ is the unique maximal element in $\mathcal{I}_L(K)$. This proves that K is a narrow ideal of L.

Now we prove the characterization of narrow ideals associated to I/J. Let K such an ideal; then, by definition, we have

$$I + M_L(K) = K + J \tag{3.1}$$

By this equation it follows that $I + M_L(K) \geq K$, hence, by Lemma 3.1, we have $I \geq K$. Moreover, again by relation ((3.1)), we deduce $K \not\leq J$, since otherwise $J \geq I$.
Conversely, let K be a narrow ideal of L that is contained in I but not in J. Certainly, $K + J = I$ since I/J is a chief factor and $K + J > J$. Hence, since

$$K/(K \cap J) \cong (K + J)/J = I/J$$

we deduce that $K/(K \cap J)$ is a chief factor, which implies $K \cap J = M_L(K)$. It remains to prove that $K/(K \cap J)$ and I/J are associated: for, as $K \cap J \leq I$, we have certainly $K + J = I + K \cap J$; moreover $K \cap J + J < K + I$, hence the thesis. □

4. Just infinite profinite Lie algebras

First we state a technical lemma. Given a profinite Lie algebra L, its *Jacobson radical* $J(L)$ is the intersection of all of its maximal closed (indeed, open) ideals.

Lemma 4.1. *Let L be a profinite Lie algebra, let I be a closed ideal of L and let S be a closed subalgebra of L containing I.*
Then, the Jacobson radical of S contains the Jacobson radical of I.

Proof. If each maximal ideal of S contains I, then certainly the Jacobson radical of S contains that of I. Otherwise, if we consider a closed maximal ideal M of S such that M does not contain I, we have

$$S/M = (I + M)/M \cong I/(I \cap M)$$

so $I/(I \cap M)$ is simple, hence $I \cap M$ is maximal in I. This proves the statement. □

Lemma 4.2. *Let L be a just infinite profinite Lie algebra that is not virtually abelian and let S be an open subalgebra. Then, the Jacobson radical $J(S)$ of S is not trivial.*

Proof. Each open subalgebra S contains an open ideal I, therefore, by Lemma 4.1, the Jacobson radical of S contains that of I; hence, it is sufficient to prove the statement for an open ideal.
For our purpose, we partition the set of open maximal ideals of S as follows: let $\mathcal{A}$ be the set of open maximal ideals of S such that the corresponding quotient is simple abelian and let $\mathcal{B}$ be the set of open maximal ideals of S such that the corresponding quotient is simple non-abelian. We also denote by A, respectively B, the intersection of the elements in $\mathcal{A}$, respectively in $\mathcal{B}$.

Suppose by contradiction that the Jacobson radical is trivial. Then, at least one of A and B must have infinite index in S.

If $A = 0$, then S decomposes as a Cartesian product of simple abelian Lie algebras and so L is a finite extension of an abelian Lie algebra, a case that we have excluded by hypothesis.

If $B = 0$, then S embeds in a Cartesian product of simple non-abelian factors, and so it is a subcartesian product since $\pi_M(S) = S/M$ for every open maximal ideal $M \triangleleft S$. Hence, by Lemma 2.2, the subalgebra S itself is a Cartesian product of finite simple Lie algebras. By Lemma 2.3 each component of this Cartesian product is a finite ideal of L, hence each such component is trivial by just infiniteness of L, and so $S = 0$, contradicting the fact that S is open.

If both A and B are not trivial, then necessarily B has infinite index; in fact A contains $[S, S]$, that is the smallest ideal of S that makes the quotient abelian and that, in addition, is an ideal in L because of the Jacobi identity. Thus, A must have finite index in S and $S \cong S/(A \cap B)$ embeds in $S/A \times S/B$, that, in turn, embeds in a Cartesian product of simple factors among which only finitely many are abelian. Note that $[S, S]$, that is in turn an open subalgebra of L, decomposes as a Cartesian product of finite simple Lie algebras, since certainly the abelian terms are killed by the Lie bracket; hence, applying Lemma 2.3 to $[S, S]$, each finite simple term is a finite ideal of L, so every such term is trivial. Therefore, S embeds in S/A which is finite. Thus, this case cannot hold either.

In conclusion, assuming the triviality of the Jacobson radical we deduce a contradiction, and our statement is proved. □

Lemma 4.3. *Let L be a just infinite profinite Lie algebra. Then the Jacobson radical of each open ideal I has finite index in I if and only if the Jacobson radical of each open Lie subalgebra S has finite index in S.*

Proof. We have only to prove that if the condition holds for open ideals, then it also holds for open subalgebras, as the converse implication trivially holds.

Thus, let S be an open subalgebra of L; of course S contains an open ideal I of L. By Lemma 4.1, we have $J(I) \leq J(S)$. Therefore

$$|S : J(S)| \leq |S : J(I)| = |S : I||I : J(I)|$$

Thus we deduce the finiteness of $|S : J(S)|$ by the finiteness of both $|S : I|$ and $|I : J(I)|$. □

Lemma 4.4. *Let L be a just infinite profinite residually solvable but not solvable Lie algebra and let S be an open subalgebra. Then, the Jacobson radical $J(S)$ has finite codimension in S.*

Proof. By Lemma 4.3 we can assume without loss of generality that S is an ideal.

The derived ideal $[S, S]$ is an ideal in L, by Jacobi identity. Moreover, since L is residually solvable, S is residually solvable too, hence the ideal $[S, S]$ is properly contained in S; otherwise, the derived series of S would not have trivial intersection. Furthermore, since the Lie algebra is not solvable, it is also not virtually abelian, hence the abelian quotient $S/[S,S]$ must be finite, or, equivalently, $[S,S]$ must have finite codimension in S. □

We recall König's Lemma, which will be useful in the subsequent result. We remind the reader that a *directed graph* is an ordered pair (V,E) where V is a set of vertices and E is a set of ordered pairs of vertices. Moreover, a *simple path* is a sequence of distinct vertices with the property that each vertex in the sequence is adjacent to the vertex next to it. A graph is *locally finite* if each vertex is adjacent to finitely many vertices.

Lemma 4.5 (König's lemma)**.** *Let Γ be a directed locally finite infinite graph. Then Γ contains an infinite simple path.*

Lemma 4.6. *Let L be a just infinite profinite residually solvable but not solvable Lie algebra. Let $\mathcal{I}$ be an infinite set of open ideals and suppose that the following property holds: for every ideal $I_1 \in \mathcal{I}$, if $I_2 \geq I_1$ then also $I_2 \in \mathcal{I}$.*
Then, there exists a strictly descending sequence $(I_i)_{i\in\mathbb{N}^+}$ of open ideals of L such that $I_i \in \mathcal{I}$ for every i.

Proof. We build a directed graph Γ whose vertices are elements of $\mathcal{I}$ and whose edges are pairs of ideals (I_1, I_2) such that $I_2 < I_1$ and such that there is not any ideal properly contained between them. If $(I, I_1) \in E(\Gamma)$, then I_1 contains the Jacobson radical of I. By Lemma 4.4 there are finitely many such ideals I_1, so Γ is a locally finite directed graph and by König's lemma it contains an infinite path; thus, there is an infinite descending chain of open ideals in L. □

5. The main theorems

Let L be a profinite Lie algebra and let S be a closed subalgebra of L. We denote by $\mathcal{I}_S$ the set of open ideals of L which are not contained in S. Then we define the *obliquity subalgebra* of S as

$$\mathrm{Ob}_L(S) = S \cap \bigcap \mathcal{I}_S$$

We can now state the following results, that relate the just infiniteness of a profinite residually solvable but not solvable Lie algebra with the finiteness of the index of the obliquity subalgebras of the open subalgebras.

Theorem 5.1. *Let L be a profinite residually solvable Lie algebra which is not virtually abelian. The following conditions are equivalent:*

1. *L is just infinite;*
2. *for every open subalgebra S the set of ideals*

$$\mathcal{I}_S = \{I \lhd_o L \mid I \not\leq S\}$$

is finite;

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(3) *there exists a family $\mathcal{F}$ of open subalgebras in L with trivial intersection such that $\mathcal{I}_S$ is finite for every $S \in \mathcal{F}$.*

Proof. (1) ⇒ (2) Suppose that the set $\mathcal{I}_S$ is infinite. Then, by Lemma 4.6 we could find an infinite descending chain in $\mathcal{I}_S$. Now, by Lemma 3.2 the intersection of the elements in this family would be a non-trivial closed ideal having infinite index in L, contradicting the hypothesis.
(2) ⇒ (3) This implication is clear.
(3) ⇒ (1) Let I be a closed nontrivial ideal of L: we have to prove that the quotient L/I is finite. Since the intersection of all the open subalgebras is trivial, there exists an open subalgebra T such that $I \nleq T$. Now, by definition of the profinite topology, the ideal I is the intersection of all the open ideals of L containing it; moreover, this intersection is extended only over the open ideals containing I and not contained in T: otherwise, in fact, I would be contained in T. Hence,

$$\mathrm{Ob}_L(T) = T \cap \bigcap \mathcal{I}_T = T \cap \bigcap_{\substack{J \lhd_o L \\ J \nleq T}} J \;\leq\; T \cap \bigcap_{\substack{I \leq J \lhd_o L \\ J \nleq T}} J \;=\; T \cap I \;<\; I$$

Now, $\mathrm{Ob}_L(T)$ has finite index, since, by hypothesis, it is the intersection of an open subalgebra and finitely many finite-index ideals, then also I has finite index, therefore the thesis holds. □

Corollary 5.2. *Let L be a profinite residually solvable Lie algebra which is not virtually abelian. L is just infinite if and only if the index of $Ob_L(T)$ is finite for every open subalgebra T.*

Proof. If L is a just infinite Lie algebra, then $\mathrm{Ob}_L(T)$ is an intersection of finitely many ideals of finite index, so it also has finite index.
Conversely, if L is not just infinite, then, by Theorem 5.1, there exists a subalgebra T such that the set $\mathcal{I}_T$ is infinite, and so $\mathrm{Ob}_L(T)$ has infinite index. □

Corollary 5.3. *Let L be a just infinite profinite residually solvable but not solvable Lie algebra and let S be an open subalgebra. Then $I \leq S$ for all but finitely many chief factors I/J in L.*

Proof. By Theorem 5.1 the family of open ideals that are not contained in S is finite, hence the thesis holds. □

We can also prove another criterion for the just infiniteness of a profinite residually solvable Lie algebra, giving some condition on the finite Lie algebras occurring in the associated inverse system.

Theorem 5.4. *Let L be a just infinite profinite residually solvable but not solvable Lie algebra and let $(\mathcal{A}_n)_{n\in\mathbb{N}}$ be a sequence of classes of finite Lie algebras such that L has infinitely many chief factors in $\mathcal{A}_n$ for all $n \in \mathbb{N}$.*
Then L is the limit of an inverse system of finite Lie algebras $(L_n)_{n\in\mathbb{N}}$ and surjective algebra morphisms $(\rho_n : L_{n+1} \to L_n)_{n\in\mathbb{N}}$ where each L_n has a subalgebra A_n such that, letting $B_n = \rho_n(A_{n+1})$, we have

(1) *$A_n > B_n > 1$;*

(2) *A_n is a narrow subalgebra in L_n;*
(3) *every ideal contains B_n or it is contained in A_n;*
(4) *B_n is a minimal ideal for L_n;*
(5) $B_n \in \mathcal{A}_n$.

Conversely, every surjective inverse system satisfying conditions (1)-(3) (for some choice of A_n) for all but finitely many n has a just infinite limit.

Proof. First we prove that a just infinite profinite Lie algebra as in the hypothesis satisfies properties (1)-(5). We will obtain a descending chain $(K_n)_{n\in\mathbb{N}}$ of narrow subalgebras of L and we will build the required inverse system through this sequence.
The first term of the sequence is $K_0 = L$. Afterwards, for the definition of the general term, suppose we have built K_n. Consider then $S := \mathrm{Ob}_L(K_n)$, which has finite index in L. Thence, pick a chief factor I/J such that $I \leq S$ and $I/J \in \mathcal{A}_n$: such a chief factor exists, since by hypothesis each class $\mathcal{A}_n$ contains infinitely many chief factors and since, by Corollary 5.3, S contains all but finitely many chief factors. We then pick K_{n+1} to be a narrow ideal associated to I/J, whose existence is justified by Lemma 3.3. Such an ideal is contained in S by Lemma 3.1.
After we have constructed the sequence of narrow subalgebras, if we define

$$L_n = L/M_L(K_{n+1}), \quad A_n = K_n/M_L(K_{n+1}), \quad B_n = K_{n+1}/M_L(K_{n+1})$$

all the conditions are satisfied.
Now we prove that every surjective inverse system satisfying properties (1)-(3) for some choice of A_n has a just infinite limit. Let I be a closed nontrivial ideal of L. For n sufficiently large, if $\pi_n : L \to L/I_n$ is the surjective map associated to the inverse limit, we have that $\pi_n(I)$ is not contained in A_n, so it must contain B_n by condition (3). Since $M_L(A_{n+1})$ contains $\ker \rho_n$, by condition (1), we have that $\pi_{n+1}(I)$ contains A_{n+1} and in particular $\pi_{n+1}(I)$ contains $\ker \rho_n$.
Since this applies for all n sufficiently large, we have that I contains $\ker \pi_n$ for some n, hence it has finite index. □

As we have previously explained, we are not able to provide analougous results for a general profinite Lie algebra L. The main obstacle to this is the lack of a result which generalizes Lemma 4.4. Such property is trivial in the case of groups, since the intersection of all the maximal closed normal subgroups of a normal subgroup is normal in the entire group; by contrary, in the case of Lie algebras the intersection of the maximal closed ideals of an ideal of L need not be an ideal of L even if, unfortunately, we are not able to provide an explicit counterexample.

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