International Journal of Group Theory
ISSN (print): 2251-7650, ISSN (on-line): 2251-7669
Vol. x No. x (202x), pp. xx-xx.

http://ijgt.ui.ac.ir
www.ui.ac.ir

ON NEUMANN'S BFC-THEOREM AND FINITE-BY-NILPOTENT PROFINITE GROUPS

WÁLLEF DA SILVA

Abstract. Let $\gamma_n = [x_1, \ldots, x_n]$ be the nth lower central word and $X_n(G)$ the set of γ_n-values in a group G. Suppose that G is a profinite group where, for each $g \in G$, there exists a positive integer $n = n(g)$ such that the set $g^{X_n(G)} = \{g^y \mid y \in X_n(G)\}$ contains less than $2^{\aleph_0}$ elements. We prove that G is a finite-by-nilpotent group.

1. Introduction

Let G be a group and $x \in G$ an element of G. We denote x^G for the conjugacy class containing x. A group is said to be a BFC-group if its conjugacy classes are finite and of bounded size. It is known by B. H. Neumann's theorem that in a BFC-group the commutator subgroup G' is finite [12]. Observe that if $|x^G| \leq m$ for each $x \in G$, then the order of G' is bounded by a number depending only on m. A first explicit bound for the order of G' was found by J. Wiegold [18], and the best known was obtained in [11] by Guralnick and Maroti (see also [13] and [15]). Groups in which conjugacy classes containing commutators are bounded were investigated in the works [1], [9] and [4]. A theorem of Shalev [16] states that in a profinite group with all conjugacy classes finite, the commutator subgroup is finite and this was generalized to multilinear commutator words in [8].

Communicated by Attila Maroti.

MSC(2010): Primary: 20F24; Secondary: 20E45, 20E18.

Keywords: Conjucagy classes, verbal subgroups, profinite groups, FC-groups.

Article Type: Research Paper.

Received: 18 August 2022, Accepted: 17 Nowember 2022.

http://dx.doi.org/10.22108/IJGT.2022.134785.1802 .

We say that a group G has order $(a, b, \ldots)$-bounded if a number depending only on the parameters $a, b, \ldots$, bounds the order of G. The following result was proved in [1]:

Theorem 1.1. [1, Theorem 1.2] *Let m, n be positive integers and G a group. If $|x^{\gamma_n(G)}| \leq m$ for any $x \in G$, then $\gamma_{n+1}(G)$ has finite (m, n)-bounded order.*

An analogue of Theorem 1.1 for profinite groups was established [3]. More specifically the following was proved:

Theorem 1.2. [3, Theorem 1.4] *Let n be a positive integer and G a profinite group. If $x^{\gamma_n(G)}$ is finite for any $x \in G$, then $\gamma_{n+1}(G)$ has finite order.*

We observe that in profinite groups certain subsets of the group are finite under the weaker hypothesis that the same sets have at most countably many elements, or in some cases less than $2^{\aleph_0}$ elements (see [5] or [2]). For example, the following holds:

Lemma 1.3. [2, Lemma 2.2] *Let H be a profinite group and $x \in H$. If the conjugacy class x^H contains less than $2^{\aleph_0}$ elements, then it is finite.*

This implies, in particular, that if G is a profinite group, $x \in G$ and the conjugacy class $x^{\gamma_n(G)}$ contains less than $2^{\aleph_0}$ elements, then $x^{\gamma_n(G)}$ is finite. An immediate consequence of this fact is the following generalization of Theorem 1.2, established in [3].

Theorem 1.4. [3, Theorem 1.6] *Let n be a positive integer and G a profinite group. If $x^{\gamma_n(G)}$ contains less than $2^{\aleph_0}$ elements for any $x \in G$, then $\gamma_{n+1}(G)$ has finite order.*

Using the concept of verbal conjugacy classes, introduced in [10], it was obtained in [3] a generalization of the Theorem 1.2. Let $X_n(G)$ be a set of γ_n-values in a group G. It was shown in [10] that if the set $x^{X_n(G)} = \{x^y \mid y \in X_n(G)\}$ is finite for each $x \in G$, then $x^{\gamma_n(G)}$ is finite. Hence, in [3] it was obtained the following:

Corollary 1.5. [3, Corollary 1.7] *Let n be a positive integer and G a profinite group. If $x^{X_n(G)}$ is finite for any $x \in G$, then $\gamma_{n+1}(G)$ has finite order.*

Another result which is straightforward from Theorem 1.2 is the following characterization of finite-by-nilpotent profinite groups.

Theorem 1.6. [3, Theorem 1.8] *A profinite group G is finite-by-nilpotent if and only if there is a positive integer n such that $x^{\gamma_n(G)}$ contains less than $2^{\aleph_0}$ elements, for any $x \in G$.*

In this paper we generalize Theorem 1.6 in the following way:

Theorem 1.7. *A profinite group G is finite-by-nilpotent if and only if, for each element $g \in G$, there exists a positive integer $n = n(g)$ such that $g^{X_n(G)}$ contains less than $2^{\aleph_0}$ elements.*

Considering certain conditions on the set $X_n(H)$ of γ_n-values of an open normal subgroup H of G, we prove the following proposition that help us in this assignment:

Proposition 1.8. *Let G be a profinite group and $g \in G$. Suppose that there are a positive integer $n = n(g)$ such that $g^{X_n(G)}$ contains less than $2^{\aleph_0}$ elements and an open normal subgroup H of G such that $X_n(H) \subseteq C_{\gamma_n(G)}(g)$. Then $g^{X_n(G)}$ is finite.*

2. **Preliminary Results**

Throughout the text G will denote a profinite group. We will sometimes use the notation $H \trianglelefteq_o G$ to denote that H is an open normal subgroup of G. The lower central word is given by $\gamma_1 = x_1$, $\gamma_n = [\gamma_{n-1}, x_n]$ for $n \geq 2$. We will denote by $X_n(G) = \{[g_1, g_2, \ldots, g_n] \,|\, g_i \in G, 1 \leq i \leq n\}$ the set of all γ_n-values in a group G and by $g^{X_n(G)}$ the set $\{y^{-1}gy \,|\, y \in X_n(G)\}$ for each $g \in G$. The verbal subgroup of G generated by $X_n(G)$ is the closed subgroup $\gamma_n(G)$, the n-th term of the lower central series of G. The notation $|X|$ will be used for the cardinality of X. For $A_1, A_2, \ldots, A_n$ subsets of G, we denote by $X_n(A_1, A_2, \ldots, A_n)$ the set of all γ_n-values $[a_1, a_2, \ldots, a_n]$ where $a_i \in A_i$ for $1 \leq i \leq n$.

Consider $I \subseteq \{1, 2, \ldots, n\}$ and $\bar{I} = \{1, 2, \ldots, n\} \backslash I$. For families of variable $(y_i)_{i \in I}, (z_i)_{i \in \bar{I}}$ we define

$$[y_i; z_i]_I = [u_1, u_2, \ldots, u_n], \text{ where } u_s = \begin{cases} y_s & \text{if } s \in I; \\ z_s & \text{if } s \notin I. \end{cases}$$

The notation extends to families $(A_i)_{i \in I}$, $(B_i)_{i \in \bar{I}}$ of subsets of G naturally by defining $X_n(A_i; B_i)_I$ as the set of all γ_n-values

$$[a_i; b_i]_I = [u_1, u_2, \ldots, u_n], \text{ where } u_s = \begin{cases} a_s \in A_s & \text{if } s \in I; \\ b_s \in B_s & \text{if } s \notin I. \end{cases}$$

The following results are consequences of [6, Corollary 2.6] and [7, Lemma 4.1].

http://dx.doi.org/10.22108/IJGT.2022.134785.1802

Lemma 2.1. *Let G be a profinite group and H be an open normal subgroup of G. If there exists elements $g, h, b_1, b_2, \dots, b_n \in G$ such that*

$$X_n(b_1H, \dots, b_nH) \subseteq C_{\gamma_n(G)}(g)h,$$

then, $X_n(H) \subseteq C_{\gamma_n(G)}(g)$.

Proof. Since

$$X_n(b_1H, \dots, b_nH) \subseteq C_{\gamma_n(G)}(g)h = hC_{\gamma_n(G)}(g)^h,$$

by [6, Corollary 2.6] we have that $X_n(H) \subseteq C_{\gamma_n(G)}(g)^h$. As $X_n(H)$ is closed under conjugation, we conclude that $X_n(H) \subseteq C_{\gamma_n(G)}(g)$. □

Lemma 2.2. *Let G be a profinite group, $g \in G$ and let H be an open normal subgroup of G. Suppose that I is a subset of $\{1, 2, \dots, n\}$ and assume that for every proper subset J of I we have*

$$X_n(G; H)_J \subseteq C_{\gamma_n(G)}(g).$$

Then, for every $g_i \in G$, $i \in I$, and $h_1, h_2, \dots, h_n \in H$, we have

$$[g_ih_i; h_i]_I \in C_{\gamma_n(G)}(g)[g_i; h_i]_I.$$

Proof. Let N be the normal subgroup generated by $X_n(G; H)_J$ for every proper subset J of I. By hypothesis, $N \leq C_{\gamma_n(G)}(g)$ so we can apply [7, Lemma 4.1] and the result follows. □

Baire's Category Theorem [14, Proposition 2.3.1] proves to be very useful in situations where we work with enumerable sets. Such applications can be found, for example, in [5, 17] and related articles. For simplicity and to steer clear of the continuum hypothesis, we will use the next proposition:

Proposition 2.3. [2, Proposition 2.1] *Let $\varphi : X \to Y$ be a continuous map between non-empty profinite spaces that is nowhere locally constant, that is, there exists no non-empty open subset $U \subseteq X$ such that $\varphi|_U$ is constant. Then $|\varphi(X)| \geq 2^{\aleph_0}$.*

The following results are consequences of [2, Lemma 3.3] and [2, Lemma 3.4].

Lemma 2.4. *Let G be a profinite group, $g \in G$ and let H be an open normal subgroup of G. Suppose that $|g^{X_n(G)}| < 2^{\aleph_0}$ for some positive integer n. If $I \subsetneq \{1, 2, \dots, n\}$ and for every proper subset $J \subsetneq I$ we have*

$$X_n(G; H)_J \subseteq C_{\gamma_n(G)}(g), \tag{2.1}$$

http://dx.doi.org/10.22108/IJGT.2022.134785.1802

then, for each arbitrary family $\mathbf{g} = (g_i)_{i \in I}$ *in* G*, there exists an open normal subgroup* $U = U_{\mathbf{g}}$ *of* G*, with* $U \subseteq H$*, such that*

$$X_n(g_i; U)_I \subseteq C_{\gamma_n(G)}(g).$$

Proof. Let $g \in G$ and n be a positive integer such that $|g^{X_n(G)}| < 2^{\aleph_0}$. Consider $\mathbf{g} = (g_i)_{i \in I}$ an arbitrary family in G and the following continuous map between non-empty profinite spaces

$$\varphi : H \times \cdots \times H \to \frac{\gamma_n(G)}{C_{\gamma_n(G)}(g)}, \ \ (h_1, \ldots, h_n) \mapsto C_{\gamma_n(G)}(g)[g_i h_i; h_i]_I.$$

By hypothesis, the image of φ contains less than $2^{\aleph_0}$ elements. By Proposition 2.3, there exists elements $b_1, b_2, \ldots, b_n \in H$ and an open normal subgroup $U = U_{\mathbf{g}}$ of G with $U \subseteq H$, such that, for every $u_1, \ldots, u_n \in U$, we have

$$[g_i b_i u_i; b_i u_i]_I \in C_{\gamma_n(G)}(g)[g_i b_i; b_i]_I.$$

In other words,

$$X_n(g_i b_i U; b_i U)_I \subseteq C_{\gamma_n(G)}(g)[g_i b_i; b_i]_I.$$

As $I \subsetneqq \{1, 2, \ldots, n\}$ we have by [6, Lemma 2.5]

$$X_n(g_i b_i U; U)_I \subseteq C_{\gamma_n(G)}(g). \tag{2.2}$$

On the other hand, note that $b_i U \subseteq H$, thus, it follows from (2.1) and by Lemma 2.2 that, for all $u_i \in U \subseteq H$ with $i = 1, 2, \ldots, n$, we have

$$[g_i b_i u_i; u_i]_I \in C_{\gamma_n(G)}(g)[g_i; u_i]_I. \tag{2.3}$$

By (2.2) and (2.3) we conclude that $X_n(g_i; U)_I \subseteq C_{\gamma_n(G)}(g)$, for each arbitrary family $\mathbf{g} = (g_i)_{i \in I}$ in G and $U = U_{\mathbf{g}} \trianglelefteq_o G$. □

3. Proof of the main results

We will start proving the Proposition 1.8 which we restate here for the reader's convenience.

Let G *be a profinite group and* $g \in G$*. Suppose that there are a positive integer* $n = n(g)$ *such that* $g^{X_n(G)}$ *contains less than* $2^{\aleph_0}$ *elements and an open normal subgroup* H *of* G *such that* $X_n(H) \subseteq C_{\gamma_n(G)}(g)$*. Then* $g^{X_n(G)}$ *is finite.*

http://dx.doi.org/10.22108/IJGT.2022.134785.1802

Proof. Let $g \in G$ and $n = n(g)$ be a positive integer such that $|g^{X_n(G)}| < 2^{\aleph_0}$. We will construct $V \trianglelefteq_\circ G$ such that, for every proper subset $I \subsetneqq \{1, 2, \dots, n\}$, we have

$$X_n(G; V)_I \subseteq C_{\gamma_n(G)}(g). \tag{3.1}$$

So, if T is a transversal to V in G, then it follows from (3.1) and by Lemma 2.2 that, for every $g_i \in T, v_i \in V, i = 1, 2, \dots, n$, we have

$$[g_1v_1, g_2v_2, \dots, g_nv_n] \in C_{\gamma_n(G)}(g)[g_1, g_2, \dots, g_n]. \tag{3.2}$$

But this shows that $g^{X_n(G)}$ is finite, because $G = \bigcup\{gV \mid g \in T\}$.

Thus, let $I \subsetneqq \{1, 2, \dots, n\}$. We prove by induction on $|I|$ that there exists $U_I \trianglelefteq_\circ G$ satisfying (3.1) and V will be the intersection of all theses subgroups U_I.

Note that, for $I = \emptyset$ we have $U_\emptyset = H$ and $X_n(H) \subseteq C_{\gamma_n(G)}(g)$ by hipothesis. Now, consider $|I| \geq 1$ and suppose that for each $J \subsetneqq I$ there exists $U_J \trianglelefteq_\circ G$ such that

$$X_n(G; U_J)_J \subseteq C_{\gamma_n(G)}(g).$$

Then, for $U = \bigcap\{U_J \mid J \subsetneqq I\} \trianglelefteq_\circ G$ we obtain

$$X_n(G; U)_J \subseteq C_{\gamma_n(G)}(g), \text{ for all } J \subsetneqq I. \tag{3.3}$$

Let R be a transversal for U in G and $\mathbf{g} = (g_i)_{i\in I}$ an arbitrary family of elements in R. By Lemma 2.4, there exists a subgroup $U_\mathbf{g} \trianglelefteq_\circ G$, with $U_\mathbf{g} \subseteq U$, such that $X_n(g_i; U_\mathbf{g})_I \subseteq C_{\gamma_n(G)}(g)$. Intersecting the finitely many groups $U_\mathbf{g}$ parameterized by $\mathbf{g}$, we obtain $U_I \trianglelefteq_\circ G$, with $U_I \subseteq U$, such that $X_n(g_i; U_I)_I \subseteq C_{\gamma_n(G)}(g)$ for all families $(g_i)_{i\in I}$ in R. Therefore, considering whatever $u_i \in U$ and $v_i \in U_I \subseteq U$, we have from (3.3) and by Lemma 2.2 that

$$[g_iu_i; v_i]_I \in C_{\gamma_n(G)}(g)[g_i; v_i]_I = C_{\gamma_n(G)}(g),$$

for all family $\mathbf{g} = (g_i)_{i\in I}$ in R. In other words,

$$\bigcup_{\mathbf{g}} X_n(g_iU; U_I)_I \subseteq C_{\gamma_n(G)}(g).$$

Since $G = \bigcup\{gU \mid g \in R\}$, we deduce that

$$X_n(G; U_I)_I = \bigcup_{\mathbf{g}} X_n(g_iU; U_I)_I \subseteq C_{\gamma_n(G)}(g).$$

The result follows. □

http://dx.doi.org/10.22108/IJGT.2022.134785.1802

Proposition 3.1. *Let G be a profinite group, $g \in G$ and n a positive integer such that $g^{X_n(G)}$ contains less than $2^{\aleph_0}$ elements. Then $g^{X_n(G)}$ is finite.*

Proof. Let $\mathbf{G} = G \times \cdots \times G$ and $g \in G$ and n a positive integer such that $g^{X_n(G)}$ contains less than $2^{\aleph_0}$ elements. Consider the following continuous map between profinite spaces

$$\varphi : \mathbf{G} \to \frac{\gamma_n(G)}{C_{\gamma_n(G)}(g)}, \ (g_1, g_2, \dots, g_n) \mapsto C_{\gamma_n(G)}(g)[g_1, g_2, \dots, g_n].$$

By hypothesis, the image of φ contains less than $2^{\aleph_0}$ elements. Then, by Proposition 2.3, there exists elements $b_1, b_2, \dots, b_n \in G$ and an open normal subgroup H of G such that, for every $h_1, h_2, \dots, h_n \in H$, we have

$$[b_1h_1, b_2h_2, \dots, b_nh_n] \in C_{\gamma_n(G)}(g)[b_1, b_2, \dots, b_n].$$

In other words,

$$X_n(b_1H, b_2H, \dots, b_nH) \subseteq C_{\gamma_n(G)}(g)[b_1, b_2, \dots, b_n].$$

By Lemma 2.1 we conclude $X_n(H) \subseteq C_{\gamma_n(G)}(g)$. It follows by Proposition 1.8 that $g^{X_n(G)}$ is finite, as we claimed. □

Now we are ready to prove the following theorem:

Theorem 3.2. *Let G be a profinite group and n a positive integer. If $g^{X_n(G)}$ contains less than $2^{\aleph_0}$ elements, for any $g \in G$, then $\gamma_{n+1}(G)$ is finite.*

Proof. By Proposition 3.1 we have that $g^{X_n(G)}$ is finite for any $g \in G$. Therefore, by Corollary 1.5 we conclude that $\gamma_{n+1}(G)$ is finite. □

The Theorem 3.2 can still be generalized in the sense that the integer n depends on the choice of each element $g \in G$. To prove this statement we need of the following two results.

Lemma 3.3. *Let G be a profinite group, m and n positive integers and $g \in G$ any element. Consider the set $\Delta_{m,n} = \{g \in G \mid |g^{X_n(G)}| \leq m\}$. Then $\Delta_{m,n}$ is closed.*

Proof. Let $l = \binom{m+1}{2}$ and $\mathbf{G}$ be the cartesian product of l copies of G. Define $Y = \{(g_1, g_2, \dots, g_l) \in \mathbf{G} \mid \exists\, i \text{ such that } g_i = 1\}$ and note that Y is closed because it is an union of kernels of the canonical projections. Now, for each $(m+1)$-tuple $(w_1, \dots, w_{m+1})$ of γ_n-values of G define the continuous map $\varphi = \varphi_{(w_1,\dots,w_{m+1})}$ from G to $\mathbf{G}$ by the rule

$$\varphi(g) = ([g, w_1w_2^{-1}], \dots, [g, w_1w_{m+1}^{-1}], [g, w_2w_3^{-1}], \dots, [g, w_mw_{m+1}^{-1}]).$$

We observe that the set $\Delta_{m,n}$ is the intersection of $\varphi^{-1}_{(w_1,\ldots,w_{m+1})}(Y)$ for all $(w_1,\ldots,w_{m+1})$. In fact, if $g\in\Delta_{m,n}$, then $|g^{X_n(G)}|\le m$. But this occurs if, and only if, for every $(m+1)$-tuple $(w_1,\ldots,w_{m+1})$ of γ_n-values of G we have at least one pair of distinct positive integers i,j such that $C_{\gamma_n(G)}(g)w_i = C_{\gamma_n(G)}(g)w_j$, that is, if and only if, $\varphi_{(w_1,\ldots,w_{m+1})}(g)\in Y$ for all $(w_1,\ldots,w_{m+1})$. □

Proposition 3.4. *Let G be a profinite group. Suppose that for each $g\in G$ there exists a positive integer $n=n(g)$ such that $g^{X_n(G)}$ is finite. Then G is a finite-by-nilpotent group.*

Proof. Let $g\in G$, $n=n(g)$ and $m=m(g)$ be positive integers such that $|g^{X_n(G)}|=m$. Consider the following set:

$$\Delta_{m,n}=\{g\in G\,|\,|g^{X_n(G)}|\le m\}.$$

By Lemma 3.3, $\Delta_{m,n}$ is closed. Since $G=\bigcup_{m,n}\Delta_{m,n}$, it follows by Baire Category Theorem [14, Proposition 2.3.1] that there are integers i,j such that $\Delta_{i,j}$ has non-empty interior. Hence, there are $a\in G$ and $H\trianglelefteq_\circ G$ with $Ha\subseteq\Delta_{i,j}$. Therefore, we deduce $|a^{X_j(G)}|\le i$ and $|(ha)^{X_j(G)}|\le i$ for all $h\in H$, in particular, $|h^{X_j(G)}|\le i^2$. As H has finite index in G, we can consider $g_1,g_2,\ldots,g_r$ as a right transversal for H in G. By hypothesis, for each $s=1,2,\ldots,r$, there are positive integers $n_s=n_s(g_s)$ such that $g_s^{X_{n_s}(G)}$ is finite. For $q=max\{n_s,s=1,2,\ldots,r\}$ we observe that $X_q(G)\subseteq X_{n_s}(G)$ for all $s=1,2,\ldots,r$. Hence $g_s^{X_q(G)}$ is finite for every $s=1,2,\ldots,r$. We can suppose that $q\ge j$, because if $q<j$ then $X_j(G)\subseteq X_q(G)$ and so $g_s^{X_j(G)}$ is finite for all $s=1,2,\ldots,r$. Since for each $x\in G$ there are $h\in H$ and $s\in\{1,2,\ldots,r\}$ such that $x=hg_s$, then $x^{X_q(G)}$ is finite because $(hg_s)^{X_q(G)}$ is finite since $X_q(G)\subseteq X_j(G)$ and $|h^{X_j(G)}|\le i^2$. Therefore, G is a profinite group such that there exists a positive integer q where $x^{X_q(G)}$ is finite for all $x\in G$. By Corollary 1.5 we conclude that $\gamma_{q+1}(G)$ is finite. In particular, G is finite-by-nilpotent as we claim. □

Now, we are ready to prove the following theorem:

Theorem 3.5. *Let G be a profinite group where for each $g\in G$ there exists a positive integer $n=n(g)$ such that $g^{X_n(G)}$ contains less than $2^{\aleph_0}$ elements. Then, G is finite-by-nilpotent.*

Proof. By Proposition 3.1 we have that $g^{X_n(G)}$ is finite for each $g\in G$ and $n=n(g)$. Therefore, by Proposition 3.4 we obtain that G is finite-by-nilpotent. □

A straightforward from Theorem 3.5 is the Theorem 1.7 that say:

A profinite group G is finite-by-nilpotent if, and only if, for each element $g\in G$, there exists a positive integer $n=n(g)$ such that $g^{X_n(G)}$ contains less than $2^{\aleph_0}$ elements.

http://dx.doi.org/10.22108/IJGT.2022.134785.1802

Acknowledgments

The author wish to thank CAPES-Brazil by support.

References

[1] E. Detomi, G. Donadze, M. Morigi and P. Shumyatsky, On finite-by-nilpotent groups, *Glasgow Mathematical Journal*, **63** (2021) 54–58.

[2] E. Detomi, B. Klopsch, P. Shumyatsky, Strong conciseness in profinite groups, *Journal of the London Math. Society*, **102** (2020), 977-993.

[3] E. Detomi and M. Morigi, On finite-by-nilpotent profinite groups, *Int. J. Group Theory*, **9** no. 4 (2020) 223–229.

[4] E. Detomi, M. Morigi and P. Shumyatsky, BFC-theorems for higher commutator subgroups, *Q. J. Math.*, **70** (2019) 849–858.

[5] E. Detomi, M. Morigi and P. Shumyatsky, On conciseness of words in profinite groups, *J. Pure Appl. Algebra*, **220** (2016) 3010–3015.

[6] E. Detomi, M. Morigi and P. Shumyatsky, On profinite groups with commutators covered by countably many cosets, *J. Algebra*, **508** (2018) 431–444.

[7] E. Detomi, M. Morigi and P. Shumyatsky, On profinite groups with word values covered by nilpotent subgroups, *Isr. J. Math.*, **226** (2018) 993–1008.

[8] E. Detomi, M. Morigi and P. Shumyatsky, Profinite groups with restricted centralizers of commutators, *Proc. R. Soc. Edinb., Sect. A, Math.*, **150** (2020) 2301–2321.

[9] G. Dierings and P. Shumyatsky, Groups with boundedly finite conjugacy classes of commutators, *Q. J. Math.*, **69** (2018) 1047–1051.

[10] S. Franciosi, F. De Giovani and P. Shumyatsky, On groups with finite verbal conjugacy classes, *Houston J. Math.*, **28** (2002) 683–689.

[11] R. M. Guralnick and A. Maroti, Average dimension of fixed point spaces with applications, *Adv. Math.*, **226** (2011) 298–308.

[12] B. H. Neumann, Groups covered by permutable subsets, *J. London Math. Soc.*, **29** (1954) 236–248.

[13] P. M. Neumann and M. R. Vaughan-Lee, An essay on BFC groups, *Proc. Lond. Math. Soc., III. Ser.*, **35** (1977) 213–237.

[14] L. Ribes and P. Zalesskii, *Profinite groups*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Berlin: Springer, (2010).

[15] D. Segal and A. Shalev, On groups with bounded conjugacy classes, *Q. J. Math., Oxf. II. Ser.*, **50** (1999) 505–516.

[16] A. Shalev, Profinite groups with restricted centralizers, *Proc. Amer. Math. Soc.*,**122** (1994) 1279–1284.

[17] P. Shumyatsky and W. da Silva, Engel conditions for soluble and pronilpotent groups, *Advances in Group Theory and Applications*, accepted for publication, (2022).

[18] J. Wiegold, Groups with boundedly finite classes of conjugate elements, *Proc. R. Soc. Lond., Ser. A*, **238** (1957) 389–401.

Wállef da Silva

Department of Mathematics, University of Brasília, 70910-900, Brasília-DF, Brazil

Email: wallef.silva@gmail.com

http://dx.doi.org/10.22108/IJGT.2022.134785.1802